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Regularity results and new perspectives for degenerate Kolmogorov equations

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Abstract

In this work, we are concerned with the regularity theory of strongly degenerate Kolmogorov equations and we also study a relativistic generalization of such equations. The Kolmogorov equation was first introduced by Kolmogorov in 1934 to study the time evolution of the density of a Brownian test particle in the phase space. It is a linear strongly degenerate second order PDE whose diffusion part is governed by the Laplace operator in a subset of the variables (*velocity variables*) coupled with a transport term that contains the directions of missing ellipticity (*position variables*). Such a drift term makes the equation non-symmetric, but at the same time it is responsible for the hypoelliptic properties of the operator.

The first part of this thesis is devoted to the investigation of Kolmogorov-type operators with regular coefficients. In Chapter 1, we revise some results on the classical regularity theory for Kolmogorov operators with constant or continuous coefficients, with a particular emphasis on their connection to Hörmander's theory of hypoellipticity. In Chapters 2 and 3, the regularity theory introduced in Chapter 1 is developed in two unexplored directions. On one hand, in Chapter 2 we extend a fundamental result of the classical regularity theory, namely Schauder estimates. More precisely, we prove that, if the operator satisfies Hörmander's hypoellipticity condition, and the right-hand side and the diffusion coefficients are Dini continuous, then the second order derivatives of the solution are Dini continuous as well. Additionally, we establish a new Taylor formula for classical solutions under minimal regularity assumptions. On the other hand, Chapter 3 is devoted to the proof of new pointwise regularity results for solutions to degenerate second order partial differential equations with constant coefficients.

The second part of this dissertation focuses on the weak regularity theory of degenerate Kolmogorov equations with discontinuous coefficients, which is nowadays the main focus of the research community. As the most recent developments in this framework have been established in the particular case of the kinetic Kolmogorov-Fokker-Planck equation, the aim of Chapter 4 is to extend some of these results to the ultraparabolic setting. More precisely, in Chapter 4 we prove a Harnack inequality and the Hölder continuity for weak solutions to the Kolmogorov equation with measurable coefficients, integrable lower order terms and nonzero source term. We then introduce a functional space, suitable for the study of weak solutions to Kolmogorov-type equations, that allows us to prove a new (weak) Poincaré inequality. The Harnack inequality contained in Chapter 4 is also crucial in Chapter 5, where we prove the existence of a fundamental solution Γ associated to the Kolmogorov operator, together with Gaussian upper and lower bounds for Γ.

Finally, in the last part of this work, we address a possible generalization of the kinetic Kolmogorov-Fokker-Planck equation, which is in accordance with the theory of special relativity. In particular, we explain why the operator proposed is the suitable relativistic generalization of the Fokker-Planck operator and we describe it as a Hörmander operator which is invariant with respect to Lorentz transformations. We subsequently start its systematic study in its appropriate framework of PDE and Hörmander's theory. The main results of this part are a Lorentz-invariant Harnack-type inequality and accurate asymptotic lower bounds for positive solutions to the equation. As a consequence, we finally obtain a lower bound for the density of the stochastic process associated to the relativistic operator.

Riassunto

In questa tesi ci concentriamo sulla teoria della regolarità di equazioni di Kolmogorov degeneri e studiamo una possibile generalizzazione relativistica di tali equazioni. L'equazione di Kolmogorov è stata introdotta da Kolmogorov nel 1934 per studiare l'evoluzione temporale della densità di una particella Browniana nello spazio delle fasi. Si tratta di una equazione differenziale alle derivate parziali lineare e fortemente degenere la cui diffusione è governata dal Laplaciano in un insieme di variabili (le cosiddette *variabili velocità*) accoppiato con un termine di trasporto che contiene le direzioni non ipoellittiche (le cosiddette *variabili posizione*). Tale termine di trasporto rende l'equazione non simmetrica ma è allo stesso tempo responsabile dell'ipoellitticità dell'operatore.

La prima parte di questo elaborato si concentra sullo studio di equazioni di tipo Kolmogorov con coefficienti regolari. Nel Capitolo 1, richiamiamo alcuni risultati riguardanti la teoria della regolarità classica di equazioni di Kolmogorov con coefficienti costanti o continui, con particolare enfasi sulla loro connessione alla teoria dell'ipoellitticità di Hörmander. In questa tesi, sviluppiamo tale teoria della regolarità in due direzioni inesplorate. Nel Capitolo 2, estendiamo un risultato fondamentale della teoria della regolarità classica, ossia le stime di Schauder. Più precisamente, dimostriamo che, se l'operatore verifica la condizione di ipoellitticità di Hörmander, e il membro di destra e i coefficienti di diffusione sono Dini continui, allora le derivate seconde della soluzione sono anch'esse Dini continue. Stabiliamo inoltre una nuova formula di Taylor per soluzioni classiche nelle ipotesi di regolarità minime. Il Capitolo 3 è invece dedicato alla dimostrazione di nuove stime di regolarità puntuale per soluzioni di equazioni di Kolmogorov con coefficienti di diffusione costanti.

La seconda parte di questa tesi si concentra sulla teoria della regolarità debole per equazioni di Kolmogorov con coefficienti discontinui, teoria che è al centro della ricerca attuale. Poiché i principali sviluppi in questo ambito riguardano il caso particolare dell'equazione di Fokker-Planck, l'obiettivo del Capitolo 4 è estendere tali risultati al caso dell'equazioni ultraparaboliche. In particolare, nel quarto capitolo, dimostriamo una disuguaglianza di Harnack e l'Hölderianità delle soluzioni deboli dell'equazione di Kolmgorov con coefficienti misurabili, termini del primo ordine integrabili e membro di destra non nullo. Successivamente, introduciamo uno spazio funzionale, appropriato per lo studio di soluzioni deboli di equazioni di tipo Kolmogorov, che ci permette di dimostrare una nuova disuguaglianza di Poincarè debole. La disuguaglianza di Harnack dimostrata nel Capitolo 4 si rivela cruciale anche nel capitolo successivo, dove dimostriamo l'esistenza di una soluzione fondamentale Γ associata all'operatore di Kolmogorov, insieme a stime gaussiane dal basso e dall'alto per Γ.

Nell'ultima parte di questo elaborato, studiamo infine una possibile generalizzazione dell'equazione di Fokker-Planck, che è in accordo con la teoria della relatività speciale. In particolare, spieghiamo perché l'operatore proposto rappresenta l'appropriata controparte relativistica dell'operatore di Fokker-Planck e lo descriviamo come un operatore di Hörmander che risulta essere invariante rispetto alle trasformazioni di Lorentz. Successivamente cominciamo lo studio sistematico di tale operatore nel contesto appropriato della teoria delle equazioni differenziali alle derivate parziali. I risultati principali di questa ultima parte sono una disuguaglianza di Harnack e un'accurata stima dal basso per soluzioni positive dell'equazione. Come conseguenza, otteniamo una stima dal basso per la densità del processo stocastico associato all'operatore relativistico.

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Contents

Introduction

In this thesis, we study a class of degenerate Partial Differential Equations of Kolmogorov-type employing new techniques. The simplest class of equations we are interested in was introduced by Kolmogorov [67] in the following form

$$\mathscr{K} u(p,y,t) := \Delta_p u(p,y,t) - \langle p, D_y u(p,y,t)\rangle - \partial_t u(p,y,t) = 0, \tag{1}$$

to describe the density u of particles having position $y \in \mathbb{R}^m$ and momentum $p \in \mathbb{R}^m$ at time t. Equation (1) is usually referred to as kinetic Kolmogorov equation or frictionless Fokker-Planck equation in the kinetic literature. It is derived from Langevin dynamics, as it is the partial differential equation satisfied by the transition density of the stochastic process solving

$$\begin{cases} dP_t = \sqrt{2}\, dW_t, \\ dY_t = P_t\, dt, \end{cases} \tag{2}$$

where $(W_t)_{t\geq 0}$ denotes an $m-$dimensional Wiener process.

The kinetic Kolmogorov equation (1) is a linear strongly degenerate second order PDE whose diffusion part is governed by the Laplace operator in some set of variables (the *velocity variables*) coupled with a transport term that contains the directions of missing ellipticity (the *position variables*). Such a drift term makes the equation non-symmetric, but at the same time it is responsible for the hypoelliptic properties of the operator.

In this thesis, we consider several generalizations of the operator in (1), which have applications in research areas as diverse as kinetic theory, probability theory and finance (see [16, 58, 97]).

In addition, the study of the above mentioned Kolmogorov-type operators is also very interesting from a mathematical point of view because of their connection to Hörmander theory, which establishes a link between the regularity of the operator and the Lie Group structure that leaves the operator invariant. Indeed, the prototype operator (1) lies in the class of Hörmander operators [54], which have been widely studied in the literature since the '60s. More precisely, the operators considered by Hörmander in his seminal work [54] are of the kind

$$\mathscr{L} = \sum_{k=1}^{m} X_k^2 + Y, \tag{3}$$

where m is a natural integer and X_k are smooth vector fields of the form

$$X_k = \sum_{j=1}^{N+1} b_{jk}(z)\,\partial_{z_j}, \qquad Y = \sum_{j=1}^{N+1} b_{jm+1}(z)\,\partial_{z_j} \quad k = 1, \dots, m, \tag{4}$$

with $b_{jk} \in C^\infty(\Omega)$ for every $j = 1, \dots, N+1$, $k = 1, \dots, m+1$ and Ω is any open subset of $\mathbb{R}^{N+1}$.

The main result of [54] is a sufficient condition for the hypoellipticity of (3), which has a quite intuitive geometric interpretation and can be understood as follows: If the missing directions in operator $\mathscr{L}$ can be recovered by the commutators of the generators X_i and Y, then, if the right-hand side of $\mathscr{L}u = f$ is smooth, the generators ensure that the solution u is also smooth in every direction. As the regularity properties of Hörmander's operators (3) are related to a Lie algebra, it became clear that the natural framework for the regularity theory of such operators is the non-Euclidean setting of Lie groups. After the work of Kolmogorov [67] where (1) was introduced, and Hörmander's celebrated article [54] on the hypoellipticity of second order degenerate linear operators, the regularity theory for operators that are invariant with respect to a Lie group structure has been widely developed by many authors. We quote here the seminal works by Folland [45], Folland and Stein [46], Rotschild and Stein [107], Nagel, Stein and Wainger [90]. We refer to the more recent monograph by Bonfiglioli, Lanconelli and Uguzzoni [20] for a comprehensive treatment of the recent achievements of the theory.

Another property that is essential for the study of the operator $\mathscr{K}$ in (1) is the invariance with respect to the non-commutative translation

$$(p, y, t) \circ (p_0, y_0, t_0) = (p_0 + p, y_0 + y + tp_0, t_0 + t), \qquad (p, y, t), (p_0, y_0, t_0) \in \mathbb{R}^{2m+1}. \tag{5}$$

Indeed, if $w(p, y, t) = u(p_0 + p, y_0 + y + tp_0, t_0 + t)$ and $g(p, y, t) = f(p_0 + p, y_0 + y + tp_0, t_0 + t,)$, then

$$\mathscr{K} u = f \quad \iff \quad \mathscr{K} w = g \quad \text{for every} \quad (p_0, y_0, t_0) \in \mathbb{R}^{2m+1}. \tag{6}$$

In several applications, where the couple (p, y) denotes the momentum and the position of a particle, the above operation is also known as *Galilean* change of variable.

Another remarkable property of operator $\mathscr{K}$ is its dilation invariance. More precisely, the operator $\mathscr{L}$ is invariant with respect to the following family of dilations

$$\delta_r(p, y, t) := (rp, r^3 y, r^2 t), \qquad r > 0, \tag{7}$$

in the following sense: if we define $w(p, y, t) = u(rp, r^3 y, r^2 t)$ and $g(p, y, t) = f(rp, r^3 y, r^2 t)$ we have that

$$\mathscr{K} u = f \quad \iff \quad \mathscr{K} w = r^2 g \quad \text{for every} \quad r > 0.$$

We remark that the dilatation in (7) has also a quite natural physical interpretation. Indeed, as in classical mechanics the velocity is proportional to the momentum, the term r^3 in front of y is due to the fact that the velocity v is the derivative of the position y with respect to time t.

As we will see in the sequel, this underlying invariance property plays a fundamental role in the study of operator $\mathscr{K}$, even though it does not hold true for every Kolmogorov operator (see Chapter 1), in contrast to what happens in the family of uniformly parabolic operators.

For a more exhaustive description of the mathematical properties of Kolmogorov operators, and of their applications, we refer to the survey article [8] by Anceschi and Polidoro and to its bibliography. Summarizing the above, the study of Kolmogorov-type operators is not only worthwhile because of their important applications but it is also highly interesting from a mathematical point of view, as discussed in Chapter 1.

In the first part of this thesis, which is constituted by Chapters 1-3, we mainly focus on the following class of Hörmander's operator

$$\mathscr{L}u := \sum_{j,k=1}^{m} a_{jk}\partial^2_{x_j x_k} u(x,t) + \sum_{j,k=1}^{N} b_{jk} x_k \partial_{x_j} u(x,t) - \partial_t u(x,t), \tag{8}$$

where $z = (x,t) = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}$, $A = (a_{jk})_{j,k=1,\ldots,m}$ and $B = (b_{jk})_{j,k=1,\ldots,N}$ are real valued matrices with constant coefficients, with A symmetric and strictly positive. In this thesis, we are mainly interested in the genuinely degenerate setting, i.e. in the case where $m < N$. Moreover, throughout this work, we will always assume that Hörmander's condition is satisfied. As we will see in Chapter 1, this boils down to assuming that the matrix B takes a suitable block form. Some other known results about Kolmogorov operators of the form (8) and their underlying Lie Group structure are presented in Chapter 1.

Constant-coefficients Kolmogorov operators like the one in (8) are naturally associated to linear stochastic differential equations. Specifically, let σ be a $N \times m$ constant matrix, B as in (8), and let $(W_t)_{t\geq 0}$ be as above a m-dimensional Wiener process. Then, if we denote by $(X_t)_{t\geq 0}$ the solution to the following N-dimensional Stochastic Differential Equation (SDE in short)

$$\begin{cases} dX_t = -BX_t\,dt + \sigma\,dW_t \\ X_{t_0} = x_0, \end{cases}$$

the *backward Kolmogorov operator* $\mathcal{K}_b$ of $(X_t)_{t\geq 0}$ acts on sufficiently regular functions u as follows

$$\mathcal{K}_b u(y,s) = \partial_s u(y,s) + \sum_{i,j=1}^{N} a_{ij}\partial^2_{y_i y_j} u(y,s) - \sum_{i,j=1}^{N} b_{ij} y_i \partial_{y_j} u(y,s).$$

where

$$A = \tfrac{1}{2}\sigma\sigma^T.$$

Moreover, the *forward Kolmogorov operator* $\mathcal{K}_f$ of $(X_t)_{t\geq 0}$ is the adjoint $\mathcal{K}_b^*$ of $\mathcal{K}_b$, that is

$$\mathcal{K}_f v(x,t) = -\partial_t v(x,t) + \sum_{i,j=1}^{N} a_{ij}\partial^2_{x_i x_j} v(x,t) + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} v(x,t) + \mathrm{tr}(B) v(x,t),$$

for sufficiently regular functions v. Note that $\mathcal{K}_f$ operator agrees with $\mathscr{L}$ in (8) up to a multiplication of the solution by $\exp(t\,\mathrm{tr}(B))$.

The regularity theory for classical solutions to Kolmogorov equations with regular coefficients like the one in (8) had been widely developed during the years, starting from the work by Lanconelli and Polidoro in [73]. In this thesis, we develop such a regularity theory in two unexplored directions.

On one hand, in Chapter 2, we deal for the first time with Dini continuous diffusion coefficients and Dini continuous right-hand side. In this setting, we derive Schauder estimates that extend the classical ones, where intrinsic Hölder continuous functions are considered. Moreover, we establish an instrinsic Taylor formula for solutions to $\mathscr{L}u = f$, which, besides being a key step in proving our Schauder estimates, is of independent interest, since it is derived under minimal regularity assumptions on u. In particular, we show that, in order to be approximated by its intrinsic Taylor polynomial of degree 2, u needs to satisfy the following requirements.

Space $C^2_{\mathscr{L}}(\Omega)$. *Let Ω be an open subset of $\mathbb{R}^{N+1}$. We say that a function u belongs to $C^2_{\mathscr{L}}(\Omega)$ if u, its derivatives $\partial_{x_i} u, \partial_{x_i x_j} u$ $(i,j = 1,\dots,m)$ and the Lie derivative Yu defined in* (1.1.5) *are continuous functions in Ω. We also require, for $i = 1,\dots,m$, that*

$$\lim_{s\to 0} \frac{\partial_{x_i} u(\exp(sB)x, t-s) - \partial_{x_i} u(x,t)}{|s|^{1/2}} = 0, \tag{9}$$

uniformly for every $(x,t) \in K$, where K is a compact set $K \subset \Omega$.

We remark that (9) can be interpreted as a condition on the second order mixed derivative of the form $Y^{1/2}\partial_{x_i} u$, since, mirroring the uniformly parabolic case, we regard the time derivative, here generalized by the Lie derivative Y, as a second order operator. It is clear that, if the derivative $Y\partial_{x_i} u$ exists, then the fractional derivative $Y\partial^{1/2}_{x_i} u$ is equal to 0.

Besides their intrinsic interest, the regularity estimates presented in Chapter 2 will also play a crucial role in proving the existence of a weak fundamental solution in the second part of this work. Finally, we remark that the results contained in Chapter 2 were presented for the first time by the author, Stroffolini and Polidoro in the paper [105].

On the other hand, in Chapter 3 we establish new pointwise regularity results for solutions to equation (8) in the case where the diffusion term is simply the Laplacian in velocity, following the recent paper [62] by the author and Ipocoana. In particular, for the first time in the ultraparabolic literature, we introduce a *pointwise modulus of mean oscillation* and we show that if the modulus of L^p-mean oscillation of $\mathscr{L}u$ at a point $z \in \mathbb{R}^{N+1}$ is Dini, then z is a Lebesgue point of continuity in L^p average for the second order derivatives

$\partial^2_{x_i x_j} u, i, j = 1, \ldots, m$, and the Lie derivative Yu. The method we follow in Chapter 3 has the advantage of being quite flexible, as shown in [77,86], where it was applied to study new regularity results for obtacle problem for the Laplace equation and the heat equation. The obstacle problem associated to (8) is not only fascinating for theoretical purposes but also for multiple applications. For example, this comes as an interest in mathematical finance to determine the arbitrage free price of options of American-type (see [96]). In recent years, many attempts have been made to study the existence and regularity of solutions to the obstacle problem in the framework of PDE (see [92] and the references therein). However, in the promising aforementioned results, they could only deal with classical solutions and continuous obstacles. For this reason, the results established in Chapter 3 aim at constituting a first step towards developing the weak regularity theory for solutions to the obstacle problem associated to Kolmogorov-type equations.

Besides its interesting mathematical structure and its connection to Hörmander's theory, operator (1) is also important as it serves as a prototype for a family of evolution equations arising in the kinetic theory of gases which take the following general form

$$\sum_{j=1}^{m} p_j \partial_{y_j} u(p,y,t) + \partial_t u(p,y,t) =: Yu(p,y,t) = \mathcal{J}(u). \tag{10}$$

In this case, Yu is the so called total derivative with respect to time in the phase space and $\mathcal{J}(u)$ is the collision operator, which can be either linear or non-linear. For instance, in the usual Fokker-Planck equation (cf. [37]) we have a linear collision operator of the form

$$\mathcal{J}(u) = \sum_{i,j=1}^{m} a_{ij}(p,y,t)\, \partial^2_{p_i,p_j} u(p,y,t) + \sum_{i=1}^{m} b_i(p,y,t)\, \partial_{p_i} u(p,y,t) + c(p,y,t)u(p,y,t). \tag{11}$$

In the second part of my thesis, which consists of Chapters 4-5, we focus on more general Kolmogorov operators which also include the usual Fokker-Planck equation, i.e. the collision operator (11). Specifically, we study Kolmogorov equations of the form

$$\begin{aligned} \mathscr{L}u(x,t) := & \sum_{i,j=1}^{m} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j} u(x,t)\right) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t) \\ & + \sum_{i=1}^{m} b_i(x,t)\partial_{x_i} u(x,t) + c(x,t)u(x,t) = f(x,t), \end{aligned} \tag{12}$$

where $z = (x,t) = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}$ and $1 \le m \le N$. We remark that we recover the Fokker-Planck equation from (12) by choosing $N = 2m$ and

$$B = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ -\mathbb{I}_m & \mathbb{O} \end{pmatrix}.$$

Another step, with which we further generalize the setting of the first part of this dissertation, is that we consider weak solutions to equation (12) in the sense of the following definition.

Weak solution. *We denote by $x^{(0)}$ the hypoelliptic variables and we let $\Omega = \Omega_m \times \Omega_{N-m+1} \subset \mathbb{R}^{N+1}$, where Ω_m is a bounded Lipschitz domain of $\mathbb{R}^m$ and Ω_{N-m+1} is a bounded Lipschitz domain of $\mathbb{R}^{N-m+1}$. We define $H^1_{x^{(0)}}$ as the Sobolev space of functions $u \in L^2(\Omega_m)$ with distribution gradient $D_m u$ lying in $(L^2(\Omega_m))^m$, i.e.*

$$H^1_{x^{(0)}} := \left\{ u \in L^2(\Omega_m) : \, D_m u \in (L^2(\Omega_m))^m \right\},$$

and we let $\mathcal{W}$ denote the closure of $C^\infty(\overline{\Omega})$ in the norm

$$\|u\|^2_{\mathcal{W}} = \|u\|^2_{L^2\left(\Omega_{N-m_0+1}; H^1_{x^{(0)}}\right)} + \|Yu\|^2_{L^2\left(\Omega_{N-m_0+1}; H^{-1}_{x^{(0)}}\right)}. \tag{13}$$

A function $u \in \mathcal{W}$ is a weak solution to (12) with source term $f \in L^2(\Omega)$ if for every non-negative test function $\varphi \in C^\infty_c(\Omega)$, we have

$$\int_\Omega -\langle ADu, D\varphi \rangle - uY\varphi + \langle b, Du \rangle \varphi + cu\varphi = \int_\Omega f\varphi,$$

where the operator $\mathscr{L}$ is written in its compact form, given that

- *the matrix $A(x,t) = (a_{ij}(x,t))_{1 \le i,j \le N}$ has real measurable entries, where a_{ij}, for every $i,j = 1, \ldots, m$, are the coefficients appearing in (12), while $a_{ij} \equiv 0$ whenever $i > m$ or $j > m$;*
- *the vector $b(x,t) := (b_1(x,t), \ldots, b_m(x,t), 0, \ldots, 0)$ contains the coefficients appearing in front of one of the lower order terms and the drift is defined as above, namely $Y = \sum_{i,j=1}^N b_{ij} x_j \partial_{x_i} - \partial_t$.*

The analysis of the regularity theory of degenerate Kolmogorov equations in divergence form with discontinuous coefficients has been an open problem for decades and is nowadays the main focus of the research community. The most recent developments in this framework have been established in the particular case of the kinetic Kolmogorov-Fokker-Planck equation. The aim of Chapter 4 is to extend some of these results to the ultraparabolic setting. In particular, following the results presented by the author and Anceschi in [11], we prove a Harnack inequality and the Hölder continuity for weak solutions to the Kolmogorov equation (12) with measurable coefficients, integrable lower order terms and nonzero source term. To the best of our knowledge, the Harnack inequality contained in Chapter 4 is the first result of this kind available for weak solutions to (12). We then introduce a suitable functional space $\mathcal{W}$ (see (13)), which mirrors the classical H^1 space for uniformly elliptic equations and therefore seems to be the most natural framework for studying the weak regularity theory for operator $\mathscr{L}$. In particular, we extend the functional setting first formally proposed by Armstrong and Mourrat in [3] for the study of the kinetic Kolmogorov-Fokker-Planck equation. To be more precise, we show that it is sufficient to require that the drift term Yu belongs to the space $L^2_{y,t}H^{-1}_{x^{(0)}}$ to derive a new (weak) Poincaré inequality. Here $x^{(0)}$ denotes, as above, the hypoelliptic variables, while y describes the directions of missing hypoellipticity. In this sense, we extend the classical ultraparabolic literature, where the drift Yu was always assumed to be in L^2.

Another motivation behind our studies is the need to determine which are the lowest possible integrability assumptions for c, b and f that allow us to prove $L^2 - L^\infty$ estimates and a Harnack inequality for weak solutions. Indeed, it is still an open problem in the ultraparabolic literature whether the optimal regularity for c, b and f is the hypoelliptic counterpart of the parabolic homogeneous dimension $\frac{N}{2}$, namely $\frac{Q+2}{2}$ where Q is defined in (1.1.17). In particular, our attention is focused on the behavior of the term b, which plays an important role in some applications, such as the Mean Field Games theory. Indeed, a Harnack inequality for weak solutions is the fundamental ingredient in the analysis of the maximal L^p regularity and well-posedness theory for Mean Field systems with degenerate diffusion, which were studied in the parabolic setting [34] and only very recently there has been a first attempt to consider the ultraparabolic setting [44].

Harnack inequalities are also important due to their connection to Gaussian lower bounds for the fundamental solution. Indeed, since the work of Moser [87], it became clear that the proof of the lower bound relies on the repeated application of the Harnack inequality. In Chapter 5, we therefore take advantage of the Harnack inequality established in Chapter 4 to prove Gaussian lower bounds for a fundamental solution Γ associated to operator $\mathscr{L}$. Following then Aronson's method [14], we establish the corresponding Gaussian upper bound. Finally, we conclude Chapter 5 by proving the existence of a weak fundamental solution relevant to operator (12). Our approach is based on a limiting procedure whose convergence is ensured by the Schauder types estimates obtained in Chapter 2. The results are obtained by the author in collaboration with Anceschi and are contained in the recent paper [12].

Finally, in the last part of this work, namely Chapter 6, we address a possible relativistic generalization of equation (1), namely

$$\mathscr{L}u(p,y,t) = \sqrt{|p|^2+1}\,\mathrm{div}_p\left(\mathcal{D}\,D_p u\right) - \langle p, D_y u\rangle - \sqrt{|p|^2+1}\,\partial_t u = 0, \tag{14}$$

where $(p,y,t) \in \mathbb{R}^{2m+1}$ and $\mathcal{D}$ is the *relativistic diffusion matrix* given by

$$\mathcal{D} = \frac{1}{\sqrt{|p|^2+1}}\left(\mathbb{I}_m + p\otimes p\right).$$

Here and in the following, $\mathbb{I}_m$ denotes the $m \times m$ identity matrix and $p \otimes p = (p_i p_j)_{i,j=1,\dots,m}$. Indeed, a questionable feature of (1) is that its diffusion term $\Delta_p u$ operates with infinite velocity, as in classical mechanics the velocity is proportional to the momentum. In this case, if the initial distribution $u(p,y,0)$ is compactly supported in y, there would be instantaneously a non-zero probability to find particles everywhere in space. This feature is clearly incompatible with the law that prevents particles from moving faster than light.

We believe that the operator introduced in (14) is the suitable generalization of (1) in view of its invariance properties. Indeed, mirroring the invariance of the non-relativistic operator with respect to Galilean transformations (5), operator $\mathscr{L}$ in (14) is invariant with respect to the equivalent relativistic transformations, namely the Lorentz transformations.

As a complete mathematical characterization of the operator in (14) is not yet available, Chapter 6 is dedicated to its systematic study in the appropriate framework of PDE theory. In particular, we place for the first time operator (14) in Hörmander's theory and construct its invariance group. Relying on classical results for operator (1), we are then able to derive asymptotic bounds for the fundamental solution associated to (14). Those results are presented by the author, Anceschi and Polidoro in the recent preprint [10].

Part I

Regular coefficients

The first part of this thesis addresses the study of Kolmogorov operators of the form

$$\mathscr{L} u := \sum_{i,j=1}^{m} a_{ij}(x,t)\partial^2_{x_i x_j} u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t), \tag{I.1}$$

where $z = (x,t) = (x_1, \dots, x_N, t) \in \mathbb{R}^{N+1}$ and $1 \le m \le N$, $B := (b_{ij})_{i,j=1,\dots,N}$ has real constant entries and the coefficients a_{ij}'s are regular, meaning that they are at least Dini continuous. Moreover, we always assume that B takes a suitable block form (see equation (1.1.2)) that ensures that the first order part of $\mathscr{L}$ induces a strong regularizing property, namely that $\mathscr{L}$ is hypoelliptic.

When the right-hand side of (I.1) is continuous, the solutions to $\mathscr{L} u = f$ are called classical because their degree of regularity corresponds to the one imposed by the equation. The study of the classical regularity theory of the class of operators (I.1) was investigated in depth over the years. In Chapter 1, we revise some known results about operators of the form (I.1) that will be useful in the forthcoming chapters. In particular, we explain how to write the operators in (I.1) with constant coefficients as Hörmander's operators (3) and we construct their invariance group. Finally, Section 1.2 of Chapter 1 collects some classical results about Kolmogorov-type operators in the more general case where the coefficients $a_{ij}(x,t)$ are assumed to be Hölder continuous. As in the parabolic setting, the classical regularity theory was developed for coefficients belonging to suitable Hölder spaces (see Definition 1.2.1 below).

In Chapter 2, we relax the regularity assumptions on the second order coefficients, by introducing a new definition of Dini continuity naturally associated to the Lie Group structure that leaves operator (I.1) invariant. Under these assumptions, we extend a fundamental result of the classical regularity theory, namely Schauder estimates. To be more precise, we prove that, if operator $\mathscr{L}$ satisfies Hörmander's hypoellipticity condition, and f and a_{ij}'s are Dini continuous functions, then the second order derivatives of the solution u to the equation $\mathscr{L} u = f$ are Dini continuous functions as well. Additionally, we establish a new Taylor formula for classical solutions to $\mathscr{L} u = f$ under minimal regularity assumptions on u. These results are the outcome of a scientific collaboration that started in 2020 between the author, Polidoro and Stroffolini and are published in [105].

In Chapter 3, we present some regularity results obtained in collaboration with Ipocoana. In contrast to Chapter 2, we assume the coefficients to be constant but we relax the regularity of the right-hand side, allowing it to be in L^p. More precisely, we establish new pointwise regularity results for solutions to degenerate second order partial differential equations of the form (I.1), when A is the identity matrix. In particular, we show that if the modulus of L^p-mean oscillation of $\mathscr{L} u$ at the origin is Dini, then the origin is a Lebesgue point of continuity in L^p average for the second order derivatives $\partial^2_{x_i x_j} u$, $i, j = 1, \dots, m$, and the Lie derivative $\left(\sum_{i,j=1}^N b_{ij} x_j \partial_{x_i} - \partial_t\right) u$. Moreover, we provide a Taylor-type expansion up to second order with an estimate of the rest in L^p norm. We observe that, although we consider the regularity problem for weak solutions to Kolmogorov operators in the framework of the Sobolev spaces, our procedure is basically pointwise and our approach follows the lines of regularity theory for classical solutions rather than the ones for weak solutions. We finally remark that the results of Chapter 3 are presented in the paper [62].

Chapter 1

Kolmogorov-type operators: an overview and some classical results

This introductory chapter collects some known results about degenerate Kolmogorov-type equations and provides an insight into their applications. Moreover, we explain the importance of Kolmogorov-type operators in connection to Hörmander's theory of hypoellipticity. Throughout this chapter, we mainly focus on Kolmogorov operators with constant diffusion coefficients. Indeed, such operators, together with their invariant Lie group structure, are the starting point to study operators with variables coefficients, which will be the content of the forthcoming chapters. Finally, we recall some known results about Kolmogorov-type operators with Hölder continuous coefficients.

The simplest Kolmogorov equation was introduced by Kolmogorov [67] in 1934 as follows

$$\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial p^2}(p,y,t) = p\frac{\partial u}{\partial y}(p,y,t) + \frac{\partial u}{\partial t}(p,y,t), \qquad (p,y,t) \in \mathbb{R}^3, \tag{1.0.1}$$

and it is strongly degenerate as the second order derivative is only taken with respect to the momentum variable. In the Introduction of this work, we have already introduced equation (1.0.1) for $\sigma = \sqrt{2}$ and discussed its link to kinetic theory of gases. We here explain in more detail its physical meaning and its connection to Langevin processes. From the physical point of view, Fokker-Planck equations like the one in (1.0.1) provide a continuous description of the dynamics of the distribution of Brownian test particles immersed in a fluid in thermodynamical equilibrium. More precisely, the distribution function u of a test particle evolves according to the linear Fokker-Planck equation defined in (1.0.1), provided that the test particle is much heavier than the molecules of the fluid and that there is no friction. In particular, equation (1.0.1) is the backward Kolmogorov equation of the Langevin process, i.e. the particle whose location in the phase space is (P_t, Y_t) evolves as

$$\begin{cases} P_t = p_0 + \sigma W_t, \\ Y_t = y_0 + \int\limits_0^t P(s)ds, \end{cases} \tag{1.0.2}$$

where $(W_t)_{t\geq 0}$ denotes a 1−dimensional Wiener process. We recall that (1.0.2) is usually referred to as *time-integrated Brownian motion.*

In his seminal paper [67], Kolmogorov provided us with the explicit expression of the density $\Pi = \Pi(t, p, y, p_0, y_0)$ of (1.0.1)

$$\Pi(t,p,y,p_0,y_0) = \frac{\sqrt{3}}{2\pi t^2} \exp\left(-\frac{(p-p_0)^2}{t} - 3\frac{(p-p_0)(y-y_0-tp_0)}{t^2} - 3\frac{(y-y_0-ty_0)^2}{t^3}\right) \qquad t>0, \tag{1.0.3}$$

and pointed out that it is a smooth function despite the strong degeneracy of equation (1.0.1). As suggested by the smoothness of the density Π, operator $\mathscr{L}$ associated to equation (1.0.1)

$$\mathscr{L} := \tfrac{1}{2}\sigma^2\partial_{vv} - v\partial_y - \partial_t, \tag{1.0.4}$$

is hypoelliptic, in the sense of the following definition, that we state for a general second order differential operator $\mathscr{L}$ acting on an open subset Ω of $\mathbb{R}^N$.

Definition 1.0.1 (Hypoellipticity)**.** An operator $\mathscr{L}$ is *hypoelliptic* if, for every distributional solution $u \in L^1_{\text{loc}}(\Omega)$ to equation $\mathscr{L}u = f$, we have that

$$f \in C^\infty(\Omega) \quad \Rightarrow \quad u \in C^\infty(\Omega). \tag{1.0.5}$$

Hörmander considered the operator $\mathscr{L}$ defined in (1.0.4) as a prototype for the family of hypoelliptic operators studied in his seminal work [54]. Specifically, the operators considered by Hörmander are the ones that can be written in form (3), i.e. as a sum of squares of smooth vector fields plus a drift term. As already mentioned in the Introduction of this work, in [54] Hörmander studied a sufficient condition for the hypoellipticity of $\mathscr{L}$. We here recall its precise statement, which requires some notation. Given two vector fields Z_1, Z_2, the commutator of Z_1 and Z_2 is the vector field:

$$[Z_1, Z_2] = Z_1\, Z_2 - Z_2\, Z_1.$$

Moreover, we recall that $\text{Lie}(X_1, \ldots, X_m, Y)$ is the Lie algebra generated by the vector fields $X_1, \ldots, X_m, Y$ and their commutators.

Hörmander's Rank Condition. *Suppose that*

$$\operatorname{rank} \text{Lie}(X_1, \ldots, X_m, Y)(z) = N + 1 \qquad \textit{for every} \quad z \in \Omega. \tag{1.0.6}$$

Then the operator $\mathscr{L}$ defined in (3) *is hypoelliptic in* Ω*,*

Let us consider again the operator $\mathscr{L}$ defined in (1.0.4) with $\sigma = \sqrt{2}$ to simplify the notation. $\mathscr{L}$ can be written in the form (3) if we choose

$$X = \partial_p \sim (0,1,0)^T, \qquad Y = -p\partial_y - \partial_t \sim (-1,0,p)^T,$$

and the Hörmander's rank condition is satisfied, as

$$[X,Y] = XY - YX = \partial_y \sim (0,0,1)^T.$$

1.1 Kolmogorov-type operators with constant coefficients

In the first part of this thesis, we consider Kolmogorov-type operators with constant coefficients of the form

$$\mathscr{L} := \sum_{i,j=1}^{m} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i} - \partial_t, \tag{1.1.1}$$

where $(x,t) \in \mathbb{R}^{N+1}$, and $1 \le m \le N$. The matrices $A := (a_{ij})_{i,j=1,\dots,m}$ and $B := (b_{ij})_{i,j=1,\dots,N}$ have real constant entries.

As in the simplest case (1.0.1), operator $\mathscr{L}$ can be strongly degenerate, meaning $m < N$. The hypoellipticity of the more general operator $\mathscr{L}$ in (1.1.1) can be stated in terms of suitable structural conditions on the matrices A and B. Indeed, by [73, Propositions 2.1 and 2.2], Hörmander's rank condition (1.0.6) is equivalent to the following assumption.

[H.1] The matrix A is symmetric and strictly positive, up to a change of basis the matrix B has the form

$$B = \begin{pmatrix} B_{0,0} & B_{0,1} & \dots & B_{0,\kappa-1} & B_{0,\kappa} \\ B_1 & B_{1,1} & \dots & B_{1,\kappa-1} & B_{1,\kappa} \\ \mathbb{O} & B_2 & \dots & B_{2,\kappa-1} & B_{2,\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & B_{\kappa,\kappa} \end{pmatrix} = \begin{pmatrix} * & * & \dots & * & * \\ B_1 & * & \dots & * & * \\ \mathbb{O} & B_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & * \end{pmatrix} \tag{1.1.2}$$

where every block B_j is an $m_j \times m_{j-1}$ matrix of rank m_j with $j = 1, 2, \dots, \kappa$. Moreover, the m_js are positive integers such that

$$m_0 \ge m_1 \ge \ldots \ge m_\kappa \ge 1, \quad \text{and} \quad m_0 + m_1 + \ldots + m_\kappa = N.$$

We agree to let $m_0 := m$ to have a consistent notation, moreover $\mathbb{O}$ denotes a block matrix whose entries are zeros, whereas the coefficients of the blocks "$*$" are arbitrary.

We remark that, if $\mathscr{L}$ is uniformly parabolic (i.e. $m = N$ and $B \equiv \mathbb{O}$), then assumption [**H.1**] is clearly satisfied. Indeed, in this case operator $\mathscr{L}$ is simply the heat operator, which is known to be hypoelliptic. However, in this thesis, we are mainly interested in the case where $m < N$ and B is non-trivial.

Under assumption **[H.1]**, operator $\mathscr{L}$ in (1.1.1) can be written in the form (3). Indeed, if the constant matrix A is symmetric and positive, then there exists a symmetric and positive matrix $A^{1/2} = (\overline{a}_{ij})_{i,j=1,\dots,m}$ such that $A = A^{1/2}A^{1/2}$. As a consequence, we can write $\mathscr{L}$ in terms of vector fields as follows

$$\mathscr{L} = \sum_{i=1}^{m} X_i^2 + Y, \tag{1.1.3}$$

where

$$X_i := \sum_{j=1}^{m} \overline{a}_{ij}\partial_{x_j}, \quad i = 1, \dots, m, \qquad Y := \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i} - \partial_t = \langle Bx, D\rangle - \partial_t. \tag{1.1.4}$$

Here and in the sequel, Yu will be understood as the *Lie derivative*

$$Yu(x,t) := \lim_{s\to 0} \frac{u(\exp(sB)x, t-s) - u(x,t)}{s}. \tag{1.1.5}$$

We observe that Yu is the derivative of u along the characteristic trajectory of Y, if we identify the directional derivative Y with the vector valued function $Y(x,t) = (Bx, -1)$. In the sequel, Y will also be regarded as *drift term*.

1.1.1 Lie Group Invariance

In this subsection, we focus on the non-Euclidean structure associated to hypoelliptic Kolmogorov operators of the form (1.1.1). Indeed, it is known that the natural geometry when studying operator $\mathscr{L}$ is determined by a suitable homogeneous Lie group structure on $\mathbb{R}^{N+1}$. More precisely, as first observed by Lanconelli and Polidoro in [73], operator $\mathscr{L}$ is invariant with respect to left translation in the group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$, where the group law is defined by

$$(x,t) \circ (\xi,\tau) = (\xi + E(\tau)x, t+\tau), \qquad (x,t), (\xi,\tau) \in \mathbb{R}^{N+1}, \tag{1.1.6}$$

and

$$E(s) = \exp(-sB), \qquad s \in \mathbb{R}. \tag{1.1.7}$$

Then $\mathbb{K}$ is a non-commutative group with zero element $(0,0)$ and inverse

$$(x,t)^{-1} = (-E(-t)x, -t). \tag{1.1.8}$$

For a given $\zeta \in \mathbb{R}^{N+1}$ we denote by ℓ_ζ the left traslation on $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$ defined as follows

$$\ell_\zeta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}, \quad \ell_\zeta(z) = \zeta \circ z. \tag{1.1.9}$$

Then the vector fields $X_1, \dots, X_m$ and Y are left-invariant, with respect to the group law (1.1.6), in the sense that

$$X_j \left(u(\zeta \circ \cdot)\right) = \left(X_j u\right)(\zeta \circ \cdot), \quad j = 1, \dots, m, \qquad Y\left(u(\zeta \circ \cdot)\right) = \left(Y u\right)(\zeta \circ \cdot), \tag{1.1.10}$$

for every $\zeta \in \mathbb{R}^{N+1}$ and every u sufficiently smooth. Hence, in particular,

$$\mathscr{L} \circ \ell_\zeta = \ell_\zeta \circ \mathscr{L} \quad \text{or, equivalently,} \quad \mathscr{L}\left(u(\zeta \circ \cdot)\right) = \left(\mathscr{L} u\right)(\zeta \circ \cdot).$$

Among the class of Kolmogorov operators satisfying assumption **[H.1]**, a central role is played by the ones which are additionally invariant with respect to a certain family of dilations $(\delta_r)_{r>0}$. We say that an operator $\mathscr{L}$ satisfying assumption **[H.1]** is invariant with respect to $(\delta_r)_{r>0}$ if

$$\mathscr{L}\left(u \circ \delta_r\right) = r^2 \delta_r \left(\mathscr{L} u\right), \quad \text{for every} \quad r > 0, \tag{1.1.11}$$

for every function u sufficiently smooth. As with the hypoellipticity, this invariance property can also be read in the expression of the matrix in B in (1.1.2). More precisely, in [73, Proposition 2.2], it is proved that operator $\mathscr{L}$ is invariant with respect to $(\delta_r)_{r>0}$ if, and only if, the matrix B in (1.1.2) agrees with B_0 defined as:

$$B_0 = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & \mathbb{O} \end{pmatrix} \tag{1.1.12}$$

In other words, every block denoted by $*$ in (1.1.2) has zero entries. In this case, the dilation is defined for every positive r as

$$\delta_r := \text{diag}(r\mathbb{I}_m, r^3\mathbb{I}_{m_1}, \dots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}, r^2), \tag{1.1.13}$$

where $\mathbb{I}_k$, $k \in \mathbb{N}$, is the k-dimensional unit matrix. It is also useful to denote by $\left(\delta_r^0\right)_{r>0}$ the family of spatial dilations defined as

$$\delta_r^0 = \text{diag}(r\mathbb{I}_m, r^3\mathbb{I}_{m_1}, \dots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}) \quad \text{for every } r > 0. \tag{1.1.14}$$

In the sequel, we also work with operators which are invariant with respect to (1.1.13), namely with operators that satisfy the following assumption.

[H.2] $\mathscr{L}_0$ is hypoelliptic and δ_r-homogeneous of degree two with respect to the family of dilations $(\delta_r)_{r>0}$ defined in (1.1.13).

Owing to (1.1.13), we now recall the definition of *Homogeneous Lie group.*

Definition 1.1.1 (Homogeneous Lie group.)**.** *If the matrix B has the form* (1.1.12), *we say that the following structure*

$$\mathbb{K}_0 = \left(\mathbb{R}^{N+1}, \circ, (\delta_r)_{r>0}\right) \tag{1.1.15}$$

is a homogeneous Lie group. In this case, the following distributive property holds

$$\delta_r(\zeta \circ z) = (\delta_r(\zeta)) \circ (\delta_r(z)), \qquad \delta_r(z^{-1}) = (\delta_r(z))^{-1}. \tag{1.1.16}$$

Remark 1.1.2. The presence of the exponents $1, 3, \ldots, 2\kappa+1, 2$ in the matrix δ_r in (1.1.13) can be understood as follows. The dilation is clearly uniformly parabolic in the first m coordinates of $\mathbb{R}^N$ and in time, as $\mathscr{L}$ is non-degenerate with respect to $x_1, \ldots, x_m$. The exponents relevant to the other coordinates can be explained while checking Hörmander's condition. For instance, let us consider the Kolmogorov operator

$$\mathscr{L} = \partial^2_{x_1x_1} + x_1\partial_{x_2} + x_2\partial_{x_3} - \partial_t = X_1^2 + Y.$$

Hörmander condition is satisfied if we have $\kappa = 2$ commutators $\partial_{x_2} = [X_1, Y] = X_1Y - YX_1$ and $\partial_{x_3} = [[X_1, Y], Y]$. Because Y needs to be considered as a second order derivative, we have that ∂_{x_2} and ∂_{x_3} are derivatives of order 3 and 5, respectively. On the other hand, the matrices A, B and δ^0_r associated to this operator are

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \delta^0_r = \begin{pmatrix} r & 0 & 0 \\ 0 & r^3 & 0 \\ 0 & 0 & r^5 \end{pmatrix}.$$

The same argument can be applied to operators that need $\kappa > 2$ steps to satisfy Hörmander's rank condition.

Example 1.1.3. A very simple example of a Kolmogorov operator which is not dilation invariant is given by the Ornstein-Uhlenbeck operator $\mathscr{L} = \Delta - \langle x, D\rangle - \partial_t$. In this case, we usually take advantage of the parabolic dilations $\delta_r(x,t) = (rx, r^2t)$.

The integer numbers

$$Q := m_0 + 3m_1 + \ldots + (2\kappa+1)m_k, \quad \text{and} \quad Q+2 \tag{1.1.17}$$

will be named *spatial homogeneous dimension of* $\mathbb{R}^N$ *with respect to* $(\delta^0_r)_{r>0}$, and *homogeneous dimension of* $\mathbb{R}^{N+1}$ *with respect to* $(\delta_r)_{r>0}$, because we have that

$$\det\, \delta^0_r = r^Q \quad \text{and} \quad \det\, \delta_r = r^{Q+2} \qquad \text{for every } r > 0.$$

We remark that the homogeneous dimension Q in (1.1.17) is the hypoelliptic counterpart of the space dimension N usually considered in the parabolic setting, see for instance [91].

We conclude this subsection introducing a semi-norm associated to the family of dilations $(\delta_r)_{r>0}$ in (1.1.13) and a quasi-distance which is invariant with respect to the group operation in (1.1.6). We first rewrite the matrix δ_r with the equivalent notation

$$\delta_r := \mathrm{diag}(r^{\alpha_1}, \dots, r^{\alpha_N}, r^2), \tag{1.1.18}$$

where $\alpha_1, \dots, \alpha_{m_0} = 1, \alpha_{m_0+1}, \dots, \alpha_{m_0+m_1} = 3, \alpha_{N-m_\kappa}, \dots, \alpha_N = 2\kappa + 1$.

Definition 1.1.4. For every $(x,t) \in \mathbb{R}^{N+1}$ we set

$$\|(x,t)\|_{\mathbb{K}} := \max\left\{ |x_1|^{\frac{1}{\alpha_1}}, \dots, |x_N|^{\frac{1}{\alpha_N}}, |t|^{\frac{1}{2}} \right\}. \tag{1.1.19}$$

We observe that the semi-norm is homogeneous of degree 1 with respect to the family of dilations $(\delta_r)_{r>0}$, namely $\|\delta_r(x,t)\|_{\mathbb{K}} = r\|(x,t)\|_{\mathbb{K}}$ for every $r > 0$ and $(x,t) \in \mathbb{R}^{N+1}$. Moreover, the following pseudo-triangular inequality holds: for every bounded set $H \subset \mathbb{R}^{N+1}$ there exists a positive constant $\mathbf{c}_H$ such that

$$\|(x,t)^{-1}\|_{\mathbb{K}} \le \mathbf{c}_H \|(x,t)\|_{\mathbb{K}}, \qquad \|(x,t) \circ (\xi,\tau)\|_{\mathbb{K}} \le \mathbf{c}_H (\|(x,t)\|_{\mathbb{K}} + \|(\xi,\tau)\|_{\mathbb{K}}), \tag{1.1.20}$$

for every $(x,t), (\xi,\tau) \in H$. Starting from the homogeneous norm in (1.1.19), we now want to define a *quasi-distance* $d_{\mathbb{K}}$ which is invariant with respect to the left translation (1.1.9). To this end, we set

$$d_{\mathbb{K}}((x,t),(\xi,\tau)) := \|(\xi,\tau)^{-1} \circ (x,t)\|_{\mathbb{K}}, \qquad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}. \tag{1.1.21}$$

It is clear that, by definition, the distance in (1.1.21) satisfies

$$d_{\mathbb{K}}((x,t),(\xi,\tau)) = d_{\mathbb{K}}((y,s)\circ(x,t),(y,s)\circ(\xi,\tau)), \qquad (y,s),(x,t),(\xi,\tau) \in \mathbb{R}^{N+1},$$

i.e. it is left invariant with respect to translations on the group $\mathbb{K}$. In addition, we remark that, from (1.1.20), it directly follows

$$d_{\mathbb{K}}((x,t),(\xi,\tau)) \le \mathbf{c}_H (d_{\mathbb{K}}((x,t),(y,s)) + d_{\mathbb{K}}((y,s),(\xi,\tau))),$$

for every $(x,t),(\xi,\tau),(y,s) \in \mathbb{R}^{N+1}$.

Remark 1.1.5. Since every norm is equivalent to any other in $\mathbb{R}^{N+1}$, other definitions have been used in the literature. For instance, in [82], the following one is chosen. For every $z = (x_1, \dots, x_N, t) \in \mathbb{R}^{N+1} \setminus \{0\}$ the norm of z is the unique positive solution r to the following equation

$$\frac{x_1^{\alpha_1}}{r^{2\alpha_1}} + \frac{x_2^{\alpha_2}}{r^{2\alpha_2}} + \ldots + \frac{x_N^{\alpha_N}}{r^{2\alpha_N}} + \frac{t^2}{r^4} = 1, \tag{1.1.22}$$

where the numbers α_j were defined in (1.1.18).

Another equivalent definition is the following: for every $z = (x,t) \in \mathbb{R}^{N+1}$ we set

$$\|z\|_{\mathbb{K}} = |t|^{\frac{1}{2}} + |x|_{\mathbb{K}}, \qquad |x|_{\mathbb{K}} = \sum_{j=1}^{N} |x_j|^{\frac{1}{\alpha_j}}. \tag{1.1.23}$$

In the sequel, we make use of these equivalent definitions of semi-norm $\| \cdot \|_{\mathbb{K}}$, choosing the one which is more suitable to the context.

Remark 1.1.6. An important property linking the two structures (translations and dilations) is the following (see [73, Remark 2.1])

$$E_0(r^2 t) = \delta_r^0\, E_0(t)\, \delta_{1/r}^0, \quad \forall r > 0, t \in \mathbb{R},$$

where $E_0(s) = \exp(-sB_0)$ with B_0 defined in (1.1.12). The previous equality implies that

$$\det E_0(r^2 t) = \det E_0(t), \quad \forall r > 0, t \in \mathbb{R},$$

and, for $r \to 0$, we have

$$1 = \det E_0(0) = \det E_0(t), \quad \forall t \in \mathbb{R}.$$

Then it is straightforward to see that, for a fixed ζ, the mappings

$$\begin{aligned} &z \mapsto z \circ \zeta \\ &z \mapsto \zeta \circ z \\ &z \mapsto z^{-1} \end{aligned}$$

have Jacobian determinant equal to 1 and therefore preserve the Lebesgue measure.

Example 1.1.7. We show how the kinetic Kolmogorov operator (1) presented in the Introduction of this work belongs to the class of operators (1.1.1). Indeed, we can write operator (1) as follows

$$\mathscr{K} := \sum_{j=1}^{m} \partial_{x_j}^2 - \sum_{j=1}^{m} x_j \partial_{x_{m+j}} - \partial_t = \Delta_p - \langle p, D_y \rangle - \partial_t, \tag{1.1.24}$$

for $(x,t) = (p,y,t) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$. It is then clear that operator $\mathscr{K}$ can be written in the form (1.1.1) with $\kappa = 1, m_1 = m$, and

$$A = \begin{pmatrix} \mathbb{I}_m & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \qquad B = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ -\mathbb{I}_m & \mathbb{O} \end{pmatrix} \tag{1.1.25}$$

In this setting, the Lie group introduced in (1.1.6) has a quite natural intepretation, as the composition law (1.1.6) agrees with the *Galilean* change of variables (5).

Moreover, the matrix B in (1.1.25) is in the form (1.1.12) and therefore $\mathscr{K}$ is invariant with respect to the dilation $\delta_r(v, y, t) := (rp, r^3y, r^2t)$ in (7). We remark that the dilation acts as the usual parabolic scaling with respect to the variable p and t.

1.1.2 Principal part operator

In this subsection, we show that dilation-invariant operators are the blow-up limit of operators satisfying assumption [**H.1**]. This technique will be fundamental in the forthcoming Chapter 2, where we consider a general operator $\mathscr{L}$ satisfying [**H.1**].

Here and in the sequel, we denote by $\mathscr{L}_0$ the *principal part operator of* $\mathscr{L}$, obtained from $\mathscr{L}$ by replacing its matrix B with B_0 in (1.1.12). As stated in Subsection 1.1.1 above, $\mathscr{L}_0$ satisfies (1.1.11).

As in the proof of the main result of Chapter 2 we rely on a blow-up argument, we will also apply dilation (1.1.13) to the general operator $\mathscr{L}$ satisfying [**H.1**]. Specifically, we define $\mathscr{L}_r$ as the *scaled operator* of $\mathscr{L}$ in terms of $(\delta_r)_{r>0}$ as follows

$$\mathscr{L}_r := r^2(\delta_r \circ \mathscr{L} \circ \delta_{\frac{1}{r}}), \tag{1.1.26}$$

and we write its explicit expression in terms of the matrix B and (δ_r) as

$$\mathscr{L}_r = \sum_{i,j=1}^{m} a_{ij}\partial^2_{x_i x_j} + Y_r, \quad r \in (0, 1] \tag{1.1.27}$$

where

$$Y_r := \langle B_r x, D\rangle - \partial_t \tag{1.1.28}$$

and $B_r := r^2 \delta_r B \delta_{\frac{1}{r}}$, i.e.,

$$B_r = \begin{pmatrix} r^2 B_{0,0} & r^4 B_{0,1} & \dots & r^{2\kappa} B_{0,\kappa-1} & r^{2\kappa+2} B_{0,\kappa} \\ B_1 & r^2 B_{1,1} & \dots & r^{2\kappa-2} B_{1,\kappa-1} & r^{2\kappa} B_{1,\kappa} \\ \mathbb{O} & B_2 & \dots & r^{2\kappa-4} B_{2,\kappa-1} & r^{2\kappa-2} B_{2,\kappa} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & r^2 B_{\kappa,\kappa} \end{pmatrix}. \tag{1.1.29}$$

Clearly, $\mathscr{L}_r = \mathscr{L}$ for every $r > 0$ if and only if $B = B_0$, and the principal part operator $\mathscr{L}_0$ is obtained as the limit of (1.1.26) as $r \to 0$.

Setting $E_r(t) = \exp(-tB_r)$, we define the translation group related to $\mathscr{L}_r$ as

$$(x, t) \circ_r (\xi, \tau) = (\xi + E_r(\tau)x, t + \tau), \quad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1}. \tag{1.1.30}$$

Remark 1.1.8. As it will be useful in the blow-up limit procedure in Chapter 2, we point out that the composition law defined in (1.1.30) depends continuously on $r \in (0, 1]$. Moreover, taking $r = 0$ in (1.1.29) we find the matrix B_0 and " $\circ_r$" in (1.1.30) simply becomes the composition law related to the dilation-invariant operator $\mathscr{L}_0$. Thus, " $\circ_r$" is a continuous function on the compact set $[0, 1]$.

1.1.3 Fundamental Solution

In this subsection, we collect some known results concerning the fundamental solution Γ of $\mathscr{L}$. First, we recall that, under the hypothesis of hypoellipticity, Hörmander in [54] constructed the fundamental solution of $\mathscr{L}$ as

$$\Gamma(z, \zeta) = \Gamma(\zeta^{-1} \circ z, 0), \quad \forall z, \zeta \in \mathbb{R}^{N+1},\ z \neq \zeta,$$

where

$$\Gamma((x, t), (0, 0)) = \begin{cases} \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\langle C^{-1}(t)x, x\rangle - t\, tr(B)\right), & \text{if } t > 0, \\ 0, & \text{if } t < 0, \end{cases} \tag{1.1.31}$$

and

$$C(t) = \int_0^t E(s) \begin{pmatrix} A & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix} E^T(s)\, ds. \tag{1.1.32}$$

We recall that assumption [**H.1**] implies that $C(t)$ is strictly positive for every $t > 0$ (see [73, Proposition A.1]) and therefore Γ in (1.1.31) is well-defined.

As a fundamental solution to $\mathscr{L}$, the following representation formula holds true: for every $u \in C_0^\infty(\mathbb{R}^{N+1})$ we have

$$u(z) = -\int_{\mathbb{R}^{N+1}} [\Gamma(z, \cdot)\mathscr{L}(u)](\zeta) d\zeta. \tag{1.1.33}$$

In order to state another crucial property of the fundamental solution, we recall the following definition.

Definition 1.1.9 (Homogeneous function)**.** We say that a function u defined on $\mathbb{R}^{N+1}$ is *homogeneous of degree* $\alpha \in \mathbb{R}$ if

$$u(\delta_r(z)) = r^\alpha u(z) \qquad \text{for every } \ z \in \mathbb{R}^{N+1}.$$

A differential operator X will be called homogeneous of degree $\beta \in \mathbb{R}$ with respect to $(\delta_r)_{r \geq 0}$ if

$$Xu(\delta_r(z)) = r^\beta\, (Xu)\, (\delta_r(z)) \qquad \text{for every } \ z \in \mathbb{R}^{N+1},$$

and for every sufficiently smooth function u. Note that, if u is homogeneous of degree α and

X is homogeneous of degree β, then Xu is homogeneous of degree $\alpha - \beta$.

As far as we are concerned with the vector fields of the Kolmogorov operator $\mathscr{L}_0$ under the invariance assumption (1.1.11), we have that $X_1, \dots, X_m$ are homogeneous of degree 1 and Y is homogeneous of degree 2 with respect to $(\delta_r)_{r \geq 0}$. In particular, $\mathscr{L}_0 = \sum_{j=1}^m X_j^2 + Y$ is homogeneous of degree 2, and its fundamental solution Γ_0 is a homogeneous function of degree $-Q$, that is

$$\Gamma_0(\delta_r(z)) = r^{-Q}\Gamma_0(z), \quad \text{for every } z \in \mathbb{R}^{N+1} \setminus \{0\},\ r > 0. \tag{1.1.34}$$

As a direct consequence, the estimate $\Gamma_0(z,\zeta) \leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^Q}$ holds for every $z, \zeta \in \mathbb{R}^{N+1}$, with $z \neq \zeta$. Analogous bounds hold for the first order and second order derivatives of Γ_0, as they are homogeneous of degree $-Q-1$ and $-Q-2$, respectively.

Concerning the fundamental solution Γ of the non-dilation-invariant operator $\mathscr{L}$, we recall the following estimates (see [41, Proposition 2.7]). We assume that all the eigenvalues of the matrix A belong to some interval $[\lambda, \Lambda] \subset \mathbb{R}^+$. Then for every $T > 0$, there exists a positive constant c, only depending on T, λ, Λ and on the matrix B, such that the following bounds hold

$$\Gamma(z,\zeta) \leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^Q}, \tag{1.1.35}$$

$$\begin{aligned}
|\partial_{x_j}\Gamma(z,\zeta)| &\leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+1}}, & |\partial_{\xi_j}\Gamma(z,\zeta)| &\leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+1}},\\
|\partial_{x_i x_j}\Gamma(z,\zeta)| &\leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+2}}, & |\partial_{\xi_i \xi_j}\Gamma(z,\zeta)| &\leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+2}},\\
|Y\Gamma(z,\zeta)| &\leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+2}}, & |Y^*\Gamma(z,\zeta)| &\leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+2}},
\end{aligned} \tag{1.1.36}$$

for every $i, j = 1, \dots, m$, $z, \zeta \in \mathbb{R}^N \times [-T, T]$ with $z \neq \zeta$. Here Y^* denotes the transposed operator of Y, defined as follows

$$\int_{\mathbb{R}^{N+1}} \varphi(x,t) Y^* \psi(x,t)\, dxdt = \int_{\mathbb{R}^{N+1}} \psi(x,t) Y \varphi(x,t)\, dxdt,$$

for every ψ, $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$.

A similar result holds for the derivatives $\partial_{x_j}\Gamma(z,\zeta)$ and $\partial_{\xi_j}\Gamma(z,\zeta)$ for $j = m+1, \dots, N$. These functions need to be considered as derivatives of order α_j, where the integer α_j has been introduced in (1.1.18). We have

$$\begin{gathered}
|\partial_{x_j}\Gamma(z,\zeta)| \leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+\alpha_j}}, \qquad |\partial_{\xi_k}\Gamma(z,\zeta)| \leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+\alpha_k}},\\
|\partial_{x_j}\partial_{\xi_k}\Gamma(z,\zeta)| \leq \frac{c}{\|\zeta^{-1} \circ z\|_{\mathbb{K}}^{Q+\alpha_j+\alpha_k}},
\end{gathered} \tag{1.1.37}$$

for every $j, k = 1, \ldots, N$, $z, \zeta \in \mathbb{R}^N \times [-T, T]$ with $z \neq \zeta$. Note that, as $\alpha_1 = \cdots = \alpha_m = 1$, the bounds in the first line of (1.1.37) agree with the first line of (1.1.36). The proof of (1.1.37) directly follows from the bound (2.59) and (2.60) in [41].

1.2 Kolmogorov operators with Hölder continuous coefficients

In this section, we show how to take advantage of the Lie Group structure introduced in Subsection 1.1.1 to study more general Kolmogorov-type operators. In particular, we consider Kolmogorov operators in trace form in $\mathbb{R}^{N+1}$

$$\mathscr{L} = \sum_{i,j=1}^{m} a_{ij}(x,t)\partial^2_{x_i x_j} + \sum_{j=1}^{m} b_j(x,t)\partial_{x_j} + \langle Bx, D\rangle - \partial_t, \qquad \text{for } (x,t) \in \mathbb{R}^{N+1} \tag{1.2.1}$$

with variable coefficients a_{ij}'s and b_j's. As in the parabolic case, the classical theory for degenerate Kolmogorov operators is developed for spaces of Hölder continuous functions. As we rely on the non-Euclidean structure defined in Subsection 1.1.1, we need to consider functions which are Hölder continuous with respect to the quasi-distance in (1.1.21), i.e. functions which are Hölder continuous *intrinsically*. More precisely, here and in the sequel we rely on the following definition.

Definition 1.2.1 (Intrinsic Hölder continuous functions)**.** Let α be a positive constant, $\alpha \leq 1$, and let Ω be an open subset of $\mathbb{R}^{N+1}$. We say that a function $f : \Omega \longrightarrow \mathbb{R}$ is *Hölder continuous with exponent* α in Ω with respect to the group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$, defined in (1.1.6), (in short: Hölder continuous with exponent α) if there exists a positive constant $M > 0$ such that

$$|f(z) - f(\zeta)| \leq M \, d_{\mathbb{K}}(z,\zeta)^{\alpha} \qquad \text{for every } z, \zeta \in \Omega, \tag{1.2.2}$$

where $d_{\mathbb{K}}$ is the distance defined in (1.1.21). If a function f satisfies (1.2.2) we write $f \in C^{\alpha}_{L}(\Omega)$, where the subscript L refers to the fact that we deal with functions which are Hölder continuous with respect to the group law that leaves operator $\mathscr{L}$ invariant.

To every bounded function $f \in C^{\alpha}_{L}(\Omega)$ we associate the semi-norm

$$[f]_{C^{\alpha}(\Omega)} = \sup_{\substack{z,\zeta\in\Omega \\ z\neq\zeta}} \frac{|f(z) - f(\zeta)|}{d_{\mathbb{K}}(z,\zeta)^{\alpha}}.$$

Moreover, we say a function f is locally Hölder continuous, and we write $f \in C^{\alpha}_{L,\mathrm{loc}}(\Omega)$, if $f \in C^{\alpha}_{L}(\Omega')$ for every compact subset Ω' of Ω.

We remark that Definition 1.2.1 relies on the Lie group $\mathbb{K}$ in (1.1.6), that is an invariant structure for the *constant-coefficients* operators. Even though the *non-constant-coefficients* operator in (1.2.1) is not invariant with respect to $\mathbb{K}$, we rely on the Lie group invariance of the model operator

$$\Delta_m + Y = \sum_{j=1}^{m} \partial^2_{x_j} \, + \, \langle Bx, D\rangle - \, \partial_t, \tag{1.2.3}$$

associated to $\mathscr{L}$. Indeed, this is a standard procedure in the study of uniformly parabolic operators. We observe that, in the uniformly parabolic setting, the model operator (1.2.3) is replaced by the heat operator and we have that $d_{\mathbb{K}}\big((\xi,\tau),(x,t)\big) = |\xi - x| + |\tau - t|^{1/2}$, so that we are considering the parabolic modulus of continuity.

In this section, we always rely on the following assumption.

(C) We always assume that hypothesis **[H.1]** holds. In addition, we assume that the matrix A satisfies the following uniform ellipticity condition: there exist two positive constants λ and Λ such that

$$\lambda \sum_{i=1}^{m} |\xi_i|^2 \leq \sum_{i,j=1}^{m} a_{ij}(z)\xi_i\xi_j \leq \Lambda \sum_{i=1}^{m} |\xi_i|^2$$

for every $(\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ and $z \in \mathbb{R}^{N+1}$. Finally, we require that the coefficients a_{ij}'s and b_j's belong to the space C^α_L introduced in Definition 1.2.1, for every $i, j = 1, \dots, m$.

We now introduce the definition of *classical solutions* to equation $\mathscr{L}u = 0$.

Definition 1.2.2. A function u is a *classical solution* to equation $\mathscr{L}u = 0$ in a domain Ω of $\mathbb{R}^{N+1}$ under assumption **(C)** if the derivatives $\partial_{x_i}u, \partial^2_{x_ix_j}u$, for $i, j = 1, \dots, m$, and the Lie derivative Yu exist as continuous functions in Ω, and the equation $\mathscr{L}u(x,t) = 0$ is satisfied at any point $(x,t) \in \Omega$. Finally, we say that u is a *classical super-solution* to $\mathscr{L}u = 0$ if $\mathscr{L}u \leq 0$. We say that u is a *classical sub-solution* if $-u$ is a classical supersolution.

The natural functional setting to study classical solutions to equation $\mathscr{L}u = 0$, with $\mathscr{L}$ as in (1.2.1), is the space

$$C^{2,\alpha}_L(\Omega) = \Big\{ u \in C^\alpha_L(\Omega) \ \mid \ \partial_{x_i}u, \partial^2_{x_ix_j}u, Yu \in C^\alpha_L(\Omega), \quad \text{for } i, j = 1, \dots, m \Big\}, \tag{1.2.4}$$

where $C^\alpha_L(\Omega)$ is given in Definition 1.2.1. Moreover, if $u \in C^{2,\alpha}_L(\Omega)$, we define the norm

$$|u|_{2+\alpha,\Omega} := |u|_{\alpha,\Omega} \, + \, \sum_{i=1}^{m} |\partial_{x_i}u|_{\alpha,\Omega} \, + \, \sum_{i,j=1}^{m} |\partial^2_{x_ix_j}u|_{\alpha,\Omega} \, + \, |Yu|_{\alpha,\Omega}. \tag{1.2.5}$$

Clearly, the definition of $C^{2,\alpha}_{L,\mathrm{loc}}(\Omega)$ follows straightforwardly from the definition of $C^\alpha_{L,\mathrm{loc}}(\Omega)$. A definition of the space $C^{k,\alpha}_L(\Omega)$ for every positive integer k is given and discussed in the work [93] by Pagliarani, Pascucci and Pignotti, where a proof of the Taylor expansion for $C^{k,\alpha}_L(\Omega)$ functions is given. It is worth noting that the authors of [93] require weaker regularity assumptions for the definition of the space $C^{2,\alpha}_L$ than the ones considered here in (1.2.4). Finally, we recall that a complete characterization of the intrinsic Hölder spaces is provided by Pagliarani, Pascucci and Pignotti in [93].

As in the uniformly elliptic and parabolic case, Schauder estimates constitute a fundamental result in the classical regularity theory. As Schauder-type estimates will be one of the main results of Chapter 2, we here revise the current literature on this subject. In particular, we recall that Schauder estimates for the dilation-invariant Kolmogorov operator (i.e. where the matrix $B = B_0$) with Hölder continuous coefficients were proved by Manfredini in [82, Theorem 1.4]. Manfredini's result was then extended by Di Francesco and Polidoro in [41, Theorem 1.3] to the non-dilation invariant case as follows.

Theorem 1.2.3. *Let us consider an operator $\mathscr{L}$ of the type* (1.2.1) *satisfying assumption **(C)** with $\alpha < 1$. Let Ω be an open subset of $\mathbb{R}^{N+1}$, $f \in C^{\alpha}_{L,\mathrm{loc}}(\Omega)$ and let u be a classical solution to $\mathscr{L}u = f$ in Ω. Then for every $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ there exists a positive constant C, depending only on the constants λ, Λ, on the Hölder-norm of the coefficients of $\mathscr{L}$ and on the diameter of Ω, such that*

$$|u|_{2+\alpha,\Omega'} \le C\Big(\sup_{\Omega''} |u| \; + \; |f|_{\alpha,\Omega''} \Big).$$

A more precise estimate taking into account the distance between the point and the boundary of the set Ω can be found in [82, Theorem 1.4] for the dilation invariant case. We omit here this statement because it requires the introduction of further notation.

We also recall that Schauder estimates in the framework of semigroup theory have been proved by Lunardi [81], Lorenzi [79], Priola [106]. We also quote analogous results obtained in the framework of stochastic theory (see Menozzi [84] and its bibliography). Schauder estimates for linear kinetic Fokker–Planck equations were obtained by Imbert and Mouhot in [56] and by Henderson and Snelson in [53], while the Boltzmann fractional framework was recently studied by Imbert and Silvestre in [60]. For a comparison between the different types of Hölder spaces considered in literature we refer the reader to [60, 93].

We now present some known results concerning the fundamental solution of operator $\mathscr{L}$ in (1.2.1) with the additional term c, namely

$$\mathscr{L} = \sum_{i,j=1}^{m} a_{ij}(x,t)\partial^2_{x_i x_j} + \sum_{j=1}^{m} b_j(x,t)\partial_{x_j} + c(x,t) + \langle Bx, D\rangle - \partial_t, \qquad \text{for } (x,t) \in \mathbb{R}^{N+1}. \tag{1.2.6}$$

These results will be useful in Chapter 5, when dealing with the weak fundamental solution associated to operators of the form (1.2.6) with measurable coefficients.

We recall that, in some particular cases, variable-coefficients Kolmogorov operators were first studied by Weber [120], Il'In [55] and Sonin [111], who used the parametrix method to construct a fundamental solution. However, the authors of the aforementioned papers worked under unnecessary restrictive conditions, i.e. they required an Euclidean regularity on the coefficients a_{ij}'s, b_j's and c. Later on, Polidoro applied in [102] the Levi parametrix method for the *dilation-invariant* operator $\mathscr{L}$ (i.e. under the additional assumption that B has the form (1.1.12)), then Di Francesco and Pascucci and Di Francesco and Polidoro removed this last assumption in [40, 41].

The Levi's parametrix method is a constructive argument to prove existence and bounds for the fundamental solution (see [75]). For every $\zeta \in \mathbb{R}^{N+1}$, the parametrix $Z(\,\cdot\,,\zeta)$ is the fundamental solution, with pole at ζ, of the following operator

$$\mathscr{L}_\zeta = \sum_{i,j=1}^{m} a_{ij}(\zeta)\,\partial^2_{x_i x_j} \,+\, \langle Bx, D\rangle \,-\, \partial_t.$$

The method is based on the following property: if the coefficients a_{ij}'s are continuous and the coefficiens b_j's are bounded, then Z is a good approximation of the fundamental solution of $\mathscr{L}$ at least as z is close to the pole ζ, since

$$\mathscr{L} Z(z,\zeta) = \sum_{i,j=1}^{m} \left(a_{ij}(z) - a_{ij}(\zeta)\right)\,\partial^2_{x_i x_j} Z(z,\zeta) + \sum_{j=1}^{m} b_j(z)\,\partial_{x_j} Z(z,\zeta).$$

We look for the fundamental solution Γ as a solution of the following Volterra equation

$$\Gamma(x,t,\xi,\tau) = Z(x,t,\xi,\tau) + \int_\tau^t \int_{\mathbb{R}^N} Z(x,t,y,s) G(y,s,\xi,\tau) dy\, ds,$$

where the unknown function G is obtained by a fixed point argument. It turns out that

$$G(z,\zeta) = \sum_{k=1}^{+\infty} (\mathscr{L} Z)_k(z,\zeta),$$

where $(\mathscr{L} Z)_1(z,\zeta) = \mathscr{L} Z(z,\zeta)$ and, for every $k \in \mathbb{N}$,

$$(\mathscr{L} Z)_{k+1}(x,t,\xi,\tau) = \int_\tau^t \int_{\mathbb{R}^N} \mathscr{L} Z(x,t,y,s)(\mathscr{L} Z)_k(y,s,\xi,\tau) dy\, ds.$$

We conclude this section stating the existence result for a classical fundamental solution for an operator $\mathscr{L}$ of the form (1.2.6) proved in [40, Theorem 1.4-1.5]. The upcoming Theorem also provides us with an equivalent result for the fundamental solution Γ^* associated to the adjoint operator of (1.2.6). To this end, we recall that the formal adjoint of operator $\mathscr{L}$ in (1.2.6) is defined as

$$\begin{aligned}\mathscr{L}^* v(\xi,\tau) = &\sum_{i,j=1}^{m} \partial_{\xi_i}\left(a_{ij}(\xi,\tau)\partial_{\xi_j} v(\xi,\tau)\right) - \sum_{i=1}^{m} \partial_{\xi_i}(b_i(\xi,\tau) v(\xi,\tau)) \\ &+ (c - \mathrm{Tr}(B)) v(\xi,\tau) + Y^* v(\xi,\tau)\end{aligned} \tag{1.2.7}$$

where

$$Y^* v(\xi,\tau) := -\sum_{i,j=1}^{N} b_{ij}\xi_j \partial_{\xi_i} v(\xi,\tau) + \partial_\tau v(\xi,\tau).$$

We are now in a position to recall the following fundamental result.

Theorem 1.2.4. *Let us consider an operator $\mathscr{L}$ of the form (1.2.6) under assumption **(C)**. Then there exists a fundamental solution $\Gamma : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to \mathbb{R}$ for $\mathscr{L}$ with the following properties:*

1. *$\Gamma(\cdot,\cdot;\xi,\tau) \in L^1_{\mathrm{loc}}(\mathbb{R}^{N+1}) \cap C(\mathbb{R}^{N+1} \setminus \{(\xi,\tau)\}$ for every $(\xi,\tau) \in \mathbb{R}^{N+1}$;*
2. *$\Gamma(\cdot,\cdot;\xi,\tau)$ is a classical solution of $\mathscr{L}u = 0$ in $\mathbb{R}^{N+1} \setminus \{(\xi,\tau)\}$ for every $(\xi,\tau) \in \mathbb{R}^{N+1}$ in the sense of Definition 1.2.2;*
3. *let $\varphi \in C(\mathbb{R}^N)$ such that for some positive constant c_0 we have*

$$|\varphi(x)| \le c_0 e^{c_0|x|^2} \qquad \text{for every } x \in \mathbb{R}^N, \tag{1.2.8}$$

   then there exists

$$\lim_{\substack{(x,t)\to(x_0,\tau)\\ t>\tau}} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\varphi(\xi)d\xi = \varphi(x_0) \qquad \text{for every } x_0 \in \mathbb{R}^N;$$

4. *let $\varphi \in C(\mathbb{R}^N)$ satisfying (1.2.8). Then, for every positive $0 < T_0 < T_1$, there exists $T \in (T_0, T_1]$ such that the function*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,T_0)\,\varphi(\xi)\; d\xi\, d\tau \tag{1.2.9}$$

   is a classical solution to the Cauchy problem

$$\begin{cases} \mathscr{L}u = 0 & \text{in } S_{T_0,T}, \\ u(\cdot,T_0) = \varphi & \text{in } \mathbb{R}^N; \end{cases} \tag{1.2.10}$$

5. *the reproduction property holds. Indeed, for every $x,\xi \in \mathbb{R}^N$ and $t,\tau \in \mathbb{R}$ with $\tau < s < t$:*

$$\Gamma(x,t;\xi,\tau) = \int_{\mathbb{R}^N} \Gamma(x,t;y,s)\,\Gamma(y,s;\xi,\tau)\,dy;$$

6. *for every $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$ with $t \le \tau$ we have that $\Gamma(x,t;\xi,\tau) = 0$;*
7. *if $c(x,t) = c$ is constant, then*

$$\int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\,d\xi = e^{-c(t-\tau)}, \qquad \forall x \in \mathbb{R}^N,\ \tau < t;$$

8. *for every $\lambda^+ > \lambda$ and for every positive $0 < T_0 < T_1$, there exists a constant C^+ , only dependent on λ, B and T such that*

$$\begin{aligned} \Gamma(x,t;\xi,\tau) &\le C^+\,\Gamma^+(x,t;\xi,\tau) \\ |\partial_{x_i}\Gamma(x,t;\xi,\tau)| &\le \frac{C^+}{\sqrt{t-\tau}}\,\Gamma^+(x,t;\xi,\tau) \\ |\partial^2_{x_ix_j}\Gamma(x,t;\xi,\tau)| &\le \frac{C^+}{t-\tau}\,\Gamma^+(x,t;\xi,\tau) \\ |Y\Gamma(x,t;\xi,\tau)| &\le \frac{C^+}{t-\tau}\,\Gamma^+(x,t;\xi,\tau) \end{aligned} \tag{1.2.11}$$

for any $i,j = 1,\dots,m$ and $(x,t),(\xi,\tau) \in \mathbb{R}^N \times (T_0,T_1)$, and where Γ^+ denotes the fundamental solution of $\mathscr{L}^{\lambda^+}$, defined as follows

$$\mathscr{L}^{\lambda^+}u(x,t) := \frac{\lambda^+}{2}\sum_{i=1}^{m_0}\partial^2_{x_i}u(x,t) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t).$$

Owing to (1.1.31), *it is clear that the explicit expression of Γ^+ is given by*

$$\Gamma^+((x,t);(\xi,\tau)) = \Gamma^+((\xi,\tau)^{-1}\circ(x,t);0,0),$$

for every $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$, with $(x,t) \neq (\xi,\tau)$ and

$$\Gamma^+(x,t;0,0) = \begin{cases} \frac{(2\pi\lambda^+)^{-\frac{N}{2}}}{\sqrt{detC(t)}}\exp\left(-\frac{1}{2\lambda}\langle C^{-1}(t)x,x\rangle - t\,\mathrm{tr}(B)\right), & \textit{if } t>0, \\ 0, & \textit{if } t \le 0. \end{cases} \tag{1.2.12}$$

Moreover, there exists a fundamental solution Γ^* to $\mathscr{L}^*$ (1.2.7) *satisfying the dual properties of this statement and such that $\Gamma^*(x,t;\xi,\tau) = \Gamma(\xi,\tau;x,t)$ for every $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$, $(x,t) \neq (\xi,\tau)$.*

1.3 Applications of Kolmogorov equations

We conclude this chapter discussing some applications of Kolmogorov-type equations. First of all, as mentioned in the Introduction, the process (1.0.2) is the solution to the Langevin equation

$$\begin{cases} dV_t = dW_t \\ dY_t = V_t\,dt, \end{cases}$$

and therefore Kolmogorov equations are connected to every stochastic process satisfying Langevin equation. In particular, several mathematical models involving linear and non linear Kolmogorov-type equations have also appeared in finance [5], [15] and [38]. Indeed, equations of the form (1.0.1) appear in various models for pricing of path-dependent financial instruments (cf., for instance, [16]). For example the equation

$$\partial_t P + \tfrac{1}{2}\sigma^2 S^2 \partial_S^2 P + (\log S)\partial_A P + r(S\partial_S P - P) = 0, \qquad S > 0,\ A, t \in \mathbb{R} \tag{1.3.1}$$

arises in the Black and Scholes option pricing problem

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ dA_t = S_t\, dt, \end{cases}$$

where σ is the volatility of the stock price S, r is the interest rate of a riskless bond and $P = P(S, A, t)$ is the price of the Asian option depending on the price of the stock S, the geometric average A of the past price and the time to maturity t. For the applications of operators in the form $\mathscr{L}$ to finance and to the stochastic theory we refer to the monograph [97] by Pascucci.

Moreover, as already mentioned in the introduction of this work, the Kolmogorov equation is the prototype for a family of evolution equations arising in kinetic theory of gases of the form

$$\partial_t u + \langle p, D_y u\rangle = \mathcal{J}(u). \tag{1.3.2}$$

Beside the collision operator (11) already considered in the introduction, $\mathcal{J}(u)$ can also occur in divergence form

$$\mathcal{J}(u) = \sum_{i,j=1}^{n} \partial_{p_i}(a_{ij}\, \partial_{p_j} u + b_i u) + \sum_{i=1}^{n} a_i \partial_{p_i} u + au.$$

In particular, we mention the following nonlinear kinetic Fokker-Planck equation

$$\partial_t u + \langle p, D_y u\rangle = \rho[u]\mathrm{div}_p\left(D_p u + p\, u\right), \tag{1.3.3}$$

with periodic condition with respect to the space variable and where $\rho[u] := \int_{\mathbb{R}^m} u(p, y, t)dp$. We refer the reader to [49, 56] for a recent treatment of the nonlinear operator in (1.3.3), in particular in connection to the linear Fokker-Planck operator.

Equation (1.3.3) plays an important role in kinetic theory as it shares some similarities with the equation introduced by Landau [74] in 1936 to describe plasmas made up of ions and electrons which interact by Coulombian forces. In the case of the Landau equation, the collision operator $\mathcal{J}(u)$ in (1.3.2) takes the following form

$$\mathcal{J}(u) = \mathrm{div}_p(A[u]D_p u + B[u]u),$$

where

$$\begin{aligned} A[u](v) &= \alpha_{m,\gamma} \int_{\mathbb{R}^m} \left(\mathbb{I}_m - \frac{w}{|w|} \otimes \frac{w}{|w|} \right) |w|^{\gamma+2} u(v-w) dw, \\ B[u](v) &= \beta_{m,\gamma} \int_{\mathbb{R}^m} |w|^{\gamma} w u(v-w) dw, \qquad \gamma \in [-m, 0] \text{ and } \alpha_{m,\gamma} > 0, \end{aligned} \tag{1.3.4}$$

as described in the article [49]. The collision operator in (1.3.3) clearly corresponds the case where the coefficients in (1.3.4) take the simpler form $A[u] = \rho[u]\mathbb{I}_m$ and $B[u] = \rho[u]\, p$. In particular, the integral quantities involving the solution appearing in (1.3.4) are replaced by their averages in the nonlinear model (1.3.3). It is worth mentioning that, even if it has a simpler structure, the nonlinear model in (1.3.3) shares the same Gaussian steady state as the Landau collision operator. For further applications of kinetic theory to the Landau equation, we also refer the reader to [28], [89] and [110].

Finally, for the description of wide classes of stochastic processes and kinetic models leading to equations of the previous type, we refer to the classical monographies [29], [30] and [36] and to the recent survey article [58] by Imbert and Silvestre, and to its bibliography.

Chapter 2

Schauder-type estimates for degenerate Kolmogorov equations with Dini continuous coefficients

2.1 Statement of the problem

In this chapter, we study the local regularity of solutions to the second order linear differential equation

$$\mathscr{L}u := \sum_{i,j=1}^{m} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u - \partial_t u = f, \tag{2.1.1}$$

where $(x,t) \in \mathbb{R}^{N+1}$, and $1 \le m \le N$ and f is Dini continuous. Our final aim is to extend a fundamental result of the classical regularity theory, namely Schauder estimates. In particular, we prove that, if operator $\mathscr{L}$ satisfies Hörmander's hypoellipticity condition, and f and a_{ij}'s are Dini continuous functions, then the second order derivatives of the solution u to the equation $\mathscr{L}u = f$ are Dini continuous functions as well. A key step in our proof is a Taylor formula for classical solutions to $\mathscr{L}u = f$ that we establish under minimal regularity assumptions on u. We finally remark that the results presented in this chapter are published in [105] and are obtained in collaboration with Polidoro and Stroffolini.

2.1.1 Assumptions

Throughout this chapter, we require that the matrices A and B satisfy the structural assumption **[H.1]** in (1.1.2), which implies that operator $\mathscr{L}$ is hypoelliptic, as discussed in Chapter 1. We point out that we do not need the more restrictive assumption **[H.2]** to derive our main theorems, i.e. all the results presented in this chapter hold true for non-dilation-invariant Kolmogorov operators.

As the problem we address here is the regularity of solutions to (2.1.1), we require as few conditions as possible for the definition of $\mathscr{L}u$. In particular, in this chapter, we rely on the following definition of classical solutions to equation (2.1.1).

Definition 2.1.1. Let Ω be an open subset of $\mathbb{R}^{N+1}$. We say that a function u belongs to $C^2_{\mathscr{L}}(\Omega)$ if u, its derivatives $\partial_{x_i} u, \partial_{x_i x_j} u$ $(i, j = 1, \dots, m)$ and the Lie derivative Yu defined in (1.1.5) are continuous functions in Ω. We also require, for $i = 1, \dots, m$, that

$$\lim_{s \to 0} \frac{\partial_{x_i} u(\exp(sB)x, t-s) - \partial_{x_i} u(x,t)}{|s|^{1/2}} = 0, \tag{2.1.2}$$

uniformly for every $(x,t) \in K$, where K is a compact set $K \subset \Omega$.

Let f be a continuous function defined in Ω. We say that a function u is a classical solution to $\mathscr{L}u = f$ in Ω if u belongs to $C^2_{\mathscr{L}}(\Omega)$, and the equation $\mathscr{L}u = f$ is satisfied at every point of Ω.

As it is customary in the heat operator framework, we regard the time derivative, here generalized by the Lie derivative Y (see (1.1.5)), as a second order operator (we refer to Definition 1.1.9 for a formal justification of this fact). As a consequence, (2.1.2) can be interpreted as a condition on the second order mixed derivative of the form $Y^{1/2}\partial_{x_i} u$. Indeed, if the derivative $Y\partial_{x_i} u$ exists, then the fractional derivative $Y\partial_{x_i}^{1/2} u$ is equal to 0. Thus, condition (2.1.2) is not demanding and it is the weakest assumption we need in order to prove that u is approximated by its intrinsic Taylor polynomial of degree 2 (see Theorem 2.1.2 below). Indeed, in order to carry out the proof of Theorem 2.1.2, the regularity of the derivatives $\partial_{x_i} u, \partial_{x_i x_j} u$, for $i, j = 1, \dots, m$, and of the Lie derivative Yu is not enough. Additionally, we need to require that the first order derivatives $\partial_{x_i} u$, $i, j = 1, \dots, m$, are continuous along the integral curve of the drift Y. Without this additional requirement, it is not possible to carry out the proof of Theorem 2.1.2 (see in particular equation (2.3.14) below). On the other hand, condition (2.1.2) seems natural as it mirrors the one found in [93], where the authors additionally require that the first order derivatives $\partial_{x_i} u$, $i, j = 1, \dots, m$, are Hölder continuous functions with respect to Y to ensure the existence of the second order polynomial of u.

2.1.2 Main results

Our first main result, and a key step in proving the Schauder estimates presented in Theorem 2.1.4, concerns the intrinsic second order Taylor polynomial. We recall that the *nth*-order intrinsic Taylor polynomial of a function u (differentiable up to order n) around the point z is defined as the unique polynomial function $P^n_z u$ of order n such that

$$u(\zeta) - P^n_z u(\zeta) = o(d_{\mathbb{K}}(\zeta, z)^n) \quad \text{as} \quad \zeta \to z,$$

where $d_{\mathbb{K}}$ denotes the quasi-distance defined in (1.1.21).

We are now in a position to state the following result.

Theorem 2.1.2. *Let $\mathscr{L}$ be an operator of the form* (2.1.1) *satisfying hypothesis* **[H.1]**. *Let Ω be an open subset of $\mathbb{R}^{N+1}$ and let u be a function in $C^2_{\mathscr{L}}(\Omega)$. For every $z := (x,t) \in \Omega$ we* *define the second order Taylor polynomial of* u *around* z *as*

$$\begin{aligned} T_z^2 u(\zeta) &:= u(z) + \sum_{i=1}^{m} \partial_{x_i} u(z)(\xi_i - x_i) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{m} \partial^2_{x_i x_j} u(z)(\xi_i - x_i)(\xi_j - x_j) - Yu(z)(\tau - t), \end{aligned} \tag{2.1.3}$$

for any $\zeta = (\xi, \tau) \in \Omega$. *Indeed, we have*

$$u(\zeta) - T_z^2 u(\zeta) = o(d_{\mathbb{K}}(\zeta, z)^2) \quad as \quad \zeta \to z. \tag{2.1.4}$$

In order to expose our next results, we first need to introduce some preliminary notation. As a first step, we introduce the sets where our local results hold true. To this end, we take advantage of the invariant structure of the principal part operator $\mathscr{L}_0$ in the study of the regularity of $\mathscr{L}$. This fact is quite natural as $\mathscr{L}_0$ is the *blow-up limit of* $\mathscr{L}$, as explained in Subsection 1.1.2. In particular, owing to the quasi-distance introduced in (1.1.21), we define the *boxes*

$$\mathcal{Q}_r(x_0, t_0) := \{(x, t) \in \mathbb{R}^{N+1} \mid d_{\mathbb{K}}((x, t), (x_0, t_0)) < r\}. \tag{2.1.5}$$

We now a provide a definition of *modulus of continuity* and *Dini continuity* which are suitable for operator $\mathscr{L}$. More precisely, we define the *modulus of continuity* of a function f defined on any set $H \subset \mathbb{R}^{N+1}$ as follows

$$\omega_f(r) := \sup_{\substack{(x,t),(\xi,\tau) \in H \\ d_{\mathbb{K}}((x,t),(\xi,\tau)) < r}} |f(x, t) - f(\xi, \tau)|. \tag{2.1.6}$$

Definition 2.1.3. A function f is said to be *Dini-continuous* in H if

$$\int_0^1 \frac{\omega_f(r)}{r} dr < +\infty.$$

We are now in position to state our main result.

Theorem 2.1.4. *Let* $\mathscr{L}$ *be an operator of the form* (2.1.1) *satisfying hypothesis* **[H.1]**. *Let* $u \in C^2_{\mathscr{L}}(\mathcal{Q}_1(0,0))$ *be a classical solution to* $\mathscr{L} u = f$. *Suppose that* f *is Dini continuous and the coefficients* a_{ij} *are constant. Then there exists a positive constant* c, *only depending on the operator* $\mathscr{L}$, *such that:*

i)

$$|\partial^2 u(0,0)| \le c \left(\sup_{\mathcal{Q}_1(0,0)} |u| + |f(0,0)| + \int_0^1 \frac{\omega_f(r)}{r} dr \right);$$

ii) for any points (x,t) and $(\xi,\tau) \in \mathcal{Q}_{\frac{1}{4}}(0,0)$ we have

$$|\partial^2 u(x,t) - \partial^2 u(\xi,\tau)| \le c \left(d \sup_{\mathcal{Q}_1(0,0)} |u| + d \sup_{\mathcal{Q}_1(0,0)} |f| + \int_0^d \frac{\omega_f(r)}{r} dr + d \int_d^1 \frac{\omega_f(r)}{r^2} dr \right).$$

where $d := d_{\mathbb{K}}((x,t),(\xi,\tau))$ and ∂^2 stands either for $\partial^2_{x_i x_j}$, with $i,j = 1,\dots,m$, or for Y.

We emphasize that Theorem 2.1.4 fails even in the simplest Euclidean setting if we do not assume any regularity condition on the function f. Consider for instance the function

$$u(x,y) = xy(\log(x^2+y^2))^\alpha, \qquad \text{with} \quad 0 < \alpha < 1.$$

A direct computation shows that

$$\Delta u(x,y) = 8\alpha \frac{xy}{x^2+y^2}(\log(x^2+y^2))^{\alpha-1} + 4\alpha(\alpha-1)\frac{xy}{x^2+y^2}(\log(x^2+y^2))^{\alpha-2},$$

so that $f(x,y) := \Delta u(x,y)$ extends to a continuous function on $\mathbb{R}^2$, which is not Dini continuous at the point $(0,0)$. On the other hand, the derivative $\partial_x \partial_y u(x,y)$ is unbounded near the origin. We also point out that, when $\alpha = 1$, the function u is a counterexample for the L^∞ bounds of the second order derivatives of weak solutions to $\Delta u = f$. [1]

We finally consider the non-constant coefficients operator $\widetilde{\mathscr{L}}$ defined as follows

$$\widetilde{\mathscr{L}} := \sum_{i,j=1}^m a_{ij}(x,t)\partial^2_{x_i x_j} + \sum_{i,j=1}^N b_{ij} x_j \partial_{x_i} - \partial_t. \tag{2.1.7}$$

We assume that the coefficients a_{ij} are Dini continuous functions and, in order to simplify the notation, we write

$$\omega_a(r) := \max_{i,j=1,\dots,m} \sup_{\substack{(x,t),(\xi,\tau) \in H \\ d_{\mathbb{K}}((x,t),(\xi,\tau)) < r}} |a_{ij}(x,t) - a_{ij}(\xi,\tau)|. \tag{2.1.8}$$

We assume that the following condition on the matrix $A(x,t) := (a_{ij}(x,t))_{i,j=1,\dots,m}$ is satisfied.

[H.2] For every $(x,t) \in \mathbb{R}^{N+1}$, the matrix $A(x,t)$ is symmetric and satisfies

$$\lambda|\xi|^2 \le \langle A(x,t)\xi, \xi \rangle \le \Lambda |\xi|^2, \qquad \text{for every} \quad \xi \in \mathbb{R}^m, \tag{2.1.9}$$

for some positive constants λ, Λ.

[1] We acknowledge that this counterexample was pointed out to one of the authors by Andreas Minne during the Workshop "New trends in PDEs", held in Catania on 29-30 May 2018.

Theorem 2.1.5. *Let $\widetilde{\mathscr{L}}$ be an operator in the form (2.1.7) satisfying hypotheses* **[H.1]** *and* **[H.2]**. *Let $u \in C^2_{\mathscr{L}}(\mathcal{Q}_1(0,0))$ be a classical solution to $\widetilde{\mathscr{L}}u = f$. Suppose that f and the coefficients a_{ij}, $i,j = 1,\dots,m$, are Dini continuous. Then for any points (x,t) and $(\xi,\tau) \in \mathcal{Q}_{\frac{1}{2}}(0,0)$ the following holds:*

$$
\begin{aligned}
|\partial^2 u(x,t) - \partial^2 u(\xi,\tau)| \leq & c\Big(d \sup_{\mathcal{Q}_1(0,0)} |u| + d \sup_{\mathcal{Q}_1(0,0)} |f| + \int_0^d \frac{\omega_f(r)}{r} dr + d \int_d^1 \frac{\omega_f(r)}{r^2} dr\Big) \\
& + c\Big(\sum_{i,j=1}^m \sup_{\mathcal{Q}_1(0,0)} |\,\partial^2_{x_i x_j} u|\Big)\Big(\int_0^d \frac{\omega_a(r)}{r} dr + d \int_d^1 \frac{\omega_a(r)}{r^2} dr\Big).
\end{aligned}
$$

where $d = d_{\mathbb{K}}((x,t),(\xi,\tau))$ and ∂^2 stands either for $\partial^2_{x_i x_j}$,$i,j = 1,\dots,m$, or for Y.

2.1.3 Comparison with existing results

Taylor polynomial

We compare Theorem 2.1.2 with the existing literature. We specifically refer to the results proved by Pagliarani, Pascucci and Pignotti in [93, 94, 101]. The authors of the above mentioned papers considered a suitable functional space $C^{n,\alpha}_B(\Omega)$, with n nonnegative integer and $\alpha \in (0,1]$, and prove that

$$
u(\zeta) - T^n_z u(\zeta) = O(d_{\mathbb{K}}(\zeta,z)^{n+\alpha}) \quad \text{as} \quad \zeta \to z. \tag{2.1.10}
$$

In order to compare this assertion with (2.1.4), we need to consider the case where $n+\alpha = 2$. We point out that the above articles do not cover the case $n = 2$ and $\alpha = 0$, while they cover $n = 1$ and $\alpha = 1$. Thus, their main results apply to the space $C^{1,1}_B(\Omega)$ of functions u that have Lipschitz continuous first order derivatives $\partial_{x_1} u, \dots, \partial_{x_m} u$ along the directions $x_1,\dots,x_m$ and satisfy

$$
\partial_{x_i} u(\exp(sB)x, t-s) - \partial_{x_i} u(x,t) = O(|s|^{1/2}), \quad \text{as} \quad s \to 0, \tag{2.1.11}
$$

for every $(x,t) \in \Omega$ and $i = 1,\dots,m$. Moreover, the functions $u \in C^{1,1}_B(\Omega)$ are required to Lipschitz continuous along the direction of the vector field Y. In this setting, (2.1.10) reads as follows

$$
u(\zeta) - T^1_z u(\zeta) = O(d_{\mathbb{K}}(\zeta,z)^2) \quad \text{as} \quad \zeta \to z.
$$

We emphasize that, since the assumption $u \in C^{1,1}_B(\Omega)$ does not imply the existence of the second order derivatives of u, $C^{1,1}_B(\Omega)$ differs substantially from our space $C^2_{\mathscr{L}}(\Omega)$. For this reason, the proof of Theorem 2.1.2 requires slightly different arguments and the additional condition (2.1.2), which is slightly stronger than (2.1.11).

Schauder estimates

As we rely on less regular coefficients, we now compare our regularity results with the current literature on this subject, which was revised in Chapter 1, Section 1.2. In particular, as a direct consequence of Theorem 2.1.5 we have the following corollary.

Corollary 2.1.6. *Let $u \in C^2_{\mathscr{L}}(\mathcal{Q}_1(0,0))$ be a classical solution to $\widetilde{\mathscr{L}} u = f$. Suppose that f and the coefficients a_{ij}, $i,j = 1, \ldots, m$, belong to $C^{0,\alpha}_L(\mathcal{Q}_1(0,0))$. Then for any points (x,t) and $(\xi,\tau) \in \mathcal{Q}_{\frac{1}{2}}(0,0)$ the following holds:*

$$
\begin{aligned}
|\partial^2 u(x,t) - \partial^2 u(\xi,\tau)| \leq & c\, d^\alpha \Bigg(\sup_{\mathcal{Q}_1(0,0)} |u| + \frac{\|f\|_{C^\alpha_L(\mathcal{Q}_1(0,0))}}{\alpha(1-\alpha)} \\
& + \sum_{i,j=1}^m \sup_{\mathcal{Q}_1(0,0)} | \, \partial^2_{x_i x_j} u| \frac{\|a\|_{C^\alpha_L(\mathcal{Q}_1(0,0))}}{\alpha(1-\alpha)} \Bigg), \qquad \text{if} \quad \alpha < 1, \\
|\partial^2 u(x,t) - \partial^2 u(\xi,\tau)| \leq & c\, d \Bigg(\sup_{\mathcal{Q}_1(0,0)} |u| + \|f\|_{C^{0,1}_L(\mathcal{Q}_1(0,0))} |\log\, d| \\
& + \Bigg(\sum_{i,j=1}^m \sup_{\mathcal{Q}_1(0,0)} | \, \partial^2_{x_i x_j} u| \Bigg) \|a\|_{C^{0,1}_L(\mathcal{Q}_1(0,0))} |\log\, d| \Bigg), \quad \text{if} \quad \alpha = 1.
\end{aligned}
$$

We observe that, for $\alpha < 1$, Corollary 2.1.6 recovers the Schauder estimates contained in Theorem 1.2.3. Note that, in this case, an interpolation inequality allows us to state a bound for the C^α_L norm of the derivatives $\partial^2 u$ in terms of $\|a\|_{C^\alpha_L(\mathcal{Q}_1(0,0))}$, $\|f\|_{C^\alpha_L(\mathcal{Q}_1(0,0))}$, and $\sup_{\mathcal{Q}_1(0,0)} |u|$ only.

Theorems 2.1.4 and 2.1.5 improve Theorem 1.2.3 and the results presented in Section 1.2 in two directions. First of all, we weaken the regularity assumption on f and on the coefficients a_{ij}'s. Second, we are able to establish Schauder estimates for $\alpha = 1$, extending the results of the aforementioned articles, where $\alpha < 1$.

Moreover, in the case of one commutator and Hölder continuous coefficients and right-hand side, Imbert and Mouhot prove in [56] Schauder estimates for linear kinetic Fokker–Planck equations, as well as well for a toy nonlinear kinetic model. Schauder estimates for kinetic equations (and in particular for linear kinetic Fokker-Planck equations in trace-form) are also obtained by Henderson and Snelson in [53], where they are crucial in proving that weak solutions of the inhomogeneous Landau equation immediately become smooth.

Finally, we recall the following results. Wei, Jiang, and Wu adapt in [121] the method introduced by Wang [119] and prove Schauder estimates for hypoelliptic degenerate operators on the Heisenberg group. The Taylor formula used in [121] is proved by Arena, Caruso and Causa in [13]. In a different framework, Wang's method has been used by Bucur and Karakhanyan [26] in the study of fractional operators.

2.1.4 Idea of the proof

The proof of our main results is based on the method introduced by Safonov in [108] for the parabolic case, explained in Krylov's book [69]. The core idea of Safonov's argument was adopted by Wang [119] for the study of the Poisson equation with Dini continuous right-hand side and by Imbert and Mouhot for the study of kinetic Fokker–Planck equations with Höder continuous coefficients. As we also work under the assumption of Dini continuity, we briefly sketch the proof contained in [119].

Spefically, Wang considers in [119] a solution u to the equation $\Delta u = f$ in some open set Ω. Without loss of generality, he assumes that the unit ball $B_1(0)$ is contained in Ω and considers a sequence of Dirichlet problems as follows. Let $B_{r_k}(0)$ be the Euclidean ball centered at the origin and of radius $r_k = \frac{1}{2^k}$, and let u_k be the solution to the Dirichlet problem

$$\Delta u_k = f(0), \quad \text{in} \quad B_{r_k}(0), \qquad u_k = u \quad \text{in} \quad \partial B_{r_k}(0).$$

Quantitative information on the derivatives of every solution u_k is obtained by using only the elementary properties of the Laplace equation, namely the weak maximum principle, and the standard a priori estimates of the derivatives, that are obtained in [119] via mean value formulas. The bounds for the derivatives of u are obtained as the limit of the analogous bounds for u_k. The Taylor expansion in this step is fundamental to conclude the proof. More precisely, following Safonov's argument, it is crucial to show that the oscillation of the remainder of the second-order Taylor polynomial of the solution decays at rate r_k^2 in a ball of radius r_k.

In this chapter, we apply the method described above to degenerate Kolmogorov operators $\mathscr{L}$, by adapting Wang's approach to the non-Euclidean structure defined in (1.1.6). In particular, the ball $B_{r_k}(0)$ is replaced by the box $\mathcal{Q}_{r_k}(0,0)$ defined through the dilation δ_{r_k} introduced in (1.1.13). Concerning the Taylor expansion, we recall the results due to Bonfiglioli [18] and the ones proved by Pagliarani, Pascucci and Pignotti [93]. We emphasize that the authors of the above articles assume that the second order derivatives of the function u are Hölder continuous, while we only require that u belongs to the space $C^2_{\mathscr{L}}(\Omega)$ introduced in Definition 2.1.1. As the regularity of the second order derivatives of u is the very subject of this chapter, we do not assume extra conditions on them and we prove in Theorem 2.1.2 the Taylor approximation under the minimal requirement that $u \in C^2_{\mathscr{L}}(\Omega)$.

2.1.5 Outline of the chapter

This chapter is structured as follows. In Section 2.2 we prove some preliminary results. In particular, we obtain some *a priori* estimates of the derivatives of the solutions u to $\mathscr{L}u = 0$ in terms of the L^∞ norm of u. We subsequently prove a mean-value formula for u. In Section 2.3 we prove our main result on the Taylor approximation of any function $u \in C^2_{\mathscr{L}}(\Omega)$. Section 2.4 contains the proof of Theorem 2.1.4, while Section 2.5 contains the proof of Theorem 2.1.5.

2.2 Preliminary results

In this section we list some preliminary facts, which are useful in proving our main results. As a first step, we prove a corollary of estimates (1.1.35), (1.1.36) and (1.1.37), which will be useful in the sequel. Secondly, we prove a priori estimates for the derivatives of u solution to the Kolmogorov equation with right-hand side equal to 0. To this end, we represent solutions to $\mathscr{L}u = 0$ as convolutions with the fundamental solution Γ of $\mathscr{L}$ and its derivatives $\partial_{x_1}\Gamma, ..., \partial_{x_N}\Gamma$. We then prove a mean-value formula for u, which is based on the Euclidean mean-value theorem and on the homogeneity of the fundamental solution.

As the estimates in (1.1.35), (1.1.36) and (1.1.37) play a crucial role in the proof of the forthcoming Lemma 2.2.2, we briefly explain how to obtain them and we refer the reader to [41, Proposition 2.7] and [40, Theorems 1.4-1.5] for further details. We firstly observe that the estimate in (1.1.35) holds true in virtue of the homogeneity of the fundamental solution Γ_0, see (1.1.34). Indeed, as already explained in Subsection 1.1.2, homogeneous operators provide a good approximation of the non-homogeneous ones. In particular, the following result holds true.

Theorem 2.2.1. *Let $\mathscr{L}$ be an operator of the form* (1.1.1) *and let $\mathscr{L}_0$ be its principal part (see Subsection 1.1.2). As above, we denote by Γ (resp. Γ_0) the fundamental solution with pole at the origin of $\mathscr{L}$ (resp. $\mathscr{L}_0$). Then for every $b > 0$, there exists a positive constant a such that*

$$\frac{1}{a}\Gamma_0(z) \le \Gamma(z) \le a\Gamma_0(z) \tag{2.2.1}$$

for every $z \in \mathbb{R}^{N+1}$ such that $\Gamma_0(z) \ge b$. Moreover, $a = a(b) \to 1$ as $b \to +\infty$.

Proof. Let $C(t)$ and $C_0(t)$ be the matrix defined in (1.1.32) corresponding to $\mathscr{L}$ and $\mathscr{L}_0$, respectively. Then, by [73, Equation (3.14)], we have

$$\det C(t) = \det C_0(t)(1 + tO(1)), \quad \text{as } t \to 0^+, \tag{2.2.2}$$

where $O(1)$ denotes a bounded function on $\mathbb{R}^N \times (0, 1]$. On the other hand, (see [73, Lemma 3.3]), there holds

$$\exp\left(-\frac{1}{4}\langle C^{-1}(t)x, x\rangle\right) = \exp\left(-\frac{1}{4}\langle C_0^{-1}(t)x, x\rangle\right) \exp\left(t\langle C^{-1}(t)x, x\rangle O(1)\right). \tag{2.2.3}$$

Thus, if $\Gamma_0(z) \ge b$, we get

$$\langle C_0^{-1}(t)x, x\rangle \le 2Q \log\left(\left(\frac{c_N}{b}\right)^{2/Q} \frac{1}{t}\right), \tag{2.2.4}$$

where $c_N = (4\pi)^{-N/2}(\det C(1))^{-1/2}$. The thesis follows at once from (1.1.31), (2.2.2), (2.2.3) and (2.2.4). □

We now focus on the first estimate in (1.1.36) that involves the first derivatives of Γ. The other estimates in (1.1.36) can be obtained in a similar way and the detailed computations can be found in [41, Proposition 2.7]. As a first step, we set $\zeta = (\xi, \tau) = w^{-1} \circ z$ so that $\Gamma(z, w) = \Gamma(\xi, \tau)$ and we compute

$$\partial_{x_j}\Gamma(\xi,\tau) = -\frac{1}{2}\left(C^{-1}(\tau)\xi\right)_j \Gamma(\xi,\tau), \quad j = 1, \dots N. \tag{2.2.5}$$

We now claim that

$$\left|\left(C^{-1}(\tau)\xi\right)_j\right| \le \frac{c}{\tau^{\alpha_j/2}}\left|\delta^0_{1/\tau}(\xi)\right| \quad j = 1, \dots N, \tag{2.2.6}$$

where the numbers α_j were defined in (1.1.18) and the constant c only depends on T and on the matrix B. Indeed, we have

$$\begin{aligned}
\left|\left(C^{-1}(\tau)\xi\right)_j\right| &\le \left|\left((C^{-1}(\tau) - C_0^{-1}(\tau))\xi\right)_j\right| + \left|\left(C_0^{-1}(\tau)\xi\right)_j\right| \\
&= \frac{1}{\tau^{\alpha_j/2}}\left|\left(\delta^0_{\sqrt{\tau}}\left(C^{-1}(\tau) - C_0^{-1}(\tau)\right)\delta^0_{\sqrt{\tau}}\left(\delta^0_{\sqrt{1/\tau}}(\xi)\right)\right)_j\right| \\
&\quad + \frac{1}{\tau^{\alpha_j/2}}\left|\left(\delta^0_{\sqrt{\tau}}\left(C_0^{-1}(\tau)\right)\delta^0_{\sqrt{\tau}}\left(\delta^0_{\sqrt{1/\tau}}(\xi)\right)\right)_j\right| \\
&\le \frac{1}{\tau^{\alpha_j/2}}\left\|\delta^0_{\sqrt{\tau}}\left(C^{-1}(\tau) - C_0^{-1}(\tau)\right)\delta^0_{\sqrt{\tau}}\right\|\left|\delta^0_{\sqrt{1/\tau}}(\xi)\right| + \frac{1}{\tau^{\alpha_j/2}}\left|C_0^{-1}(1)\delta^0_{\sqrt{1/\tau}}(\xi)\right|.
\end{aligned} \tag{2.2.7}$$

Moreover, we recall that, by [73, Remark 2.1], for any given $T > 0$, there exists a positive constant c_T such that

$$\left\|\delta^0_{\sqrt{1/\tau}}\left(C^{-1}(\tau) - C_0^{-1}(\tau)\right)\delta^0_{\sqrt{1/\tau}}\right\| \le c_T\tau\|C_0(1)\|. \tag{2.2.8}$$

Hence, combining (2.2.7) and (2.2.8), we infer

$$\left|\left(C^{-1}(\tau)\xi\right)_j\right| \le \frac{1 + c_T\tau}{\tau^{\alpha_j/2}}\left\|C_0^{-1}(1)\right\|\left|\delta^0_{\sqrt{1/\tau}}(\xi)\right|,$$

which yields (2.2.6). On the other hand, in virue of the homogeneity of the norm $\|\cdot\|_{\mathbb{K}}$, we have

$$\|(\xi,\tau)\|_{\mathbb{K}} = \sqrt{\tau}\|(\delta^0_{\sqrt{1/\tau}}(\xi), 1)\|_{\mathbb{K}} \le c\sqrt{\tau}\left(|\delta^0_{\sqrt{1/\tau}}(\xi)|_{\mathbb{K}} + 1\right),$$

where c is a constant that only depends on B. Combining the previous inequality with (2.2.6), we obtain

$$\|(\xi,\tau)\|^{\alpha_j}_{\mathbb{K}}\left|\left(C^{-1}(\tau)\xi\right)_j\right| \le c\left(|\delta^0_{\sqrt{1/\tau}}(\xi)|_{\mathbb{K}} + 1\right)^{\alpha_j+1}, \tag{2.2.9}$$

where c is a constant that only depends on B. Then, taking advantage of (1.2.11), we get

$$\|(\xi,\tau)\|_{\mathbb{K}}^{\alpha_j} \left|\partial_{x_j}\Gamma(\xi,\tau)\right| \leq c\Gamma^+(\xi,\tau), \quad j=1,\dots,N,$$

where Γ^+ was defined in (1.2.12). In particular, for $j=1,\dots,m$, we can rewrite the previous estimate as follows

$$\|(\xi,\tau)\|_{\mathbb{K}} \left|\partial_{x_j}\Gamma(\xi,\tau)\right| \leq c\Gamma^+(\xi,\tau), \quad j=1,\dots,m.$$

The desired estimate in (1.1.36) finally follows from (1.1.35).

We are now in a position to state and prove the following result.

Lemma 2.2.2. *Assume that all the eigenvalues of the matrix A belong to some interval $[\lambda,\Lambda]\subset\mathbb{R}^+$. Then there exist two positive constants C, only depending on λ,Λ and on the matrix B, such that the following holds true. For every $R\in(0,1]$ we have that*

$$\sup\left\{\Gamma(z,\zeta) : z\in\mathcal{Q}_{\frac{R}{2}}(0), \zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)\right\} \leq \frac{C}{R^Q}. \tag{2.2.10}$$

Moreover

$$\sup\left\{\left|\partial_{x_j}\partial_{\xi_k}\Gamma(z,\zeta)\right| : z\in\mathcal{Q}_{\frac{R}{2}}(0), \zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)\right\} \leq \frac{C}{R^{Q+\alpha_j+\alpha_k}}. \tag{2.2.11}$$

and

$$\sup\left\{\left|Y\partial_{\xi_k}\Gamma(z,\zeta)\right| : z\in\mathcal{Q}_{\frac{R}{2}}(0), \zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)\right\} \leq \frac{C}{R^{Q+2+\alpha_k}}. \tag{2.2.12}$$

for every $j,k=1,\dots,N$.

Proof. We want to apply (1.1.35), (1.1.36) for two points $z,\zeta\in\mathcal{Q}_{R_0}(0)$, where $R_0>0$ is a constant such that $\|\zeta^{-1}\circ z\|_{\mathbb{K}}\leq R_0$ whenever $z\in\mathcal{Q}_{\frac{1}{2}}(0)$, and $\zeta\in\mathcal{Q}_1(0)$. The existence of such a positive number R_0 follows from the pseudo-triangular inequality (1.1.20). With this choice of R_0, we apply (1.1.35), and we find

$$\begin{aligned}\sup&\left\{\Gamma(z,\zeta) : z\in\mathcal{Q}_{\frac{R}{2}}(0), \zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)\right\} \leq \\ &c\left(\inf\left\{\|\zeta^{-1}\circ z\|_{\mathbb{K}} : z\in\mathcal{Q}_{\frac{R}{2}}(0), \zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)\right\}\right)^{-Q}.\end{aligned} \tag{2.2.13}$$

We therefore need to estimate the infimum of $\|\zeta^{-1}\circ z\|_{\mathbb{K}}$ for $z\in\mathcal{Q}_{\frac{R}{2}}(0)$ and $\zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)$. We first consider the points $\bar{z}:=\delta_{\frac{1}{R}}(z)$ and $\bar{\zeta}:=\delta_{\frac{1}{R}}(\zeta)$ which belong to $\mathcal{Q}_{\frac{1}{2}}(0)$ and $\mathcal{Q}_1(0)\setminus\mathcal{Q}_{\frac{3}{4}}(0)$, respectively. We now define the function $g(\bar{z},\bar{\zeta}):=\|\bar{\zeta}^{-1}\circ_R\bar{z}\|_{\mathbb{K}}$, which is continuous on the compact set $E:=\overline{\mathcal{Q}_{\frac{1}{2}}(0)}\times\overline{\mathcal{Q}_1(0)\setminus\mathcal{Q}_{\frac{3}{4}}(0)}\times[0,1]$, as observed in Remark 1.1.8. Thus, by Weierstrass's Theorem, g attains a minimum m on E, i.e.,

$$\|\bar{\zeta}^{-1}\circ_R\bar{z}\|_{\mathbb{K}}\geq m, \quad \forall\bar{z}\in\overline{\mathcal{Q}_{\frac{1}{2}}(0)}, \quad \forall\bar{\zeta}\in\overline{\mathcal{Q}_1(0)\setminus\mathcal{Q}_{\frac{3}{4}}(0)}, \quad \forall R\in[0,1].$$

Going back to the box of radius R, i.e. applying dilation δ_R to the points $\bar{z}$ and $\bar{\zeta}$ yields

$$\|\zeta^{-1} \circ z\|_{\mathbb{K}} \geq mR, \quad z \in \overline{\mathcal{Q}_{\frac{R}{2}}(0)}, \quad \zeta \in \overline{\mathcal{Q}_R(0) \setminus \mathcal{Q}_{\frac{3R}{4}}(0)}, \tag{2.2.14}$$

and therefore (2.2.13) becomes

$$\sup\left\{\Gamma(z,\zeta) : z \in \mathcal{Q}_{\frac{R}{2}}(0), \zeta \in \mathcal{Q}_R(0) \setminus \mathcal{Q}_{\frac{3R}{4}}(0)\right\} \leq \frac{c}{m^Q R^Q} =: \frac{C}{R^Q} \tag{2.2.15}$$

where the constant C does not depend on R.

To obtain (2.2.11) and (2.2.12), we use the bounds for the derivatives of Γ in (1.1.36) and apply the same arguments as above. □

In order to state our next result, we recall the notation introduced in (1.1.18), that is $\delta_r = \operatorname{diag}(r^{\alpha_1}, \dots, r^{\alpha_N}, r^2)$. In the sequel we assume that all the eigenvalues of the constant matrix A belong to some interval $[\lambda, \Lambda] \subset \mathbb{R}^+$. We are now in position to state our result.

Proposition 2.2.3. *Let u be a solution to $\mathscr{L}u = 0$ in $\mathcal{Q}_R(z_0)$, with $R \in]0,1]$. Then*

$$|\partial_{x_j} u|(z) \leq \frac{C}{R^{\alpha_j}} \|u\|_{L^\infty(\mathcal{Q}_R(z_0))}, \quad \text{for every} \quad z \in \mathcal{Q}_{\frac{R}{2}}(z_0), \quad j = 1, \dots, N,$$

for some positive constant C only depending on λ, Λ and on the matrix B.

Proof. Without loss of generality, we can assume $z_0 = 0$. Let $\eta_R \in C_0^\infty(\mathbb{R}^{N+1})$ be a cut-off function such that

$$\eta_R(x,t) = \chi(\|(x,t)\|_{\mathbb{K}}), \tag{2.2.16}$$

where $\chi \in C^\infty([0,+\infty),[0,1])$ is such that $\chi(s) = 1$ if $s \leq \frac{3R}{4}$, $\chi(s) = 0$ if $s \geq R$ and $|\chi'| \leq \frac{c}{R}$, $|\chi''| \leq \frac{c}{R^2}$. Then, for every $z \in \mathcal{Q}_R(0)$ and for $i = 1, \dots, N$, there exists a constant c, only depending on B, such that

$$|\partial_{x_i} \eta_R(z)| \leq \frac{c}{R^{\alpha_i}}, \qquad |\partial_t \eta_R(z)| \leq \frac{c}{R^2}. \tag{2.2.17}$$

Consequently, for every $z \in \mathcal{Q}_R(0)$ and $i,j = 1, \dots, m$, we have $|\partial^2_{x_i x_j} \eta_R(z)| \leq \frac{c}{R^2}$ and therefore we obtain a bound for the second order part of $|\mathscr{L}\eta_R(z)|$.

Since $\eta_R \equiv 1$ in $\in \mathcal{Q}_{\frac{3R}{4}}(0)$, for every $z \in \mathcal{Q}_{\frac{R}{2}}(0)$ we represent a solution u to $\mathscr{L}u = 0$ as follows

$$u(z) = (\eta_R u)(z) = -\int_{\mathcal{Q}_R(0)} [\Gamma(z,\cdot)\mathscr{L}(\eta_R u)](\zeta) d\zeta. \tag{2.2.18}$$

Since $\mathscr{L} = \text{div}(AD_m) + Y$ and $\mathscr{L}u = 0$ by assumption, (2.2.18) can be rewritten as

$$\begin{aligned} u(z) = (\eta_R u)(z) = &- \int_{\mathcal{Q}_R(0)} [\Gamma(z,\cdot)\text{div}(AD_m(\eta_R))u](\zeta)d\zeta \\ &- \int_{\mathcal{Q}_R(0)} [\Gamma(z,\cdot)Y(\eta_R)u](\zeta)d\zeta \\ &- 2\int_{\mathcal{Q}_R(0)} [\Gamma(z,\cdot)\langle D_m u, AD_m\eta_R\rangle](\zeta)d\zeta. \end{aligned} \tag{2.2.19}$$

Integrating by parts the last integral in (2.2.19), we obtain, for every $z \in \mathcal{Q}_{\frac{R}{2}}(0)$

$$\begin{aligned} u(z) = (\eta_R u)(z) = &\int_{\mathcal{Q}_R(0)} [\Gamma(z,\cdot)\text{div}(AD_m(\eta_R))u](\zeta)d\zeta \\ &- \int_{\mathcal{Q}_R(0)} [\Gamma(z,\cdot)Y(\eta_R)u](\zeta)d\zeta \\ &+ 2\int_{\mathcal{Q}_R(0)} [\langle D_m^\zeta\Gamma(z,\cdot), AD_m\eta_R\rangle u](\zeta)d\zeta, \end{aligned} \tag{2.2.20}$$

where D_m is the gradient with respect to $x_1, \ldots, x_m$ and the superscript in D_m^ζ indicates that we are differentiating w.r.t the variable ζ.

Since $z \in \mathcal{Q}_{\frac{R}{2}}(0)$ and $\partial_{x_i}\eta_R, Y(\eta_R) = 0$ $(i = 1, \ldots, m)$ in $\mathcal{Q}_{\frac{3R}{4}}(0)$, after differentiating under the integral sign (2.2.20), we find

$$\begin{aligned} \partial_{x_j}u(z) = \partial_{x_j}(\eta_R u)(z) = &\int_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)} [\partial_{x_j}\Gamma(z,\cdot)\text{div}(AD_m(\eta_R))u](\zeta)d\zeta \\ &- \int_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)} [\partial_{x_j}\Gamma(z,\cdot)Y(\eta_R)u](\zeta)d\zeta \\ &+ 2\int_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)} [\langle\partial_{x_j}D_m^\zeta\Gamma(z,\cdot), AD_m\eta_R\rangle u](\zeta)d\zeta, \end{aligned}$$

for every $j = 1, ..., N$. Thus, we obtain

$$\begin{aligned} |\partial_{x_j}u(z)| = |\partial_{x_j}(\eta_R u)(z)| \le &\int_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)} \left|[\partial_{x_j}\Gamma(z,\cdot)\text{div}(AD_m(\eta_R))u](\zeta)\right|d\zeta \\ &+ \int_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)} \left|[\partial_{x_j}\Gamma(z,\cdot)Y(\eta_R)u](\zeta)\right|d\zeta \\ &+ 2\int_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)} \left|[\langle\partial_{x_j}D_m^\zeta\Gamma(z,\cdot), AD_m\eta_R\rangle u](\zeta)\right|d\zeta \\ =: &\widetilde{I}_1(z) + \widetilde{I}_2(z) + \widetilde{I}_3(z), \end{aligned}$$

We estimate $\widetilde{I}_1(z)$ and $\widetilde{I}_2(z)$, for $z \in \mathcal{Q}_{\frac{R}{2}}(0)$. We have

$$\widetilde{I}_1(z) \le \|u\|_{L^\infty(\mathcal{Q}_R(0))} \sup_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)} |\mathrm{div}(AD_m(\eta_R))| \mathrm{meas}(\mathcal{Q}_R(0)) \sup_{\substack{z\in\mathcal{Q}_{\frac{R}{2}}(0),\\ \zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)}} \left|\partial_{x_j}\Gamma(z,\zeta)\right|,$$

$$\widetilde{I}_2(z) \le \|u\|_{L^\infty(\mathcal{Q}_R(0))} \sup_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)} |Y(\eta_R)| \mathrm{meas}(\mathcal{Q}_R(0)) \sup_{\substack{z\in\mathcal{Q}_{\frac{R}{2}}(0),\\ \zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)}} \left|\partial_{x_j}\Gamma(z,\zeta)\right|.$$

We now apply Lemma 2.2.2 and obtain

$$\sup_{\substack{z\in\mathcal{Q}_{\frac{R}{2}}(0),\\ \zeta\in\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)}} \left|\partial_{x_j}\Gamma(z,\zeta)\right| \le \frac{\widetilde{C}}{R^{Q+\alpha_j}}. \tag{2.2.21}$$

Moreover, by our choice of the cut-off function η_R, we have

$$|\mathrm{div}(AD_m(\eta_R))| \le \frac{\Lambda\, c}{R^2} \quad \text{in} \quad \mathcal{Q}_R(0), \tag{2.2.22}$$

where Λ is the largest eigenvalue of A. Finally, combining inequalities (2.2.21) and (2.2.22) with $\mathrm{meas}(\mathcal{Q}_R(0)) = R^{Q+2}\mathrm{meas}(\mathcal{Q}_1(0))$, we obtain

$$\widetilde{I}_1(z) \le \frac{C}{R^{\alpha_j}}\|u\|_{L^\infty(\mathcal{Q}_R(0))}, \quad z \in \mathcal{Q}_{\frac{R}{2}}(0), \tag{2.2.23}$$

We now estimate $|Y(\eta_R)| \le |\langle Bx, D\eta_R\rangle| + |\partial_t\eta_R|$ in $\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)$. The bound for the derivative with respect to time of η_R is obtained using (2.2.17). Moreover

$$|\langle Bx, D\eta_R(\zeta)\rangle| \le \sum_{i,k=1}^{N} |b_{ik}||x_k||\partial_{x_i}\eta_R(\zeta)| \le c\sum_{i,k=1}^{N}|b_{ik}|R^{\alpha_k-\alpha_i}, \tag{2.2.24}$$

where $\zeta \in \mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)$. Notice that in sum (2.2.24) the exponent $\alpha_k - \alpha_i$ is always greater or equal to -2, because of the form of the matrix B. Since by assumption $R \le 1$, we estimate (2.2.24) as follows

$$|\langle Bx, D\eta_R\rangle| \le \frac{C'}{R^2}, \quad \text{in } \mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0), \tag{2.2.25}$$

where C' is a constant that only depends on the matrix B and on the constant c in (2.2.17).

Finally, using again $\mathrm{meas}(\mathcal{Q}_R(0)) = R^{Q+2}\mathrm{meas}(\mathcal{Q}_1(0))$, together with (2.2.21) and (2.2.25), we obtain

$$\widetilde{I}_2(z) \le \frac{C}{R^{\alpha_j}}\|u\|_{L^\infty(\mathcal{Q}_R(0))}, \quad z \in \mathcal{Q}_{\frac{R}{2}}(0), \tag{2.2.26}$$

where C depends only on the constants c and $\widetilde{C}$ in (2.2.17) and (2.2.21) and on the matrix

B.

By the same argument we prove that, for a point $z \in \mathcal{Q}_{\frac{R}{2}}(0)$, we have

$$\widetilde{I}_3(z) \leq \|u\|_{L^\infty(\mathcal{Q}_R(0))} \frac{c}{R} \operatorname{meas}(\mathcal{Q}_R(0)) \sup_{\substack{z \in \mathcal{Q}_{\frac{R}{2}}(0), \\ \zeta \in \mathcal{Q}_R(0) \setminus \mathcal{Q}_{\frac{3R}{4}}(0)}} \left|\partial_{x_j} D_m^\zeta \Gamma(z,\zeta)\right| \leq \frac{C}{R^{\alpha_j}} \|u\|_{L^\infty(\mathcal{Q}_R(0))}.$$

where C denotes once again a constant depending only on c, $\widetilde{C}$ and B. Combining the inequality above with (2.2.23) and (2.2.26), we finally obtain

$$\|\partial_{x_j} u\|_{L^\infty(\mathcal{Q}_{\frac{R}{2}}(0))} \leq \frac{C}{R^{\alpha_j}} \|u\|_{L^\infty(\mathcal{Q}_R(0))}, \quad j = 1, ..., N.$$

□

We state a result analogous to Proposition 2.2.3, written in terms of the vector fields $X_1, \dots, X_m, Y$ introduced in (1.1.4).

Proposition 2.2.4. *Let u be a solution to $\mathscr{L} u = 0$ in $\mathcal{Q}_R(0)$, for $R \in]0,1[$, then for any $X_i, X_j \in \{X_1, ..., X_m\}$, there exists a constant C, only depending on λ, Λ and on the matrix B, such that*

$$|X_i u|(z) \leq \frac{C}{R} \|u\|_{L^\infty(\mathcal{Q}_R(0))}, \quad z \in \mathcal{Q}_{\frac{R}{2}}(0),$$
$$|X_i X_j u|(z) \leq \frac{C}{R^2} \|u\|_{L^\infty(\mathcal{Q}_R(0))}, \quad z \in \mathcal{Q}_{\frac{R}{2}}(0).$$

Similarly, we have that

$$|Yu|(z) \leq \frac{C}{R^2} \|u\|_{L^\infty(\mathcal{Q}_R(0))}, \quad z \in \mathcal{Q}_{\frac{R}{2}}(0).$$

Proof. The estimate of $X_1, \dots, X_m$ has been proved in Proposition 2.2.3. The proof of the remaining estimates is obtained by reasoning as in Proposition 2.2.3, and using estimates (2.2.11) and (2.2.12), respectively. We omit the details here. □

In the sequel, we will need to estimate the second order derivatives of a solution to $\mathscr{L} u = g$, where g is a polynomial of degree at most two. To this end, we let

$$g_1(z) = \langle v, x \rangle, \qquad g_2(z) = \langle Mx, x \rangle, \tag{2.2.27}$$

be two polynomial functions, where v and M denote a constant vector of $\mathbb{R}^N$ and a $N \times N$ constant matrix, respectively.

Lemma 2.2.5. *Let η_R be the cut-off function introduced in* (2.2.16) *and let g_1 and g_2 be the functions defined in* (2.2.27). *Then there exists a positive constant C, only depending on λ, Λ*

and on the matrix B, such that

$$\left|\partial^2_{x_ix_j}\int_{\mathcal{Q}_R(0)}\Gamma(z,\zeta)\eta_R(\zeta)d\zeta\right|\leq C, \tag{2.2.28}$$

$$\left|\partial^2_{x_ix_j}\int_{\mathcal{Q}_R(0)}\Gamma(z,\zeta)\eta_R(\zeta)g_1(\zeta)d\zeta\right|\leq CR, \tag{2.2.29}$$

$$\left|\partial^2_{x_ix_j}\int_{\mathcal{Q}_R(0)}\Gamma(z,\zeta)\eta_R(\zeta)g_2(\zeta)d\zeta\right|\leq CR^2, \tag{2.2.30}$$

for every $z\in\mathcal{Q}_{\frac{R}{2}}(0)$, $R\in]0,1]$ and for any $i,j=1,\dots,m$.

Proof. Reasoning as in the proof of Proposition 2.11 in [41], we write the right-hand side of (2.2.28) as

$$\begin{aligned}\partial^2_{x_ix_j}\int_{\mathcal{Q}_R(0)}\Gamma(\zeta^{-1}\circ z)\eta_R(\zeta)d\zeta &= \lim_{\varepsilon\to0}\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)\eta_R(\zeta)d\zeta\\ &\quad+\eta_R(z)\int_{\|\zeta\|_{\mathbb{K}}=1}\partial_{x_i}\Gamma_0(\zeta)\nu_j d\sigma(\zeta)\\ &=: \lim_{\varepsilon\to0}I^0_1(\varepsilon,z)+I^0_2(z).\end{aligned} \tag{2.2.31}$$

We rewrite $I^0_1(\varepsilon,z)$ as

$$\begin{aligned}I^0_1(\varepsilon,z)=&\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)\left(\eta_R(\zeta)-\eta_R(z)\right)d\zeta\\ &+\eta_R(z)\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)d\zeta.\end{aligned} \tag{2.2.32}$$

By the definition of η_R, we have

$$0\leq\eta_R\leq1,\quad \eta_R(\zeta)-\eta_R(z)=0,\quad \text{for any } \zeta\in\mathcal{Q}_{\frac{3R}{4}}(0),\ z\in\mathcal{Q}_{\frac{R}{2}}(0). \tag{2.2.33}$$

Thus, taking advantage of Lemma 2.2.2, we infer

$$\begin{aligned}&\left|\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)\left(\eta_R(\zeta)-\eta_R(z)\right)d\zeta\right|\\ &=\left|\int_{\mathcal{Q}_R(0)\setminus\mathcal{Q}_{\frac{3R}{4}}(0)}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)\left(\eta_R(\zeta)-\eta_R(z)\right)d\zeta\right|\leq\frac{C}{R^{Q+2}}R^{Q+2}=C\end{aligned} \tag{2.2.34}$$

Thus we find

$$I^0_2(z)+\lim_{\varepsilon\to0}\eta_R(z)\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)d\zeta=C. \tag{2.2.35}$$

Combining estimates (2.2.34) and (2.2.35) we conclude the proof of (2.2.28).

We now prove (2.2.29). Reasoning as in (2.2.31) and exploiting the definition of g_1, we can rewrite the right-hand side of (2.2.29) as

$$\begin{aligned}\partial^2_{x_ix_j}\int_{\mathcal{Q}_R(0)}\Gamma(\zeta^{-1}\circ z)\eta_R(\zeta)\langle v,\xi\rangle d\zeta &= \lim_{\varepsilon\to 0}\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)\eta_R(\zeta)\langle v,\xi\rangle d\zeta\\ &\quad+\langle v,x\rangle\eta_R(z)\int_{\|\zeta\|_{\mathbb{K}}=1}\partial_{x_i}\Gamma_0(\zeta)\nu_j d\sigma(\zeta)\\ &=:\lim_{\varepsilon\to 0}I_1^1(\varepsilon,z)+I_2^1(z).\end{aligned} \tag{2.2.36}$$

We prove that the first integral in (2.2.36) uniformly converges as $\varepsilon\to 0^+$. To this end, we first rewrite $I_1^1(\varepsilon,z)$ as follows

$$\begin{aligned}I_1^1(\varepsilon,z)&=\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)\left(\eta_R(\zeta)-\eta_R(z)\right)\langle v,\xi\rangle d\zeta\\ &\quad+\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)\left(\eta_R(z)\right)\langle v,\xi-x\rangle d\zeta\\ &\quad+\langle v,x\rangle\eta_R(z)\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)d\zeta\\ &=:I_1'(\varepsilon,z)+I_2'(\varepsilon,z)+I_3'(\varepsilon,z).\end{aligned} \tag{2.2.37}$$

To estimate $I_1'(\varepsilon,z)$ we use the same argument as in (2.2.34), with the only difference that now in the integral we have the additional term $\langle v,\xi\rangle$. We find a bound for this term observing that

$$|\langle v,\xi\rangle|\leq|v|\cdot\|\zeta\|_{\mathbb{K}}\leq|v|\cdot R, \tag{2.2.38}$$

where $|v|$ denotes the norm of v in $\mathbb{R}^N$. Therefore, we obtain

$$|I_1'(\varepsilon,z)|\leq C\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\frac{d\zeta}{\|\zeta^{-1}\circ z\|_{\mathbb{K}}^{Q+1}}\leq c\,R, \tag{2.2.39}$$

where C is a constant that depends only on λ,Λ,B and v.

We now show that the same bound holds for $I_2'(\varepsilon,z)$. We first observe that

$$|\langle v,x-\xi\rangle|\leq|v|\cdot\|\zeta^{-1}\circ z\|_{\mathbb{K}}, \tag{2.2.40}$$

As a consequence, using again (2.2.33) and (1.1.36), we infer

$$|I_2'(\varepsilon,z)|\leq\int_{\mathcal{Q}_R(0)\cap\{\|\zeta^{-1}\circ z\|_{\mathbb{K}}\geq\varepsilon\}}\frac{d\zeta}{\|\zeta^{-1}\circ z\|_{\mathbb{K}}^{Q+1}}\leq c(R-\varepsilon)\leq c\,R. \tag{2.2.41}$$

Using (2.2.39) and (2.2.41) we obtain

$$\lim_{\varepsilon\to 0^+} I_1'(\varepsilon,z)=O(R), \qquad \lim_{\varepsilon\to 0^+} I_2'(\varepsilon,z)=O(R), \quad \text{as } R\to 0. \tag{2.2.42}$$

Finally, as for $I_3'(\varepsilon,z)$, we compute

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathcal{Q}_R(0)\cap\{\varepsilon\le\|\zeta^{-1}\circ z\|_{\mathbb{K}}\le \mathbf{c}R\}} \partial^2_{x_ix_j}\Gamma(\zeta^{-1}\circ z)d\zeta\\
&=\lim_{\varepsilon\to 0}\int_{\mathcal{Q}_R(0)\cap\{\varepsilon\le\|\zeta^{-1}\circ z\|_{\mathbb{K}}\le \mathbf{c}R\}} \partial^2_{w_iw_j}\Gamma(w)e^{-\tau\,\mathrm{tr}B}dw\\
&=-\lim_{\varepsilon\to 0}\int_{\|w\|_{\mathbb{K}}=\varepsilon}\partial_{w_i}\Gamma(w)e^{-\tau\,\mathrm{tr}B}\nu_j d\sigma_j(w)+\lim_{\varepsilon\to 0}\int_{\|w\|_{\mathbb{K}}=\mathbf{c}R}\partial_{w_i}\Gamma(w)e^{-\tau\,\mathrm{tr}B}\nu_j d\sigma_j(w)\\
&=-\int_{\|w\|_{\mathbb{K}}=1}\partial_{w_i}\Gamma_0(w)\nu_j d\sigma_j(w)+\int_{\|w\|_{\mathbb{K}}=\mathbf{c}R}\partial_{w_i}\Gamma(w)e^{-\tau\,\mathrm{tr}B}\nu_j d\sigma_j(w).
\end{aligned}$$

We then obtain,

$$\begin{aligned}
I_2^1(z)+\lim_{\varepsilon\to 0}I_3'(\varepsilon,z)&=\langle v,x\rangle\eta_R(z)\int_{\|\zeta\|_{\mathbb{K}}=1}\partial_{x_i}\Gamma_0(\zeta)\nu_j d\sigma(\zeta)\\
&\quad-\langle v,x\rangle\eta_R(z)\int_{\|w\|_{\mathbb{K}}=1}\partial_{w_i}\Gamma_0(w)\nu_j d\sigma_j(w)\\
&\quad+\langle v,x\rangle\eta_R(z)\int_{\|w\|_{\mathbb{K}}=\mathbf{c}R}\partial_{w_i}\Gamma(w)e^{-\tau\,\mathrm{tr}B}\nu_j d\sigma_j(w)\\
&=\langle v,x\rangle\eta_R(z)\int_{\|w\|_{\mathbb{K}}=\mathbf{c}R}\partial_{w_i}\Gamma(w)e^{-\tau\,\mathrm{tr}B}\nu_j d\sigma_j(w).
\end{aligned}$$

Keeping in mind that

$$\lim_{R\to 0}\int_{\|w\|_{\mathbb{K}}=\mathbf{c}R}\partial_{w_i}\Gamma(w)e^{-\tau\,\mathrm{tr}B}\nu_j d\sigma_j(w)=\int_{\|w\|_{\mathbb{K}}=1}\partial_{w_i}\Gamma_0(w)\nu_j d\sigma_j(w)=c',$$

we finally find

$$I_2^1(z)+\lim_{\varepsilon\to 0}I_3'(\varepsilon,z)=O(R), \quad \text{as } R\to 0. \tag{2.2.43}$$

Identity (2.2.29) follows from (2.2.39), (2.2.41) and (2.2.43).

By the same argument, we obtain

$$\partial^2_{x_ix_j}\int_{\mathcal{Q}_R(0)}\Gamma(\zeta^{-1}\circ z)\eta_R(\zeta)\langle M\xi,\xi\rangle d\zeta=O(R^2). \tag{2.2.44}$$

We omit the details here as the procedure is analogous. □

From Proposition 2.2.3 and Lemma 2.2.5, we derive the following result.

Lemma 2.2.6. *Let w be a solution to $\mathscr{L}w=\langle v,\xi\rangle+\langle M\xi,\xi\rangle$ in $\mathcal{Q}_R(0)$, where v and M are* *as in* (2.2.27)*. Then*

$$|\partial^2_{x_ix_j} w(z)| \le \frac{C}{R^2}\|w\|_{L^\infty(\mathcal{Q}_R(0))} + CR, \tag{2.2.45}$$

for every $i,j = 1,\dots,m, 0 < R \le 1$ *and for any* $z \in \mathcal{Q}_{\frac{R}{2}(0)}$.

Proof. Reasoning as in Proposition 2.2.3, we obtain

$$\begin{aligned}
|\partial^2_{x_ix_j} w(z)| \le &\int_{\mathcal{Q}_R(0)\setminus \mathcal{Q}_{\frac{3R}{4}}(0)} \left|[\partial^2_{x_ix_j}\Gamma(z,\cdot)\mathrm{div}(AD_m(\eta_R))w](\zeta)\right| d\zeta \\
&+ \int_{\mathcal{Q}_R(0)\setminus \mathcal{Q}_{(3/4)\varrho^k}(0)} \left|[\partial^2_{x_ix_j}\Gamma(z,\cdot)Y(\eta_R)w](\zeta)\right| d\zeta \\
&+ 2\int_{\mathcal{Q}_R(0)\setminus \mathcal{Q}_{\frac{3R}{4}}(0)} \left|[\partial^2_{x_ix_j}\Gamma(z,\cdot)\langle D_m w, AD_m\eta_R\rangle](\zeta)\right| d\zeta \\
&+ \left|\left[\partial^2_{x_ix_j}\int_{\mathcal{Q}_R(0)}[\Gamma(z,\cdot)\eta_R](\zeta)\langle v,\xi\rangle d\zeta\right]\right| \\
&+ \left|\left[\partial^2_{x_ix_j}\int_{\mathcal{Q}_R(0)}[\Gamma(z,\cdot)\eta_R](\zeta)\langle M\xi,\xi\rangle d\zeta\right]\right| \\
=:\ & \overline{I}_1(z) + \overline{I}_2(z) + \overline{I}_3(z) + \overline{I}_4(z) + \overline{I}_5(z).
\end{aligned} \tag{2.2.46}$$

The terms $\overline{I}_1(z), \overline{I}_2(z), \overline{I}_3(z)$ were already estimated in Proposition 2.2.3 as

$$\overline{I}_1(z), \overline{I}_2(z), \overline{I}_3(z) \le \frac{C}{R^2}\|w\|_{L^\infty(\mathcal{Q}_R(0))}, \quad z \in \mathcal{Q}_{\frac{R}{2}(0)}.$$

Additionally, $\overline{I}_4(z)$ and $\overline{I}_5(z)$ are $O(R)$ in virtue of Lemma 2.2.5 and thus (2.2.45) is proved. □

We now prove a mean value theorem for solutions u to $\mathscr{L}u = 0$ in cylinders $\mathcal{Q}_R(\zeta)$.

Proposition 2.2.7 (Scale invariant Lipschitz estimate)**.** *Let* ζ *be any point of* $\mathbb{R}^{N+1}$*, and let* u *be a solution to* $\mathscr{L}u = 0$ *in* $\mathcal{Q}_R(\zeta)$*, with* $R \in (0,1]$*. Then the following estimate holds*

$$|u(z) - u(\zeta)| \le \frac{C}{R} d_{\mathbb{K}}(z,\zeta)\|u\|_{L^\infty(\mathcal{Q}_R(\zeta))}, \tag{2.2.47}$$

for every $z \in \mathcal{Q}_{\frac{R}{2}}(\zeta)$*. Here* C *is a constant that only depends on* λ, Λ *and on the matrix* B*.*

Proof. Thanks to the left-invariance of operator $\mathscr{L}$, it is not restrictive to assume $\zeta = 0$, then we need to prove

$$|u(z) - u(0)| \le \frac{C}{R}\|z\|_{\mathbb{K}}\|u\|_{L^\infty(\mathcal{Q}_R(0))}.$$

Consider $z = (x,t) \in \mathcal{Q}_{\frac{R}{2}}(0)$, and apply the standard mean-value theorem

$$
\begin{aligned}
|u(z) - u(0)| &= |u(x_1, \dots, x_N, t) - u(0, \dots, 0, 0)| \\
&\le \sum_{i=1}^{N} |x_i|\, |\partial_{x_i} u(\vartheta_1 x_1, \dots, \vartheta_N x_N, t)| + |t|\, |Yu(0, \dots, 0, \vartheta t),
\end{aligned} \tag{2.2.48}
$$

where $\vartheta_1, \dots, \vartheta_N, \vartheta \in (0,1[$. For every $i = 1, \dots, N$, we have $|x_i| \le \|z\|_{\mathbb{K}}^{\alpha_i} \le R^{\alpha_i}$, and $(\vartheta_1 x_1, \dots, \vartheta_N x_N, t) \in \mathcal{Q}_{\frac{R}{2}}(0)$. Then, by Proposition 2.2.3, we find

$$|\partial_{x_i} u(\vartheta_1 x_1, \dots, \vartheta_N x_N, t)| \le \frac{c}{R^{\alpha_i}} \|u\|_{L^\infty(\mathcal{Q}_R(0))}.$$

so that

$$|x_i|\, |\partial_{x_i} u(\vartheta_1 x_1, \dots, \vartheta_N x_N, t)| \le \frac{c}{R} \|z\|_{\mathbb{K}} \|u\|_{L^\infty(\mathcal{Q}_R(0))}.$$

Analogously, we have that $|\vartheta t| \le |t| \le \|z\|_{\mathbb{K}}^2 \le R^2$, and from Proposition 2.2.4 it follows that

$$|Yu(0, \dots, 0, \vartheta t)| \le \frac{c}{R^2} \|u\|_{L^\infty(\mathcal{Q}_R(0))},$$

thus

$$|t|\, |Yu(0, \dots, 0, \vartheta t)| \le \frac{c}{R} \|z\|_{\mathbb{K}} \|u\|_{L^\infty(\mathcal{Q}_R(0))}.$$

The proof of the proposition can be obtained by combining the above estimates. □

2.3 Taylor formula

In this section, we prove Theorem 2.1.2. The proof is based on the method introduced by Pagliarani, Pascucci and Pignotti in [93] for the dilation-invariant operator $\mathscr{L}_0$ and then generalized by Pagliarani and Pignotti in [94] and by Pignotti in [101] to the not dilation-invariant one.

We want to emphasize the main differences with respect to the previous approaches in the literature. The first result about a Taylor inequality in homogeneous groups goes back to the seminal book of Folland and Stein, [47]. In the proof, they used a quantitative version of the Carathéodory-Chow-Rashevsky connectivity result and a Mean Value Theorem. A slightly improved version of this result has been proved by Bonfiglioli [18] who was able to derive also a Taylor formula with integral remainder. Both approaches were assuming, for a polynomial of degree n, the differentiability up to order n in the Euclidean sense. A more intrinsic point of view has been introduced in the paper [93], where the authors considered functions regular in the intrinsic sense, i.e. with respect to the Lie Group structure introduced in Subsection 1.1.1.

In order to prove Theorem 2.1.2, we follow the same procedure introduced in [93] and

[94,101] and we point out the modifications needed to deal with our slightly different situation.

We next introduce some further notation. We define the spaces $V_0, \dots, V_\kappa$ as the vector subspaces of $\mathbb{R}^N$ which are invariant with respect to dilation $(\delta_r)_{r>0}$ introduced in (1.1.13). Specifically, for $n = 0, \dots, \kappa$, we set

$$V_n := \{0\}^{\bar{m}_{n-1}} \times \mathbb{R}^{m_n} \times \{0\}^{N-\bar{m}_n},$$

where $\bar{m}_n := m_0 + \dots m_n$, with $m_{-1} \equiv 0$. Moreover, we let $x^{[n]}$ be the projection of $x \in \mathbb{R}^N$ on V_n. Note that

$$\mathbb{R}^N = \bigoplus_{n=0}^{\kappa} V_n, \qquad x = x^{[0]} + \dots + x^{[\kappa]}, \tag{2.3.1}$$

for every $x \in \mathbb{R}^N$. Moreover, in accordance with the dilation $(\delta_r)_{r>0}$, we have

$$\delta_r(x^{[n]}) = r^{2n+1} x^{[n]}, \quad \forall x^{[n]} \in V_n, \tag{2.3.2}$$

for every $n = 0, \dots, \kappa$. In virtue of assumption **[H.1]**, the linear application $B^n : V_0 \to V_n$ is surjective; however, it is in general not injective. Thus, we define the subspaces $V_{0,n} \subset V_0$ as follows

$$V_{0,n} := \ker(B^n)^\perp.$$

The linear map $B^n : V_{0,n} \to V_n$ is now bijective.

The method of the proof relies on the construction of a finite sequence of points which connect $z = (x, t)$ and $\zeta = (\xi, \tau)$ and are located along suitable trajectories. More precisely, we start from z and choose $z_1 = (x_1, t_1)$ as the point along the integral curve of the drift Y satisfying the condition $t_1 = \tau$. We then move along the integral paths of $X_1, \dots, X_m$ to a point $z_2 = (x_2, t_2)$ such that $x_2^{[0]} = \xi^{[0]}$ and $t_2 = \tau$. This allows us to exploit the regularity of u along the vector fields $X_1, \dots, X_m, Y$ and estimate the remainder in (2.1.4) in terms of the homogeneous norm of the new points.

Since we have no apriori regularity of u with respect to other vector fields, we increment the higher level coordinates $x^{[1]}, \dots, x^{[\kappa]}$ by moving along trajectories defined as concatenations of integral curves of $X_1, \dots, X_m, Y$. Specifically, for any $z \in \mathbb{R}^{N+1}$ and $s \in \mathbb{R}$ we define iteratively the family of trajectories $(\gamma_{v,s}^{(n)}(z))_{n=0,\dots,\kappa}$ as follows

$$\begin{aligned} \gamma_{v,s}^{(0)}(z) &= e^{sX_v}(z) = (x + sv, t) \\ \gamma_{v,s}^{(n+1)}(z) &= e^{-s^2 Y}(\gamma_{v,-s}^{(n)}(e^{s^2 Y}(\gamma_{v,s}^{(n)}(z)))), \end{aligned} \tag{2.3.3}$$

where v is a suitable vector in V_0, and $X_v = v_1 \partial_{x_1} + \dots + v_m \partial_{x_m}$.

At this point we need to distinguish the dilation-invariant operators from the non dilation-inviariant ones. In the first case, the trajectories $(\gamma_{v,s}^{(n)}(z))_{n=0,\dots,\kappa}$ have the remarkable property of modifying the components $x^{[n]} + \cdots + x^{[\kappa]}$ leaving unchanged the components $x^{[0]} + \cdots + x^{[n-1]}$; thus, we reach the point ζ after κ steps. The proof of Theorem 2.1.2, for dilation-invariant operators, follows by exploiting the regularity of u with respect to $X_1, \dots, X_m, Y$, as we connect z to ζ along integral curves of the vector fields $X_1, \dots, X_m, Y$. The next example illustrates the geometric construction in the simplest case, corresponding to $\kappa = 1$.

Example 2.3.1. We consider the degenerate Kolmogorov operator

$$\mathcal{K}_0 := \partial^2_{xx} + x\partial_y - \partial_t$$

and show how to use the trajectories defined in (2.3.3) to connect an arbitrary point $z \in \mathbb{R}^3$ with the origin. In this case, we have

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and thus

$$e^{sX}(x, y, t) = (x + s, y, t), \quad e^{sY}(x, y, t) = (x, y + sx, t - s).$$

Moreover,

$$\mathbb{R}^2 = V_0 \oplus V_1 = \operatorname{span}\{e_1\} \oplus \operatorname{span}\{e_2\}, \quad V_{0,0} = V_{0,1} = \operatorname{span}\{e_1\}.$$

Let $z = (x, y, t)$ be a point in $\mathbb{R}^3$, and consider for simplicity $\zeta = (0, 0, 0)$. We first adjust the temporal component by moving along the drift Y, and we reach the point

$$z_1 = e^{tY}(z) = (x, y + tx, 0).$$

We then move along the integral curve of the vector field X to make x equal to 0:

$$z_2 = e^{s_0 X}(z_1) = (x + s_0, y + tx, 0) = (0, y + tx, 0), \quad \text{by choosing} \quad s_0 = -x.$$

We reached the point $z_2 \in V_1$ and we plan to steer it to $(0, 0, 0)$. We move along a curve defined as concatenation of integral paths of X and Y as follows:

$$\begin{aligned} z_3 &= e^{s_1 X}(z_2) = (s_1, y + tx, 0), \\ z_4 &= e^{s_1^2 Y}(z_3) = (s_1, y + tx + s_1^3, -s_1^2), \\ z_5 &= e^{-s_1 X}(z_4) = (0, y + tx + s_1^3, -s_1^2), \\ z_6 &= e^{-s_1^2 Y}(z_5) = (0, y + tx + s_1^3, 0), \end{aligned} \tag{2.3.4}$$

and we reach the point $\zeta = (0, 0, 0)$ if we choose $s_1 = (-tx - y)^{\frac{1}{3}}$.

When considering a not dilation-invariant operator $\mathscr{L}$, the method illustrated above fails.

Indeed, in this case the trajectory $(\gamma^{(n)}_{v,s}(z))$ may affect the components $x^{[0]} + \cdots + x^{[n-1]}$, as the following example shows.

Example 2.3.2. We consider the degenerate Kolmogorov operator

$$\mathcal{K} := \partial^2_{xx} + x\partial_y + x\partial_x - \partial_t. \tag{2.3.5}$$

In this case, B takes the form

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

and therefore the operator $\mathcal{K}$ is not dilation-invariant. Let us emphasize the differences with the dilation-invariant case studied in Example 2.3.1. We denote again the points in $\mathbb{R}^3$ by $z = (x, y, t)$ and consider $\zeta = (0, 0, 0)$. The first two steps of the procedure used in Example 2.3.1 allow us to move from z to some point $z_1 = (x_1, y_1, 0)$, then to some other point $z_2 = (0, y_2, 0)$. The difference with the homogeneous case arises in the third step, i.e. when we are dealing with the y-variable.

Let us suppose we want to move from any point $z = (0, y, 0) \in V_1$ to the origin $(0, 0, 0)$. If we reproduce the same construction as in (2.3.4), we find:

$$\begin{aligned} z_1 &= e^{sX}(z) = (s, y, 0), \\ z_2 &= e^{s^2Y}(z_1) = (se^{s^2}, -s + se^{s^2} + y, -s^2), \\ z_3 &= e^{-sX}(z_2) = (se^{s^2} - s, -s + se^{s^2} + y, -s^2), \\ z_4 &= e^{-s^2Y}(z_3) = (s(1 - e^{-s^2}), s(1 - e^{-s^2}) + y, 0). \end{aligned} \tag{2.3.6}$$

If we choose s such that $s(1 - e^{-s^2}) = -y$, we obtain $z_4 = (-y, 0, 0)$, so that its second component is zero but, in constrast with the previous Example 2.3.1, we have that z_4 doesn't agree with our target point $\zeta = (0, 0, 0)$.

In order to reach the point $\zeta = (0, 0, 0)$ also in the case of not dilation-invariant operators, we rely on the method introduced by Pagliarani and Pignotti in [94] and by Pignotti in [101]. In the case of the operator $\mathcal{K}$ in (2.3.5) it is sufficient to use once more the integral curve of the vector field $X = \partial_x$. In the case of more general operators a further topological argument is needed to conclude the construction. We refer to [94, 101] for a detailed description of this construction. This step is needed as the invariance properties of operator $\mathscr{L}$ are related to the structure of the matrix B in (1.1.2). For this reason, to generalize the proof of Theorem 2.1.2 to the not dilation-invariant case is necessary also when treating the constant-coefficients operator $\mathscr{L}$ (and thus Theorem 2.1.4).

We are now ready to prove our result.

Proof of Theorem 2.1.2. Let $z = (x, t), \zeta = (\xi, \tau)$ be two given points of Ω. As explained above, the proof relies on a finite sequence of integral paths of the vector fields $X_1, \dots, X_m$ and Y connecting z to ζ. We use the construction made by Pagliarani, Pascucci and Pignotti [93] for a dilation-invariant operator $\mathscr{L}$. In this case the trajectories $(\gamma^{(n)}_{v,s}(z))_{n=0,\dots,\kappa}$ defined in (2.3.3) are explicitly given and we prove that (2.1.4) holds. We then discuss the modifications needed to deal with any not dilation-invariant operator $\mathscr{L}$, as introduced by Pagliarani and Pignotti in [94] and by Pignotti in [101].

As a preliminary result, we prove our claim (2.1.4) under the assumption that the points $z = (x,t)$ and $\zeta = (\xi,\tau)$ have the same temporal component $t = \tau$, by a finite iteration on $n = 0,\dots,\kappa$. We remove this assumption in the last part of the proof.

Base case $n = 0$. In this case, we are only changing the variables x_i, for $i = 1,\dots,m$, moving along the direction $e^{s_0 X_{v_0}}$ where $v_0 = (v_{0,1},\dots,v_{0,m},0\dots,0)$ is a suitable unit vector in V_0. Thus, equation (2.1.3), with $z = (x,t)$ and $\zeta = (x + s_0 v_0, t)$, rewrites as

$$T^2_z u(\zeta) = u(x,t) + \sum_{i=1}^{m} \partial_{x_i} u(x,t) s_0 v_{0,i} + \frac{s_0^2}{2} \sum_{i,j=1}^{m} \partial^2_{x_i,x_j} u(x,t) v_{0,i} v_{0,j}. \tag{2.3.7}$$

We observe that $\|z^{-1}\circ\zeta\|_K^2 = |s_0|^2$ and therefore we want to show that

$$u(\zeta) - T^2_z u(\zeta) = o(|s_0|^2) \quad \text{as} \quad s_0 \to 0. \tag{2.3.8}$$

By the multidimensional Euclidean mean-value theorem, there exist $(\bar{v}_{i,j})_{1\le i,j\le m}$, with $\bar{v}_{i,j} \in \text{span}\{e_1,\dots,e_m\}$ and $|\bar{v}_{i,j}| \le |v_0|$ such that

$$\begin{aligned} u(\zeta) - T^2_z u(\zeta) &= \frac{s_0^2}{2} \sum_{i,j=1}^{m} (\partial^2_{x_i,x_j} u(x + s_0 \bar{v}_{i,j}, t) v_{0,i} v_{0,j} - \partial^2_{x_i,x_j} u(x,t)) v_{0,i} v_{0,j} \\ &= o(|s_0|^2) \quad \textit{as } s_0 \to 0, \end{aligned} \tag{2.3.9}$$

where we have used the continuity of the second order derivatives of u. Thus, we have proved (2.3.8) and we are done.

Let us remark that we do not need the dilation-invariance property for Y, as we do not make use of the vector field Y in this part of the construction.

Inductive step. Suppose that the thesis is true for a given nonnegative $n < \kappa$. We prove it for $n+1$. For every $z,\zeta \in \mathbb{R}^{N+1}$ we set

$$\widetilde{T}^2_z u(\zeta) := T^2_z u(\zeta) - u(z). \tag{2.3.10}$$

We define the points

$$\begin{gathered} z = (x,t), z_1 = \gamma^{(n)}_{v,s}(z), z_2 = e^{s^2 Y}(z_1) \\ z_3 = \gamma^{(n)}_{v,-s}(z_2), z_4 = e^{-s^2 Y}(z_3) = \gamma^{(n+1)}_{v,s}(z) \end{gathered}$$

where v is the unique unitary vector in $V_{0,n+1} \subset V_0$, defined as $v = \frac{w}{|w|}$, where w is the vector in $V_{0,n+1}$ such that $B^{n+1}w = \zeta^{[n+1]} - z^{[n+1]}$ and $s = |w|^{\frac{1}{2(n+1)+1}}$. We aim to prove that

$$u(z_4) - T^2_z u(z_4) = o(\|z^{-1} \circ z_4\|^2_{\mathbb{K}}) = o(|s|^2) \quad \text{as} \quad s \to 0. \tag{2.3.11}$$

We now rewrite (2.3.11) by using the notation (2.3.10) as follows

$$\begin{aligned} u(z_4) - T^2_z u(z_4) = & \boxed{u(z_4) - u(z_3)}_{(1)} \\ & + \boxed{u(z_3) - u(z_2) - \widetilde{T}^2_{z_2} u(z_3)}_{(2)} \\ & + \boxed{u(z_2) - u(z_1)}_{(3)} \\ & + \boxed{\widetilde{T}^2_{z_1} u(z) + u(z_1) - u(z)}_{(4)} \\ & + \boxed{\widetilde{T}^2_{z_2} u(z_3) - \widetilde{T}^2_{z_1} u(z)}_{(5)} \\ & - \boxed{\widetilde{T}^2_z u(z_4)}_{(6)}. \end{aligned} \tag{2.3.12}$$

By the inductive hypothesis, the second and the forth difference are $o(|s|^2)$ as $s \to 0$. Moreover, recalling (2.1.3), we have that $\widetilde{T}^2_z u(z_4) \equiv 0$, being $x^{[0]}_4 = x^{[0]}$ and $t_4 = t$.

We next apply definition (2.1.3) to the fifth difference, and we find

$$\begin{aligned} \widetilde{T}^2_{z_2} u(z_3) - \widetilde{T}^2_{z_1} u(z) = & - s \sum_{i=1}^{m} (\partial_{x_i} u(z_2) - \partial_{x_i} u(z_1)) v_i \\ & - \frac{s^2}{2} \sum_{i,j=1}^{m} (\partial^2_{x_i x_j} u(z_2) - \partial^2_{x_i x_j} u(z_1)) v_i v_j. \end{aligned} \tag{2.3.13}$$

As an immediate consequence of condition (2.1.2), we obtain the following equation

$$\partial_{x_i} u(z_2) - \partial_{x_i} u(z_1) = \partial_{x_i} u(e^{s^2 Y}(z_1)) - \partial_{x_i} u(z_1) = o(|s|). \tag{2.3.14}$$

Using the previous equation and the continuity of second order derivatives of u, we find that (2.3.13) is equal to $o(|s|^2)$.

We now observe that

$$u(z_4) - u(z_3) = u(e^{-s^2 Y}(z_3)) - u(z_3).$$

By applying the mean value theorem along the direction of the drift, we find that there exists $\bar{s}$ such that

$$u(e^{-s^2 Y}(z_3)) - u(z_3) = -s^2 Y u(e^{\bar{s} Y}(z_3)),$$

where $|\bar{s}| \leq |s|$. Similarly we obtain that

$$u(z_2) - u(z_1) = s^2 Y u(e^{\widetilde{s}Y}(z_1)),$$

where again $\widetilde{s}$ verifies $|\widetilde{s}| \leq |s|$.

By letting $s \to 0$, we find that $\bar{s}, \widetilde{s} \to 0$, and therefore, using the continuity of Yu, we have shown that the sum of the first and the third difference in (2.3.12) is again equal to $o(|s|^2)$ as $s \to 0$. This proves (2.3.11) and therefore concludes the proof of the inductive step.

As already pointed out, the construction of the trajectories in the case of not dilation-invariant operators requires the adjustments introduced in [94, 101], to deal with the fact that the term $\widetilde{T}_z^2 u(z_4)$ in (2.3.12) fails to vanish. Indeed, with the notation (2.3.1), x writes as $x = x^{[0]} + x^{[1]} + \cdots + x^{[\kappa]}$, and we have $\widetilde{T}_z^2 u(z_4) \neq 0$ whenever $x_4^{[0]} \neq x^{[0]}$. To overcome this problem, we define a new point $z_5 = (x_5, t_5)$ as follows:

$$x_5^{[0]} = x^{[0]}, \quad x_5^{[1]} = x_4^{[1]}, \ldots, x_5^{[\kappa]} = x_4^{[\kappa]}, \quad t_5 = t_4.$$

Note that in [94, 101] it is proved that

$$\left| x^{[0]} - x_4^{[0]} \right| \leq C \| z^{-1} \circ \zeta \|_{\mathbb{K}},$$

for some positive constant C only depending on the matrix B. Then

$$u(z_5) - u(z_4) = o(\| z^{-1} \circ \zeta \|_{\mathbb{K}}^2) \quad \text{as} \quad \zeta \to z. \tag{2.3.15}$$

With this modification, expression (2.3.12) is replaced by

$$u(z_5) - T_z^2 u(z_5) = u(z_5) - u(z_4) - \widetilde{T}_z^2 u(z_5) + o(\| z^{-1} \circ \zeta \|_{\mathbb{K}}^2) \quad \text{as} \quad \zeta \to z.$$

Moreover, $x_5^{[0]} = x^{[0]}$ and $t_5 = t$ yield $\widetilde{T}_z^2 u(z_5) = 0$. From (2.3.15) it then follows that

$$u(z_5) - T_z^2 u(z_5) = o(\| z^{-1} \circ \zeta \|_{\mathbb{K}}^2) \quad \text{as} \quad \zeta \to z.$$

We are now in position to prove (2.1.4). We first consider the point $\bar{z} = e^{(t-\tau)Y}(z) = (e^{(t-\tau)B}x, \tau)$ and write

$$u(\zeta) - T_z^2 u(\zeta) = u(\zeta) - T_{\bar{z}}^2 u(\zeta) + T_{\bar{z}}^2 u(\zeta) - T_z^2 u(\zeta). \tag{2.3.16}$$

Thanks to the previous steps, the first difference is $o(\| \bar{z}^{-1} \circ \zeta \|_K^2) = o(\| z^{-1} \circ \zeta \|_{\mathbb{K}}^2)$ as $\| z^{-1} \circ \zeta \|_{\mathbb{K}}^2 \to 0$, since ζ and $\bar{z}$ have the same temporal component τ. At the same time, the second difference in (2.3.16) can be rewritten as

$$
\begin{aligned}
T^2_{\bar z}u(\zeta) - T^2_z u(\zeta) = u(\bar z) - u(z) + \sum_{i=1}^{m} \left(\partial_{x_i} u(\bar z) - \partial_{x_i} u(z)\right)(\xi_i - x_i) \\
+ \frac{1}{2}\sum_{i,j=1}^{m} \left(\partial^2_{x_i x_j} u(\bar z) - \partial^2_{x_i x_j} u(z)\right)(\xi_i - x_i)(\xi_j - x_j) + Yu(z)(\tau - t).
\end{aligned}
\tag{2.3.17}
$$

Using the mean value theorem along the drift, we can rewrite difference $u(\bar z) - u(z)$ in (2.3.17) as

$$
u(e^{(t-\tau)Y}(z)) - u(z) = (t-\tau)Yu(e^{\delta Y}(z)), \tag{2.3.18}
$$

where δ is such that $|\delta| \le |t - \tau|$. Hence, we obtain

$$
u(\bar z) - u(z) - Y(z)(t-\tau) = (t-\tau)(Yu(e^{\delta Y}(z)) - Y(z)), \tag{2.3.19}
$$

which is $o(|t-\tau|) = o(\|z^{-1} \circ \zeta\|^2_{\mathbb{K}})$ as $\|z^{-1} \circ \zeta\|^2_{\mathbb{K}} \to 0$, thanks to the continuity of Yu.

We observe that we can apply condition (2.1.2) to the point z, which is not fixed, thanks to the fact that such a condition holds locally uniformly. Hence, using the aforemention condition (2.1.2), together with the continuity of the second derivatives of u, we obtain that the second and third difference in (2.3.17) are also $o(|t-\tau|) = o(\|z^{-1}\circ\zeta\|^2_{\mathbb{K}})$ as $\|z^{-1}\circ\zeta\|^2_{\mathbb{K}} \to 0$.

Combining all the previous estimates, we obtain

$$
T^2_{\bar z}u(\zeta) - T^2_z u(\zeta) = o(\|z^{-1} \circ \zeta\|^2_{\mathbb{K}}), \quad \text{as } \|z^{-1} \circ \zeta\|^2_{\mathbb{K}} \to 0. \tag{2.3.20}
$$

and therefore (2.3.16) is equal to $o(\|z^{-1} \circ \zeta\|^2_{\mathbb{K}})$ as $\|z^{-1} \circ \zeta\|^2_{\mathbb{K}} \to 0$. This concludes the proof. □

2.4 Proof of Theorem 2.1.4

We first prove a preliminary lemma, which is a straightforward consequence of the maximum principle.

Lemma 2.4.1. *Given $\varphi \in C(\partial \mathcal{Q}_R(z_0))$ and $g \in C_b(\mathcal{Q}_R(z_0))$, we assume that v solves the following Dirichlet problem*

$$
\begin{cases}
\mathscr{L} v = g, & \text{in } \mathcal{Q}_R(z_0), \\
v = \varphi, & \text{in } \partial \mathcal{Q}_R(z_0).
\end{cases}
$$

Then, the following holds

$$
\|v\|_{L^\infty(\mathcal{Q}_R(z_0))} \le \|\varphi\|_{L^\infty(\mathcal{Q}_R(z_0))} + (t_0 - t_1)\|g\|_{L^\infty(\mathcal{Q}_R(z_0))}, \tag{2.4.1}
$$

where $t_1 = t_0 - R^2$ is the time coordinate of the basis of the cylinder $\mathcal{Q}_R(z_0)$.

Proof. We introduce the function $w(x,t) := (t-t_1)\|g\|_{L^\infty(\mathcal{Q}_R(z_0))} + \|\varphi\|_{L^\infty(\mathcal{Q}_R(z_0))}$ and we let $u := v - w$. Clearly, u satisfies $\mathscr{L}u = g + \|g\|_{L^\infty(\mathcal{Q}_R(z_0))} \geq 0$ in $\mathcal{Q}_R(z_0)$. Moreover, as $v \equiv \varphi$ on the boundary of $\mathcal{Q}_R(z_0)$, we have $u = \varphi - (t-t_1)\|g\|_{L^\infty(\mathcal{Q}_R(z_0))} - \|\varphi\|_{L^\infty(\mathcal{Q}_R(z_0))} \leq 0$ in $\partial\mathcal{Q}_R(z_0)$. By the strong maximum principle, it follows that $u(x,t) \leq 0$ in $\mathcal{Q}_R(z_0)$. Replacing v by $-v$, estimate (2.4.1) follows at once. □

Proof of Theorem 2.1.4. We first prove assertion *(ii)*. We denote $\mathcal{Q}_k = \mathcal{Q}_{\varrho^k}(0)$, $\varrho = \frac{1}{2}$ and we consider the following sequence of Dirichlet problems:

$$\begin{cases} \mathscr{L}u_k = f(0), & \textit{in } \mathcal{Q}_k \\ u_k = u, & \textit{in } \partial\mathcal{Q}_k \end{cases} \tag{2.4.2}$$

For any point $z = (x,t)$ satisfying $\|z\|_{\mathbb{K}} \leq \frac{1}{2}$, we want to estimate the quantity

$$I(z) := |\partial^2 u(z) - \partial^2 u(0)|,$$

where $\partial^2 u(z)$ stands for either $\partial^2_{x_i x_j} u(z)$, with $i,j = 1,\ldots,m$, or $Yu(z)$. To this end, we write I as the sum of three terms:

$$\begin{aligned} I(z) \leq |\partial^2 u_k(z) - \partial^2 u_k(0)| + |\partial^2 u_k(0) - \partial^2 u(0)| + \\ + |\partial^2 u(z) - \partial^2 u_k(z)| =: I_1(z) + I_2(z) + I_3(z). \end{aligned}$$

We first estimate I_2. Following [119], we prove that $\left(\partial^2 u_k(0)\right)_{k\in\mathbb{N}}$ is a Cauchy sequence and that its limit agrees with $\partial^2 u(0)$. The same assertion holds for I_3 of course.

First, we let $v_k := u - u_k$ and we observe that v_k satisfies the Dirichlet boundary value problem

$$\begin{cases} \mathscr{L}v_k = f - f(0), & \textit{in } \mathcal{Q}_k \\ v_k = 0, & \textit{in } \partial\mathcal{Q}_k \end{cases} \tag{2.4.3}$$

From Lemma 2.4.1 it follows that

$$\|v_k\|_\infty \leq 4\varrho^{2k}\|f - f(0)\|_\infty \leq 4\varrho^{2k}\omega_f(\varrho^k). \tag{2.4.4}$$

Moreover, since $\mathscr{L}(u_k - u_{k+1}) = 0$ in $\mathcal{Q}_{k+1}$, we apply Proposition 2.2.4 and Lemma 2.4.1, and we find

$$\begin{aligned} \|\partial_{x_i}(u_k - u_{k+1})\|_{L^\infty(\mathcal{Q}_{k+2})} &\leq C\varrho^{-k-2} \sup_{\mathcal{Q}_{k+1}} |u_k - u_{k+1}| \\ &\leq C\varrho^{-k}\Big(\sup_{\mathcal{Q}_{k+1}} |v_k| + \sup_{\mathcal{Q}_{k+1}} |v_{k+1}|\Big) \\ &\leq C\varrho^{-k}\varrho^{2k}\omega_f(\varrho^k) = C\varrho^k\omega_f(\varrho^k), \end{aligned} \tag{2.4.5}$$

for any $i = 1, \dots, m$. In the same way, we obtain

$$\begin{aligned}\|\partial^2_{x_i x_j}(u_k - u_{k+1})\|_{L^\infty(\mathcal{Q}_{k+2})} &\le C\varrho^{-2k-4} \sup_{\mathcal{Q}_{k+1}} |u_k - u_{k+1}| \\ &\le C\varrho^{-2k}\varrho^{2k}\omega_f(\varrho^k) = C\omega_f(\varrho^k)\end{aligned} \tag{2.4.6}$$

for $i, j = 1, \dots, m$, and

$$\begin{aligned}\|Y(u_k - u_{k+1})\|_{L^\infty(\mathcal{Q}_{k+2})} &\le C\varrho^{-2k-4} \sup_{\mathcal{Q}_{k+1}} |u_k - u_{k+1}| \\ &\le C\varrho^{-2k}\varrho^{2k}\omega_f(\varrho^k) = C\omega_f(\varrho^k).\end{aligned} \tag{2.4.7}$$

Let $k \ge 1$ such that $\varrho^{k+4} \le \|z\|_{\mathbb{K}} \le \varrho^{k+3}$, then we have:

$$\sum_{l=k}^{\infty} |\partial^2 u_l(0) - \partial^2 u_{l+1}(0)| \le C \sum_{l=k}^{\infty} \omega_f(\varrho^l) \le C \int_0^{\|z\|_{\mathbb{K}}} \frac{\omega_f(r)}{r} dr. \tag{2.4.8}$$

We next identify the sum of the series $\sum_{l=k}^{\infty} \left(\partial^2 u_l(0) - \partial^2 u_{l+1}(0)\right)$ as

$$\sum_{l=k}^{\infty} \left(\partial^2 u_l(0) - \partial^2 u_{l+1}(0)\right) = \partial^2 u_k(0) - \partial^2 u(0). \tag{2.4.9}$$

To this end, we first consider the derivative $\partial^2_{x_i x_j} u_k$ and we prove that

$$\lim_{k \to +\infty} \partial^2_{x_i x_j} u_k(0) = \partial^2_{x_i x_j} T_0^2 u(0), \tag{2.4.10}$$

where $T_0^2 u(\zeta)$ is the second-order Taylor polynomial of u around the origin, computed at some point $\zeta = (\xi, \tau) \in \mathcal{Q}_k$:

$$T_0^2 u(\zeta) = u(0) + \sum_{i=1}^{m} \partial_{x_i} u(0)\xi_i + \frac{1}{2} \sum_{i,j=1}^{m} \partial^2_{x_i x_j} u(0)\xi_i\xi_j - Yu(0)\tau.$$

Thus, by applying Theorem 2.1.2 to $u \in C^2_{\mathscr{L}}(\mathcal{Q}_1(0))$, we obtain from (2.4.10) that

$$\lim_{k \to +\infty} \partial^2_{x_i x_j} u_k(0) = \partial^2_{x_i x_j} u(0). \tag{2.4.11}$$

We compute $\mathscr{L} T_0^2 u$ in $\zeta = (\xi, \tau)$ as

$$\begin{aligned}\mathscr{L} T_0^2 u(\zeta) =& \sum_{i,j=1}^{m} a_{ij} \partial^2_{\xi_i \zeta_j} u(0) - \partial_t u(0) + \sum_{j=1}^{N} \Bigg(\sum_{i=1}^{m} b_{ij} \partial_{\xi_i} u(0) \Bigg) \xi_j + \sum_{l,j=1}^{N} \Bigg(\sum_{i=1}^{m} b_{il} \partial^2_{\xi_j \xi_i} u(0) \Bigg) \xi_l \xi_j \\ =& \sum_{i,j=1}^{m} a_{ij} \partial^2_{\xi_i \zeta_j} u(0) - \partial_t u(0) + \langle v, \xi \rangle + \langle M\xi, \xi \rangle,\end{aligned}$$

where $v = (v_j)_{j=1,\dots,N} = (\sum_{i=1}^m b_{ij}\partial_{\xi_i}u(0))_{j=1,\dots,N}$ is a constant vector of $\mathbb{R}^N$ and $M = (m_{lj})_{l,j=1,\dots,N} = \left(\sum_{i=1}^m b_{il}\partial^2_{\xi_j\xi_i}u(0)\right)_{l,j=1,\dots,N}$ is a $N \times N$ constant matrix.

In addition, as $\mathscr{L}u = f$ in $\mathcal{Q}_k$, we have that

$$\sum_{i,j=1}^m a_{ij}\partial^2_{\xi_i\zeta_j}u(0) - \partial_t u(0) = \mathscr{L}_0 u(0) = \mathscr{L}u(0) = f(0) \tag{2.4.12}$$

and thus

$$\mathscr{L}T_0^2 u(\zeta) = f(0) + \langle v, \xi\rangle + \langle M\xi, \xi\rangle. \tag{2.4.13}$$

Thus, the definition of u_k in (2.4.2) gives us

$$\mathscr{L}\left(T_0^2 u - u_k\right)(\zeta) = \langle v, \xi\rangle + \langle M\xi, \xi\rangle, \quad \zeta \in \mathcal{Q}_k. \tag{2.4.14}$$

We now apply Lemma 2.2.6 to $T_0^2 u - u_k$ for $R = \varrho^k$ and infer

$$|\partial^2_{x_i x_j}(u_k - T_0^2 u)(0)| \leq C\varrho^{-2k} \sup_{\mathcal{Q}_k} |u_k - T_0^2 u| + O(\varrho^k). \tag{2.4.15}$$

Moreover, since $T_0^2 u$ is the second-order Taylor polynomial of u, we have $u(\zeta) = T_0^2 u(\zeta) + o(\|\zeta\|^2_{\mathbb{K}})$. It follows that

$$\sup_{\zeta\in\mathcal{Q}_k} |u - T_0^2 u| = o(\varrho^{2k}) \tag{2.4.16}$$

Thus, from estimates (2.4.16) and (2.4.4), we obtain

$$\sup_{\mathcal{Q}_k} |u_k - T_0^2 u| \leq \sup_{\mathcal{Q}_k} |v_k| + \sup_{\mathcal{Q}_k} |u - T_0^2 u| \leq 4\omega_f(\varrho^k)\varrho^{2k} + o(\varrho^{2k}) \leq o(\varrho^{2k}). \tag{2.4.17}$$

Estimates (2.4.15) and (2.4.17) finally yield

$$|\partial^2_{x_i x_j}(u_k - T_0^2 u)(0)| \leq C\varrho^{-2k} \sup_{\mathcal{Q}_k} |u_k - T_0^2 u| + O(\varrho^k) \leq C\varrho^{-2k} o(\varrho^{2k}) + O(\varrho^k) \leq o(1),$$

where, as usual, the indexes i and j range from 1 to m. Thus, for any $i, j = 1, \dots, m$ we have showed that (2.4.10) holds true. Repeating the same argument for the vector field Y, and using again Theorem 2.1.2, we obtain:

$$\lim_{k\to+\infty} Y u_k(0) = Y T_0^2 u(0) = Y u(0).$$

In conclusion, using (2.4.8), we obtain:

$$I_2 \leq \sum_{l=k}^{\infty} |\partial^2 u_l(0) - \partial^2 u_{l+1}(0)| \leq C \int_0^{\|z\|_{\mathbb{K}}} \frac{\omega_f(r)}{r} dr, \tag{2.4.18}$$

for $k \geq 1$ such that $\varrho^{k+4} \leq \|z\|_{\mathbb{K}} \leq \varrho^{k+3}$. Similarly, we can estimate I_3 through the solution of $\mathscr{L}v = f(z)$ in $\mathcal{Q}_j(z)$ and $v = u$ on $\partial\mathcal{Q}_j(z)$ and obtain

$$I_3 \leq \sum_{l=k}^{\infty} |\partial^2 u_l(z) - \partial^2 u_{l+1}(z)| \leq C \int_0^{\|z\|_{\mathbb{K}}} \frac{\omega_f(r)}{r} dr. \tag{2.4.19}$$

Finally, let us estimate I_1. Since $h_k = u_k - u_{k+1} \in C^\infty(\mathcal{Q}_{k+2})$, we can apply Proposition 2.2.7 to the functions $\partial^2_{x_i x_j} h_k$ and $Y h_k$:

$$|\partial^2_{x_i x_j} h_k(z) - \partial^2_{x_i x_j} h_k(0)| \leq \frac{C}{\varrho^k} \|z\|_{\mathbb{K}} \|\partial^2_{x_i x_j} h_k\|_{L^\infty(\mathcal{Q}_{k+1})}$$

and

$$|Y h_k(z) - Y h_k(0)| \leq \frac{C}{\varrho^k} \|z\|_{\mathbb{K}} \|Y h_k\|_{L^\infty(\mathcal{Q}_{k+1})},$$

for $i, j = 1, \dots, m$. We can now apply once again (2.4.6) to obtain

$$|\partial^2_{x_i x_j} h_k(z) - \partial^2_{x_i x_j} h_k(0)| \leq \frac{C}{\varrho^k} \|z\|_{\mathbb{K}} \|\partial^2_{x_i x_j} h_k\|_{L^\infty(\mathcal{Q}_{k+1})} \leq C \|z\|_{\mathbb{K}} \varrho^{-k} \omega_f(\varrho^k).$$

In addition, thanks to (2.4.7), we infer

$$|Y h_k(z) - Y h_k(0)| \leq \frac{C}{\varrho^k} \|z\|_{\mathbb{K}} \|Y h_k\|_{L^\infty(\mathcal{Q}_{k+1})} \leq C \|z\|_{\mathbb{K}} \varrho^{-k} \omega_f(\varrho^k).$$

Hence, since $u_k(z) - u_k(0) = u_0(z) - u_0(0) + \sum_{j=0}^{k-1} (h_j(0) - h_j(z))$, we have

$$\begin{aligned} I_1 &\leq |\partial^2 u_0(z) - \partial^2 u_0(0)| + \sum_{j=0}^{k-1} |\partial^2 h_j(z) - \partial^2 h_j(0)| \\ &\leq C \|z\|_{\mathbb{K}} \big(\|u_0\|_{L^\infty(\mathcal{Q}_0)} + C \sum_{j=0}^{k-1} \varrho^{-j} \omega_f(\varrho^j) \big) \\ &\leq C \|z\|_{\mathbb{K}} \big(\|u\|_{L^\infty(\mathcal{Q}_1(0))} + \|f\|_{L^\infty(\mathcal{Q}_1(0))} + C \int_{\|z\|_{\mathbb{K}}}^1 \frac{\omega_f(r)}{r^2} \big). \end{aligned}$$

Combining the above estimate with (2.4.18) and (2.4.19), we complete the proof of *(ii)*.

We now prove assertion *(i)*. We consider u_1 solution to the following Dirichlet problem

$$\begin{cases} \mathscr{L} u_1 = f(0), & \textit{in } \mathcal{Q}_{1/2}(0) \\ u_1 = u, & \textit{in } \partial\mathcal{Q}_{1/2}(0) \end{cases}$$

Then, we have

$$|\partial^2 u(0)| \leq |\partial^2 u(0) - \partial^2 u_1(0)| + |\partial^2 u_1(0)|. \tag{2.4.20}$$

Thanks to (2.4.18), we can estimate the first term in (2.4.20) as

$$|\partial^2 u(0) - \partial^2 u_1(0)| \le C \int_0^1 \frac{\omega_f(r)}{r} dr. \tag{2.4.21}$$

To estimate the second term in (2.4.20), we consider the function $v(z) := u_1(z)\eta_{1/2}(z)$, where $\eta_{1/2}$ is the cut-off function introduced in (2.2.16) with $R = \frac{1}{2}$. Reasoning as in the proof of Proposition 2.2.3, we obtain

$$\begin{aligned} u(z) = v(z) = & \int_{\mathcal{Q}_{\frac{1}{2}}(0)} [\Gamma(z,\cdot)\mathrm{div}(AD_m(\eta_{1/2}))u_1](\zeta)d\zeta \\ & - \int_{\mathcal{Q}_{\frac{1}{2}}(0)} [\Gamma(z,\cdot)Y(\eta_{1/2})u_1](\zeta)d\zeta \\ & - \int_{\mathcal{Q}_{\frac{1}{2}}(0)} [\Gamma(z,\cdot)\eta_{1/2}\mathscr{L}(u_1)](\zeta)d\zeta \\ & + 2\int_{\mathcal{Q}_{\frac{1}{2}}(0)} [\langle D_m^{\zeta}\Gamma(z,\cdot), AD_m\eta_{1/2}\rangle u_1](\zeta)d\zeta, \end{aligned}$$

where $z \in \mathcal{Q}_{\frac{1}{4}}(0)$. Thanks to Lemma 2.4.1, we estimate

$$\sup_{\mathcal{Q}_{\frac{1}{2}}(0)} |u_1| \le \sup_{\mathcal{Q}_{\frac{1}{2}}(0)} |u| + 4|f(0)|.$$

As the derivatives of $\eta_{1/2}$ vanish in $\mathcal{Q}_{3/8}(0)$, for any $i, j = 1, \dots, m$, we obtain

$$\begin{aligned} |\partial^2_{x_i x_j} u_1(z)| \le & \int_{\mathcal{Q}_{\frac{1}{2}}(0)\setminus\mathcal{Q}_{\frac{3}{8}}(0)} \left|[\partial^2_{x_i x_j}\Gamma(z,\cdot)\mathrm{div}(AD_m(\eta_{1/2}))u_1](\zeta)\right|d\zeta \\ & + \int_{\mathcal{Q}_{\frac{1}{2}}(0)\setminus\mathcal{Q}_{\frac{3}{8}}(0)} \left|[\partial^2_{x_i x_j}\Gamma(z,\cdot)Y(\eta_{1/2})u_1](\zeta)\right|d\zeta \\ & + 2\int_{\mathcal{Q}_{\frac{1}{2}}(0)\setminus\mathcal{Q}_{\frac{3}{8}}(0)} \left|[\langle\partial^2_{x_i x_j}D_m^{\zeta}\Gamma(z,\cdot), AD_m\eta_{1/2}\rangle u_1](\zeta)\right|d\zeta \\ & + \left|f(0)\Big[\partial^2_{x_i x_j}\int_{\mathcal{Q}_{\frac{1}{2}}(0)} [\Gamma(z,\cdot)\eta_{1/2}](\zeta)d\zeta\Big]\right| \\ & =: \overline{I}_1(z) + \overline{I}_2(z) + \overline{I}_3(z) + \overline{I}_4(z). \end{aligned} \tag{2.4.22}$$

Moreover, as the derivatives of $\eta_{1/2}$ are bounded, we estimate the first and second integral in (2.4.22) as

$$\begin{aligned} \overline{I}_1(z) &\le C\big[\sup_{\mathcal{Q}_{\frac{1}{2}}(0)} |u| + 4|f(0)|\big], \\ \overline{I}_2(z) &\le C\big[\sup_{\mathcal{Q}_{\frac{1}{2}}(0)} |u| + 4|f(0)|\big], \end{aligned}$$

$$\overline{I}_3(z) \le C\big[\sup_{\mathcal{Q}_{\frac{1}{2}}(0)} |u| + 4|f(0)|\big].$$

Finally, by taking advantage of (2.2.28), we obtain that $\overline{I}_4(z)$ is bounded by a constant C that only depends on B, λ and Λ.

By using the same argument we can estimate $|Yu_1(0)|$ and thus

$$|\partial^2 u_1(0)| \le C\big[\sup_{\mathcal{Q}_{\frac{1}{2}}(0)} |u| + 4|f(0)|\big]. \tag{2.4.23}$$

Combining estimates (2.4.20) and (2.4.23), we conclude the proof of Theorem 2.1.4. □

Remark 2.4.2. We observe that the iteration step in (2.4.8) is needed in order to conclude the proof and to obtain the integrals appearing on the right-hand side of Theorem 2.1.4 *i*)-*ii*). Indeed, letting k go to infinity in (2.4.6), (2.4.7) would not allow us to conclude the argument as we do not know how the modulus of continuity of f decays when $k \to \infty$.

2.5 Dini continuous coefficients

This section is devoted to the proof of Theorem 2.1.5. We therefore consider a solution u to the equation

$$\widetilde{\mathscr{L}} u = f,$$

where the operator $\widetilde{\mathscr{L}}$ does satisfy the hypotheses **[H.1]** and **[H.2]** and f is assumed to be Dini continuous, and we proceed as in the proof of Theorem 2.1.4. Specifically, we denote $\mathcal{Q}_k = \mathcal{Q}_{\varrho^k}(0)$, $\varrho = \frac{1}{2}$ and we consider the following sequence of Dirichlet problems:

$$\begin{cases} \sum\limits_{i,j=1}^{m} a_{ij}(0,0)\partial^2_{x_i x_j} u_k + Y u_k = f(0), & \text{in } \mathcal{Q}_k \\ u_k = u, & \text{on } \partial\mathcal{Q}_k. \end{cases} \tag{2.5.1}$$

Note that the bounds given in Propositions 2.2.3, 2.2.4 and 2.2.7 only depend on the constants λ, Λ in **[H.2]** and on the matrix B. Keeping in mind this fact, the proof of Theorem 2.1.5 is given by the same argument used in the proof of Theorem 2.1.4.

Proof of Theorem 2.1.5. Consider, for every $k \in \mathbb{N}$, the auxiliary function $v_k := u - u_k$, and note that it is a solution to the boundary value problem

$$\begin{cases} \sum\limits_{i,j=1}^{m} a_{ij}(0,0)\partial^2_{x_i x_j} v_k + Y v_k \\ \quad = f - f(0) + \sum\limits_{i,j=1}^{m} (a_{ij}(0) - a_{ij}(x,t))\partial^2_{x_i x_j} u, & \text{in } \mathcal{Q}_k \\ v_k = 0, & \textit{in } \partial\mathcal{Q}_k \end{cases} \tag{2.5.2}$$

In order to simplify the notation, we let

$$\eta := \max_{i,j=1,\dots,m} \|\partial^2_{x_i x_j} u\|_{L^\infty(\mathcal{Q}_1)}. \tag{2.5.3}$$

From Lemma 2.4.1 it follows that

$$\|v_k\|_{L^\infty(\mathcal{Q}_k)} \le C\varrho^{2k}[\omega_f(\varrho^k) + \omega_a(\varrho^k)\eta].$$

Hence

$$\|u_k - u_{k+1}\|_{L^\infty(\mathcal{Q}_{k+1})} \le C\varrho^{2k}[\omega_f(\varrho^k) + \omega_a(\varrho^k)\eta].$$

As already observed, we can apply Corollary 2.2.4 and obtain estimates for the second order derivatives of v_k. In fact, for any $i, j = 1, \dots, m$, we have

$$\begin{aligned}\|\partial^2_{x_i x_j}(u_k - u_{k+1})\|_{L^\infty(\mathcal{Q}_{k+2})} &\le C(\varrho^k)^{-2} \sup_{\mathcal{Q}_{k+1}} |u_k - u_{k+1}| \\ &\le C\varrho^{-2k}\varrho^{2k}[\omega_f(\varrho^k) + \omega_a(\varrho^k)\eta] = C[\omega_f(\varrho^k) + \omega_a(\varrho^k)\eta] \end{aligned} \tag{2.5.4}$$

and

$$\begin{aligned}\|Y(u_k - u_{k+1})\|_{L^\infty(\mathcal{Q}_{k+2})} &\le C(\varrho^k)^{-2} \sup_{\mathcal{Q}_{k+1}} |u_k - u_{k+1}| \\ &\le C\varrho^{-2k}\varrho^{2k}[\omega_f(\varrho^k) + \omega_a(\varrho^k)\eta] = C[\omega_f(\varrho^k) + \omega_a(\varrho^k)\eta] \end{aligned} \tag{2.5.5}$$

To estimate the second order derivatives of the function u, we apply Theorem 2.1.2 and proceed as in the proof of Theorem 2.1.4. Since there are no significant differences, we omit the details here. □

Chapter 3

Pointwise estimates for degenerate Kolmogorov equations with L^p-source term

3.1 Statement of the problem

In this chapter, we study the pointwise regularity of solutions u belonging to the Sobolev space $S^p(\Omega)$ (see Section 3.2) to the following Cauchy problem

$$\begin{cases} \mathscr{L}_0 u = f \quad \text{in} \quad \mathcal{Q}_1^- \\ f \in L^p(\mathcal{Q}_r^-) \quad \text{and} \quad f(0) = 0, \end{cases} \tag{3.1.1}$$

where $1 < p < \infty$ and $\mathcal{Q}_r^- = B_r \times (-r^2, 0)$ is the past cylinder defined through the open ball $B_r = \{x \in \mathbb{R}^N : |x|_{\mathbb{K}} \leq r\}$ and $|\cdot|_{\mathbb{K}}$ is the semi-norm, due to the nature of operator $\mathscr{L}_0$, defined in (1.1.23). More precisely, we show that if the modulus of L^p-mean oscillation of $\mathscr{L}_0 u$ at the origin is Dini, then the origin is a Lebesgue point of continuity in L^p average for the second order derivatives $\partial^2_{x_i x_j} u$, $i, j = 1, \ldots, m$, and the Lie derivative $\left(\sum_{i,j=1}^N b_{ij} x_j \partial_{x_i} - \partial_t\right) u$. Moreover, we are able to provide a Taylor-type expansion up to second order with an estimate of the rest in L^p norm. We point out that the results of this chapter are presented in the paper [62] and are obtained in collaboration with Ipocoana.

3.1.1 Assumptions and mathematical preliminaries

We suppose here that $1 < p < \infty$ and that the origin $0 = (0, 0)$ is a Lebesgue point of f, so that we are able to define $f(0)$ if needed.

Moreover, we denote by $\mathscr{L}_0$ the model Kolmogorov operator of the form

$$\mathscr{L}_0 := \sum_{i,j=1}^{m} \partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t, \tag{3.1.2}$$

where $(x, t) \in \mathbb{R}^{N+1}$, and $1 \leq m \leq N$. In this case, the diffusion matrix A in (1.1.1) is simply the identity matrix $\mathbb{I}_m$.

In this chapter, we require that assumption [**H.2**] of Chapter 1, Subection 1.1.1 holds true. We recall that, by [73, Propositions 2.1 and 2.2], the second assertion in hypothesis [**H.2**] is equivalent to assume that, for some basis on $\mathbb{R}^N$, the matrix B takes the block form (1.1.12). Hence, in the following of this chapter, we shall always assume that B has the canonical form (1.1.12).

We now define a class of polynomials that are homogeneous of degree 2 with respect to the dilations in (1.1.13). According to Definition 1.1.9, it is clear that the polynomials which are homogeneous of degree two with respect to dilation (1.1.13) are those of degree two in the first m spatial variables and one in time. For this reason, it is natural to define the following class of polynomials, which will be greatly used in the sequel. Namely,

$$\begin{aligned}\widetilde{\mathcal{P}} = \{ P : \ & \text{polynomials of degree less or equal to two in } \ x_1 \dots x_m \\ & \text{and less or equal to one in } t \} .\end{aligned} \tag{3.1.3}$$

$$\mathcal{P} \colon = \Big\{ P \in \widetilde{\mathcal{P}} : \mathscr{L}_0 P = 0 \Big\}. \tag{3.1.4}$$

$$\mathcal{P}_c \colon = \Big\{ P \in \widetilde{\mathcal{P}} : \mathscr{L}_0 P = c \Big\}. \tag{3.1.5}$$

In particular, we take P_* such that $\mathscr{L}_0 P_* = 1$ and set $\mathcal{P}_c = cP_* + \mathcal{P}$.

Finally, owing to the instrinsic geometry introduced in Chapter 1, Subsection 1.1.1, and in particular to the definition of semi-norm in (1.1.23), we introduce the *unit past cylinder*

$$\mathcal{Q}_1^- = \{ (x,t) \in \mathbb{R}^{N+1} \mid |x|_{\mathbb{K}} < 1, \quad t \in (-1,0) \}.$$

For every $(x_0, t_0) \in \mathbb{R}^{N+1}$ and $r > 0$, we set

$$\mathcal{Q}_r^-(x_0,t_0) := z_0 \circ \delta_r(\mathcal{Q}_1^-) = \{ (x,t) \in \mathbb{R}^{N+1} \mid (x,t) = (x_0,t_0) \circ \delta_r(\xi,\tau), (\xi,\tau) \in \mathcal{Q}_1^- \},$$

where "$\circ$" is the composition law defined in (1.1.6) and δ_r denotes the family of dilations in (1.1.13). We also observe that, as shown in Remark 1.1.6, the Lebesgue measure is invariant with respect to the translation group associated to $\mathscr{L}_0$, since the matrix B takes the form (1.1.12). Moreover, we have

$$\text{meas}\left(\mathcal{Q}_r^-(x_0,t_0) \right) = r^{Q+2} \text{meas}\left(\mathcal{Q}_1^-(x_0,t_0) \right), \qquad \forall\, r > 0, (x_0,t_0) \in \mathbb{R}^{N+1}.$$

3.1.2 Main results

In order to introduce the main results of this chapter, we need to give an appropriate definition of modulus of continuity. Indeed, the previous results in literature, including the ones contained in the former chapter, were derived assuming a modulus of continuity defined on some open set $\mathcal{Q}^- \subset \mathbb{R}^{N+1}$ (see (2.1.6)).

On the other hand, we here introduce a *pointwise modulus of mean oscillation.* More precisely, following [86], for $p \in (1,+\infty)$, we define the following *modulus of L^p-mean oscillation* for the function f *at the origin* as

$$\widetilde{\omega}(f;r) := \inf_{c\in\mathbb{R}} \left(\frac{1}{|\mathcal{Q}_r^-|} \int_{\mathcal{Q}_r^-} |f(x,t) - c|^p \right)^{\frac{1}{p}}. \tag{3.1.6}$$

We now set

$$\widetilde{N}(u;r) := \inf_{P\in\widetilde{\mathcal{P}}} \left(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |u-P|^p \right)^{\frac{1}{p}}, \tag{3.1.7}$$

where Q is the homogeneous dimension defined in (1.1.17) and $\widetilde{\mathcal{P}}$ is the class of polynomials introduced in (3.1.3).
Owing to (3.1.6), let c_r be the unique constant such that

$$\widetilde{\omega}(f;r) = \inf_{c\in\mathbb{R}} \left(\frac{1}{|\mathcal{Q}_r^-|} \int_{\mathcal{Q}_r^-} |f(x,t) - c|^p \right)^{\frac{1}{p}} = \left(\frac{1}{|\mathcal{Q}_r^-|} \int_{\mathcal{Q}_r^-} |f(x,t) - c_r|^p \right)^{\frac{1}{p}}. \tag{3.1.8}$$

If u is a solution of (3.1.1), we let

$$\hat{N}(u,f;r) = \inf_{P\in\mathcal{P}_{c_r}} \left(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |u-P|^p \right)^{\frac{1}{p}}. \tag{3.1.9}$$

Moreover, for $0 < a < b$, we define

$$\hat{N}(u,f;a,b) = \sup_{a\le\rho\le b} \hat{N}(u,f;\rho) \tag{3.1.10}$$

$$\widetilde{\omega}(f;a,b) = \sup_{a\le\rho\le b} \widetilde{\omega}(f;\rho) \tag{3.1.11}$$

In the sequel, we will also make use of the following notation. For a given $\lambda \in (0,1)$, we set

$$\underline{N}(r) = \hat{N}(u,f;\lambda r,r), \tag{3.1.12}$$

$$\underline{\omega}(r) = \widetilde{\omega}(f;\lambda^2 r,r). \tag{3.1.13}$$

For readers' convenience, we eventually recall the following definition.

Definition 3.1.1. A modulus of continuity ω is said Dini if it satisfies the following integral condition

$$\int_0^1 \frac{\omega(r)}{r} dr < +\infty.$$

This chapter is devoted to prove the following theorem.

Theorem 3.1.2. *Let $p \in (1,\infty)$. Then there exist constants $\beta, r_* \in (0,1]$, $\lambda \in (0,1)$ and $C > 0$, such that the following holds. If $u \in L^p(\mathcal{Q}_1^-)$ satisfies (3.1.1) with the associated $\widetilde{\omega}$ defined in (3.1.6), then we have* *i) Pointwise BMO estimate*

$$\sup_{r\in(0,1]} \widetilde{N}(u;r) \le C\left\{\left(\int_{\mathcal{Q}_1^-}|u|^p\right)^{\frac{1}{p}} + \left(\int_{\mathcal{Q}_1^-}|f|^p\right)^{\frac{1}{p}} + \sup_{r\in(0,1]}\widetilde{\omega}(f;r)\right\}. \tag{3.1.14}$$

ii) Pointwise VMO estimate

$$\left(\widetilde{\omega}(f;r)\to 0 \text{ as } r\to 0^+\right) \quad \Rightarrow \quad \left(\widetilde{N}(u;r)\to 0 \text{ as } r\to 0^+\right). \tag{3.1.15}$$

iii) Dini continuity of $\widetilde{N}(u;\cdot)$
If $\widetilde{\omega}(f;\cdot)$ is Dini, then $\widetilde{N}(u;\cdot)$ is Dini. In particular, for every $\rho\in(0,\frac{\lambda}{4})$, the following holds

$$\int_0^{4\rho}\frac{\widetilde{N}(u;r)}{r}dr \le C\left\{\left(\frac{4\rho}{\lambda}\right)^{\beta}\left(\widetilde{N}(u;1)+\widetilde{\omega}(f;1)\right) + \int_0^{4\rho}\frac{\widetilde{\omega}(f;r)}{r}dr + \rho^{\beta}\int_{4\rho}^{1}\frac{\widetilde{\omega}(f;r)}{r^{1+\beta}}dr\right\}.$$

where C is a constant that does not depend on f, u and ρ.

iv) Pointwise control on the solution
Let $\widetilde{\omega}(f;\cdot)$ be Dini. Then there exists a unique polynomial $P_0\in\mathcal{P}$, namely a solution to equation $\mathscr{L}_0 P_0 = 0$, with

$$P_0(x,t) = a + \langle b,x\rangle + \frac{1}{2}\langle cx,x\rangle + d\,t,$$

where b is a vector in $\mathbb{R}^N$ such that $b_j=0$ when $j>m$ and c is a $N\times N$ matrix such that $c_{ij}=0$ when $i>m\vee j>m$, such that for every $r\in(0,\frac{r_*}{4}]$ there holds

$$\left(\frac{1}{|\mathcal{Q}_r^-|}\int_{\mathcal{Q}_r^-}\left|\frac{u(x,t)-P_0(x,t)}{r^2}\right|^p\right)^{\frac{1}{p}} \le C\left\{\widetilde{M}_0\left(\frac{4r}{\lambda}\right)^{\beta} + \int_0^{4r}\frac{\widetilde{\omega}(f;s)}{s}ds + r^{\beta}\int_{4r}^{1}\frac{\widetilde{\omega}(f;s)}{s^{1+\beta}}ds\right\}, \tag{3.1.16}$$

with

$$\widetilde{M}_0 = \int_0^1\frac{\widetilde{\omega}(f;s)}{s}ds + \left(\int_{\mathcal{Q}_1^-}|u|^p\right)^{\frac{1}{p}} + \left(\int_{\mathcal{Q}_1^-}|f|^p\right)^{\frac{1}{p}}.$$

Moreover, we have

$$|a|+|b|+|c|+|d| \le C\widetilde{M}_0.$$

The proof of Theorem 3.1.2 is non-constructive and it is based on decay estimates, which we achieve by contradiction, blow-up and compactness results.

We observe that from Theorem 3.1.2 *iv)* (inequality (3.1.16)), the next result follows straightforwardly.

Corollary 3.1.3. *If the modulus of L^p-mean oscillation of $\mathscr{L}_0 u$ at the origin is Dini, then the origin is a Lebesgue point of continuity in L^p average for the second order derivatives $\partial^2_{x_i x_j} u$, $i, j = 1, \ldots, m$, and the Lie derivative Yu.*

We observe that a simple consequence of Theorem 3.1.2 is that the second order derivatives $\partial^2_{x_i x_j} u$, $i, j = 1, \ldots, m$, and the Lie derivative Yu are Hölder continuous in some open set $\Omega \subset \mathbb{R}^{N+1}$, when $\mathscr{L}_0 u$ is Hölder continuous with respect to the distance introduced in (1.1.21). Moreover, let us remark that Theorem 3.1.2 provides us with a Taylor-type expansion up to second order with an estimate of the rest in L^p norm. We finally emphasize that, although we consider the regularity problem for weak solutions to Kolmogorov operators in the framework of the Sobolev spaces, our procedure is basically pointwise. Indeed, we consider some L^p norm of the function $u - P_0$ on a cylinder of radius r and we obtain our result by letting r going to zero. Thus, this approach follows the lines of regularity theory for classical solutions rather than the ones for weak solutions, which does not seem to be usual when dealing with Kolmogorov-type operators.

3.1.3 Comparison with existing results

The results contained in Theorem 3.1.2 may be seen as a generalization of [86] and [77], where this kind of results are obtained for elliptic and parabolic equations, respectively. However, up to our knowledge, the case of Kolmogorov-type operators has not been investigated.
The main difficulty with respect to the previous literature lies in the fact that the regularity properties of the Kolmogorov equations on $\mathbb{R}^{N+1}$ depend strongly on the geometric Lie group structure introduced in (1.1.6). In particular, this reflects on the family of dilations we consider. Furthermore, according to (3.1.2), we here take into account also the case where $m < N$ and therefore $\mathscr{L}_0$ is strongly degenerate. We emphasize that when $m = N$ and $B \equiv \mathbb{O}$, our result recovers the one contained in [77].

Regarding the classical regularity theory of Kolmogorov operators, we recall the Schauder type estimates listed in Chapter 1, Section 1.2.

Moreover, in [99] and then in [32], a pointwise estimate for weak solutions to Kolmogorov equations with right-hand side equal to zero was proved. In order to do so, the authors adapted the Moser iterative method to the non-Euclidean framework of the (homogeneous and non-homogeneous, respectively) Lie groups. Finally, the regularity of strong solutions to the Cauchy-Dirichlet and obstacle problem for a class of Kolmogorov-type operators was studied in [64] using a blow-up technique.

3.1.4 Outline of the chapter

The structure of the chapter is the following. Some general control results are contained in Section 3.2, some from the literature and a Caccioppoli-type estimate we prove ad hoc for our problem. Finally, Section 3.3 is devoted to proving our main result Theorem 3.1.2. In particular, for sake of semplicity, we first derive some preliminary estimates in Subsection 3.3.1 in order to finally give a shorter proof of Theorem 3.1.2 in Subsection 3.3.2.

3.2 Preliminary results

We here list some general ultraparabolic estimates. Some of them are well-know from the literature, so for their proofs we will refer to source.
First, for Ω open set in $\mathbb{R}^{N+1}$, $p \in (1, +\infty)$, we define the Sobolev space

$$S^p(\Omega) = \{u \in L^p(\Omega) : \partial_{x_i} u, \partial^2_{x_i x_j} u, Yu \in L^p(\Omega), \quad i, j = 1, \dots, m\}.$$

If we set

$$\|u\|^p_{S^p(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{i=1}^m \|\partial_{x_i} u\|^p_{L^p(\Omega)} + \sum_{i,j=1}^m \|\partial^2_{x_i x_j} u\|^p_{L^p(\Omega)} + \|Yu\|^p_{L^p(\Omega)}$$

we have the following local a priori estimates in $S^p(\Omega)$ for solutions to $\mathscr{L}_0 u = f$ (see [24]).

Theorem 3.2.1. *(Ultraparabolic interior L^p-estimates)*
Assume **[H]** *holds and let u be a solution to $\mathscr{L}_0 u = f$ in Ω, where Ω is now a bounded open set in $\mathbb{R}^{N+1}$. If $\Omega_1 \subset\subset \Omega$, then we can find a constant c, only depending on B, p, Ω and Ω_1, such that*

$$\|u\|_{S^p(\Omega_1)} \leq c(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}). \tag{3.2.1}$$

We now state a general compactness result proved in [27].

Theorem 3.2.2. *Let Ω be an open set of $\mathbb{R}^{N+1}$ and let $u \in S^p(\Omega)$ be a weak solution to $\mathscr{L}_0 u = f$ in Ω with $f \in L^p_{\text{loc}}(\Omega)$. Then, for every $z_0 \in \Omega$ and $\rho, \sigma > 0$ such that $\mathcal{Q}^-_\rho(z_0)$ is contained in Ω and $\sigma < \frac{\rho}{2\mathbf{c}_H}$, with $\mathbf{c}_H$ defined in (1.1.20), we have that*

if $1 < p < Q + 2$ and $p < q < p^*$ then there exists a positive constant $\widetilde{C}_{p,q}$ such that

$$\|u(\cdot \circ h) - u\|_{L^q(\mathcal{Q}^-_\sigma(z_0))} \leq \widetilde{C}_{p,q}(\|u\|_{L^p(\mathcal{Q}^-_\rho(z_0))} + \|f\|_{L^p(\mathcal{Q}^-_\rho(z_0))})\|h\|^{(Q+2)(\frac{1}{q} - \frac{1}{p^*})}$$

where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q+2}.$$

As a preliminary result, we state and prove the following Caccioppoli-type estimate which we obtain ad hoc for our problem.

Lemma 3.2.3. *(Caccioppoli-type estimate)*
Let $P \in \mathcal{P}_{c_r}$ and let u be a solution to (3.1.1) in $\mathcal{Q}_r^-$. Let $p \in (1, +\infty)$ and let ρ, r such that $1 \le \rho < r$. Then, for $W := (u-P)|u-P|^{\frac{p}{2}-1}$, the following estimate holds:

$$\begin{aligned}&\frac{2(p-1)}{p^2}\int_{\mathcal{Q}_\rho^-}|D_m W|^2\\&\le\left(\frac{2}{(p-1)}\,\frac{c_2^2}{(r-\rho)^2}+\frac{2}{p}\,c_1\,\frac{r^{2\kappa+1}}{r-\rho}\right)\int_{\mathcal{Q}_r^-}W^2+\widetilde{\omega}(f;r)\,|\mathcal{Q}_r^-|\left(\frac{1}{|\mathcal{Q}_r^-|}\int_{\mathcal{Q}_r^-}\eta^{2p'}W^2\right)^{\frac{1}{p'}},\end{aligned}$$

where c_1, c_2 are dimensional constants, p' is such that $\frac{1}{p}+\frac{1}{p'}=1$ and D_m denotes the partial gradient in the first m variables, that is

$$D_m := (\partial_{x_1}, \dots, \partial_{x_m}).$$

Proof. On the past cylinder $\mathcal{Q}_r^-$, we have

$$-\mathscr{L}_0 u + f = 0, \quad \text{and} \quad -\mathscr{L}_0 P + c_r = 0, \tag{3.2.2}$$

since u is a solution to (3.1.1) and $P \in \mathcal{P}_{c_r}$. We set $\varphi := \eta^2 w|w|^{p-2}$, where $w := u - P$ and η is a C^∞−function with compact support, to be chosen later. Taking the difference of the two equations in (3.2.2) and multiplying it by φ, we obtain

$$-\int_{\mathcal{Q}_r^-}\eta^2 w|w|^{p-2}\mathscr{L}_0 w = -\int_{\mathcal{Q}_r^-}\eta^2 w|w|^{p-2}\left(f(x)-c_r\right). \tag{3.2.3}$$

An integration by parts shows that

$$\begin{aligned}-\int_{\mathcal{Q}_r^-}\eta^2 w|w|^{p-2}\mathscr{L}_0 w &= \int_{\mathcal{Q}_r^-}\langle AD_m w, D_m(\eta^2 w|w|^{p-2})\rangle - \int_{\mathcal{Q}_r^-}\eta^2 w|w|^{p-2}Y(w)\\ &=: I_1 + I_2,\end{aligned} \tag{3.2.4}$$

where D_m denotes the gradient with respect to $x_1, \dots, x_m$. We now observe that

$$D_m\varphi = 2\eta D_m\eta\, w|w|^{p-2} + \eta^2(p-1)|w|^{p-2}D_m w$$

and therefore we can rewrite the term I_1 on the right-hand side of (3.2.4) as

$$I_1 = 2\int_{\mathcal{Q}_r^-}\langle AD_m w, D_m\eta\rangle\eta w|w|^{p-2} + (p-1)\int_{\mathcal{Q}_r^-}\eta^2|w|^{p-2}\langle AD_m w, D_m w\rangle.$$

Taking advantage of $D_m W = \frac{p}{2}|w|^{\frac{p}{2}-1}D_m w$, for $W = w|w|^{\frac{p}{2}-1}$, the previous equation rewrites as

$$I_1 = \frac{4(p-1)}{p^2}\int_{\mathcal{Q}_r^-}\eta^2\langle AD_m W, D_m W\rangle + \frac{4}{p}\int_{\mathcal{Q}_r^-}\eta W\,\langle AD_m W, D_m\eta\rangle. \tag{3.2.5}$$

We now take care of the term I_2 in (3.2.4). We first notice that

$$Y(W) = \frac{p}{2}|w|^{\frac{p}{2}-1}Y(w),$$

which, together with the divergence theorem and the identity

$$Y(W^2\eta^2) = 2\eta W^2 Y(\eta) + 2\eta^2 W Y(W),$$

yields

$$I_2 = \frac{2}{p}\int_{\mathcal{Q}_r^-} \eta W^2 Y(\eta). \tag{3.2.6}$$

Thus, combining (3.2.5) and (3.2.6), we can rewrite identity (3.2.3) as

$$\begin{aligned} 0 = \frac{4(p-1)}{p^2}\int_{\mathcal{Q}_r^-} \eta^2 \langle AD_m W, D_m W\rangle + \frac{4}{p}\int_{\mathcal{Q}_r^-} \eta W \,\langle AD_m W, D_m \eta\rangle \\ + \frac{2}{p}\int_{\mathcal{Q}_r^-} \eta W^2 Y(\eta) + \int_{\mathcal{Q}_r^-} \eta^2 w|w|^{p-2}\left(f(x) - c_r\right). \end{aligned}$$

Now, setting $\varepsilon = \frac{p-1}{2p}$ and using the estimate

$$\eta\,|W|\,|\langle AD_m W, D_m\eta\rangle| \le \varepsilon\eta^2\langle AD_m W, D_m W\rangle + \frac{W^2}{4\varepsilon}\langle AD_m\eta, D_m\eta\rangle,$$

we finally obtain

$$\begin{aligned} &\frac{2(p-1)}{p^2}\int_{\mathcal{Q}_r^-} \eta^2\langle AD_m W, D_m W\rangle \\ &\le \frac{2}{(p-1)}\int_{\mathcal{Q}_r^-} W^2\langle AD_m\eta, D_m\eta\rangle + \frac{2}{p}\int_{\mathcal{Q}_r^-} W^2\eta|Y(\eta)| + \int_{\mathcal{Q}_r^-} |f - c_r|\eta^2|W|^{\frac{2(p-1)}{p}} \\ &\le \frac{2}{(p-1)}\int_{\mathcal{Q}_r^-} W^2\langle AD_m\eta, D_m\eta\rangle + \frac{2}{p}\int_{\mathcal{Q}_r^-} W^2\eta|Y(\eta)| \\ &\qquad + |\mathcal{Q}_r^-|\,\widetilde{\omega}(f;r)\left(\frac{1}{|\mathcal{Q}_r^-|}\int_{\mathcal{Q}_r^-} \eta^{2p'}|W|^{\frac{2p'(p-1)}{p}}\right)^{\frac{1}{p'}} \\ &\le \frac{2}{(p-1)}\int_{\mathcal{Q}_r^-} W^2\langle AD_m\eta, D_m\eta\rangle + \frac{2}{p}\int_{\mathcal{Q}_r^-} W^2\eta|Y(\eta)| \\ &\qquad + |\mathcal{Q}_r^-|\,\widetilde{\omega}(f;r)\left(\frac{1}{|\mathcal{Q}_r^-|}\int_{\mathcal{Q}_r^-} \eta^{2p'}|W|^{2}\right)^{\frac{1}{p'}}. \end{aligned} \tag{3.2.7}$$

where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. The thesis follows by making a suitable choice of the function η in (3.2.7). More precisely, we set

$$\eta(x,t) = \chi\left(\|(x,0)\|_{\mathbb{K}}\right)\chi_t(t)$$

where $\chi \in C^\infty([0,+\infty))$ is the cut-off function defined by

$$\chi(s) = \begin{cases} 0, & \text{if } s \geq r, \\ 1, & \text{if } 0 \leq s \leq \rho, \end{cases} \qquad |\chi'| \leq \frac{2}{r-\rho},$$

and $\chi_t \in C^\infty((-\infty, 0]$ is defined by

$$\chi_t(s) = \begin{cases} 0, & \text{if } s \leq -r^2, \\ 1, & \text{if } -\rho^2 \leq s \leq 0, \end{cases} \qquad |\chi_t'| \leq \frac{2}{r-\rho},$$

with $\frac{r}{2} \leq \rho < r$. We observe that

$$|Y\eta| \leq c_1 \frac{r^{2\kappa+1}}{r-\rho}, \qquad |\partial_{x_j}\eta| \leq \frac{c_2}{r-\rho} \text{ for } j = 1, \ldots, m_0,$$

where c_1 and c_2 are dimensional constants. Then, accordingly to (3.2.7), we finally obtain

$$\begin{aligned} &\frac{2(p-1)}{p^2} \int_{\mathcal{Q}_\rho^-} |D_m W|^2 \\ &\leq \frac{2}{(p-1)} \frac{c_2^2}{(r-\rho)^2} \int_{\mathcal{Q}_r^-} W^2 + \frac{2}{p} c_1 \frac{r^{2\kappa+1}}{r-\rho} \int_{\mathcal{Q}_r^-} W^2 \\ &\quad + |\mathcal{Q}_r^-| \, \widetilde{\omega}(f;r) \left(\frac{1}{|\mathcal{Q}_r^-|} \int_{\mathcal{Q}_r^-} \eta^{2p'} |W|^2 \right)^{\frac{1}{p'}} \end{aligned}$$

and this concludes the proof. □

3.3 Pointwise estimates for the Kolmogorov equation

This section is the core of the chapter and it is devoted to prove our main result, Theorem 3.1.2. Since the proof is rather convoluted, we have decomposed it in intermediate results proved in Subsection 3.3.1, which will be combined in Subsection 3.3.2 in order to give a simpler proof of Theorem 3.1.2.

3.3.1 Preliminary estimates

The following result is a useful tool in order to prove Lemma 3.3.3 below, as it allows us to rescale the L^p-norm of a given polynomial from a cylider of radius r (for a large r) to a unit cylider.

Lemma 3.3.1 (Larger cylider/smaller cylider)**.** *The following statements hold:*

(i) there exists a constant $C_2 = C_2(p, Q) > 0$ s.t. for every polynomial $P \in \widetilde{\mathcal{P}}$, for any

$r \geq 1$ it holds

$$\left(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |P|^p \right)^{\frac{1}{p}} \leq C_2 \left(\int_{\mathcal{Q}_1^-} |P|^p \right)^{\frac{1}{p}};$$

(ii) there exists a constant $\widetilde{C}_2 = \widetilde{C}_2(p, Q) > 0$ s.t. for every polynomial $P \in \widetilde{\mathcal{P}}$, for any $r < 1$ it holds

$$\left(\int_{\mathcal{Q}_1^-} |P|^p \right)^{\frac{1}{p}} \leq \widetilde{C}_2 \left(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |P|^p \right)^{\frac{1}{p}}.$$

Proof. We only carry out the proof of assertion (i), since case (ii) is totally analogous. We start by writing the polynomial P as $P(x,t) = a + \langle b, x \rangle + \frac{1}{2} \langle cx, x \rangle + d\ t$, where b is a vector in $\mathbb{R}^N$ such that $b_j = 0$ when $j > m$ and c is a $N \times N$ matrix such that $c_{ij} = 0$ when $i > m \vee j > m$. We moreover recall that $\mathcal{Q}_r^- = B_r \times (-r^2, 0)$, where $B_r = \{x \in \mathbb{R}^N : |x|_{\mathbb{K}} \leq r\}$, with $| \cdot |_{\mathbb{K}}$ as defined in (1.1.23). Then, owing to $\|(x,t)\|_K = |x|_K + |t|^{1/2}$ with in particular $|x_i| \leq r$ for $i = 1 \dots m$ and $|t| \leq r^2$, there exists a constant $C > 0$ s.t.

$$\left(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |P|^p \right)^{\frac{1}{p}} \leq C \left(\frac{|a|}{r^2} + \frac{|b|}{r} + |c| + |d| \right). \tag{3.3.1}$$

On the other hand it is possible to show by contradiction that

$$|a| + |b| + |c| + |d| \leq C \left(\int_{\mathcal{Q}_1^-} |P|^p \right)^{\frac{1}{p}}. \tag{3.3.2}$$

In order to prove (3.3.2), we first observe that, for a given polynomial $P(x,t) = a + \langle b, x \rangle + \frac{1}{2} \langle cx, x \rangle + d\ t$, it is not restrictive to assume that

$$|a| + |b| + |c| + |d| = 1. \tag{3.3.3}$$

Indeed, if (3.3.3) is not satisfied, it is sufficient to observe that, for a given $r > 0$, we have

$$\begin{aligned} &\left(\int_{\mathcal{Q}_1^-} |ra + \langle rb, x \rangle + \frac{1}{2} \langle rcx, x \rangle + d\ rt|^p \right)^{\frac{1}{p}} \\ &\qquad = r \left(\int_{\mathcal{Q}_1^-} |a + \langle b, x \rangle + \frac{1}{2} \langle cx, x \rangle + d\ t|^p \right)^{\frac{1}{p}}. \end{aligned} \tag{3.3.4}$$

Then the proof for general coefficients $(a, b, c, d) \neq (0, 0, 0, 0)$ which do not satisfy (3.3.3) immediately follows by setting $r = \frac{1}{|a|+|b|+|c|+|d|}$.

We are now in a position to prove (3.3.2) by contradiction. Indeed, if (3.3.2) is false, we have that, for every constant $K > 0$, there exists a polynomial P_K with coefficients

(a_K, b_K, c_K, d_K), that, without loss of generality satisfy (3.3.3), such that

$$1 = |a_K| + |b_K| + |c_K| + |d_K| > \frac{1}{K}\left(\int_{\mathcal{Q}_1^-} |P_K|^p\right)^{\frac{1}{p}}. \tag{3.3.5}$$

Therefore, for $K \to 0^+$, it follows that while the right hand side of (3.3.5) goes to infinity, the left hand side remains constant. This implies a contradiction, as the sum of the norms of the coefficients of P_K would be both a constant and infinity. The thesis follows by the combination of (3.3.1) and (3.3.2), with $r \geq 1$. □

Remark 3.3.2. We propose an alternative proof of inequality (3.3.2) which does not require a contradiction argument. As a first step, we fix $1 < p < \infty$ and we observe that the function

$$(a, b, c, d) \mapsto \left(\int_{\mathcal{Q}_1^-} |P|^p\right)^{\frac{1}{p}} =: F(a, b, c, d)$$

is continuous and strictly positive in $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N} \times \mathbb{R}$. Then, in virtue of Weierstrass' theorem, F admits a strictly positive minimum on the compact set

$$A := \{(a, b, c, d) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N} \times \mathbb{R} : |a| + |b| + |c| + |d| = 1\}.$$

The proof for the general case of $(a, b, c, d) \neq (0, 0, 0, 0)$ that do not satisfy (3.3.3) follows reasoning as in the proof of Lemma 3.3.1. In particular, (3.3.4) yields $F(ra, rb, rc, rd) = rF(a, b, c, d)$ and we just need to choose $r = \frac{1}{|a|+|b|+|c|+|d|}$.

We now prove the following Lemma.

Lemma 3.3.3. *(Estimates on larger cylinders)*
Let u be solution of $\mathscr{L}_0 u = f$ in $\mathcal{Q}_R^-$ for $R > 2$. Then for any $\rho \in [1, R/2]$, there exists a positive constant $C_1 = C_1(p, Q)$ s.t.

$$\Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |u - P_1|^p dx\ dt\Big)^{\frac{1}{p}} \leq C_1 \int_1^{4\rho} \frac{\hat{N}(u, f; s) + \widetilde{\omega}(f; s)}{s}\, ds, \tag{3.3.6}$$

where $P_1 \in \mathcal{P}_1$ is the polynomial realizing the infimum in definition (3.1.9) *at level one.*

Proof. We start working on the left hand side of (3.3.6). Namely, for any $\rho \geq 1$

$$\begin{aligned}&\Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |u - P_1|^p\Big)^{\frac{1}{p}}\\ \leq&\Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |u - P_\rho|^p\Big)^{\frac{1}{p}} + \Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |P_\rho - P_1|^p\Big)^{\frac{1}{p}} = \hat{N}(u, f; \rho) + I_1.\end{aligned} \tag{3.3.7}$$

where in the last line we recalled (3.1.9), and $P_\rho \in \widetilde{\mathcal{P}_{c_\rho}}$ is a polynomial realizing the infimum in the definition of $\hat{N}(u,f;\cdot)$ at the level ρ. We now estimate I_1 as follows

$$I_1 \leq \Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |P_r - P_1|^p \Big)^{\frac{1}{p}} + \sum_{j=1}^{k} \Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |P_{2^j r} - P_{2^{j-1} r}|^p \Big)^{\frac{1}{p}} =: I_2 + I_3, \quad (3.3.8)$$

where we have written $\rho \geq 1$ as $\rho = 2^k r$ for an integer $k \geq 1$ and $r \in [1/2, 1)$. In order to control I_2 and I_3 we need to achieve a more general estimate. For an arbitrary $\gamma > 1$, for any $\alpha \in [1, \gamma]$ we have that

$$\begin{aligned}
&\Big(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |P_{\alpha r} - P_r|^p \Big)^{\frac{1}{p}} \\
\leq& \Big(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |u - P_r|^p \Big)^{\frac{1}{p}} + \Big(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |u - P_{\alpha r}|^p \Big)^{\frac{1}{p}} \\
\leq& \Big(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |u - P_r|^p \Big)^{\frac{1}{p}} + \alpha^{\frac{Q+2+2p}{p}} \Big(\frac{1}{(\alpha r)^{Q+2+2p}} \int_{\mathcal{Q}_{\alpha r}^-} |u - P_{\alpha r}|^p \Big)^{\frac{1}{p}} \\
\leq& \alpha^{\frac{Q+2+2p}{p}} (\hat{N}(u,f;r) + \hat{N}(u,f;\alpha r)) \\
\leq& \gamma^{\frac{Q+2+2p}{p}} (\hat{N}(u,f;r) + \hat{N}(u,f;\alpha r)).
\end{aligned} \quad (3.3.9)$$

We take care of I_2 choosing $\alpha r = 1$, for $r \in \left[\frac{1}{\gamma}, 1\right)$ in (3.3.9). Namely, applying both case (i) and (ii) from Lemma 3.3.1, we get

$$\begin{aligned}
I_2 :=& \Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |P_r - P_1|^p \Big)^{\frac{1}{p}} \leq C_2 \Big(\int_{\mathcal{Q}_1^-} |P_1 - P_r|^p \Big)^{\frac{1}{p}} \\
\leq& C \Big(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |P_1 - P_r|^p \Big)^{\frac{1}{p}} \leq \gamma^{\frac{Q+2+2p}{p}} (\hat{N}(u,f;r) + \hat{N}(u,f;1)).
\end{aligned} \quad (3.3.10)$$

Exploiting again (3.3.9) together with case (i) from Lemma 3.3.1, we infer that for every $\rho \geq 1$, which we write as $\rho = 2^k r$ with $r \in \left[\frac{1}{\gamma}, 1\right)$, there holds

$$I_3 \leq C \sum_{j=1}^{k} \hat{N}(u,f;2^j r). \quad (3.3.11)$$

Now, collecting bounds (3.3.8), (3.3.10) and (3.3.11), we have that (3.3.7) reads

$$\begin{aligned}
&\Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |u - P_1|^p \Big)^{\frac{1}{p}} \\
\leq& C \left(\hat{N}(u,f;1) + \hat{N}(u,f;r) + \sum_{j=1}^{k} \hat{N}(u,f;2^j r) \right),
\end{aligned} \quad (3.3.12)$$

where $C = C(p, Q, \gamma)$ is a positive constant. We now want to estimate the right hand side of

(3.3.12), in particular for any $\gamma > 1$ and for $\alpha \in [1, \gamma]$ it follows that

$$\begin{aligned}
\hat{N}(u, f; \alpha r) &\le \Big(\frac{1}{(\alpha r)^{Q+2+2p}} \int_{\mathcal{Q}^-_{\alpha r}} |u - P_{\alpha r}|^p \Big)^{\frac{1}{p}} \\
&\le \Big(\frac{1}{(\gamma r)^{Q+2+2p}} \int_{\mathcal{Q}^-_{\gamma r}} |u - P_{\gamma r}|^p \Big)^{\frac{1}{p}} \Big(\frac{\gamma}{\alpha} \Big)^{\frac{Q+2+2p}{p}} \\
&\quad + \Big(\frac{1}{(\alpha r)^{Q+2+2p}} \int_{\mathcal{Q}^-_{\alpha r}} |P_{\gamma r} - P_{\alpha r}|^p \Big)^{\frac{1}{p}} \\
&\le \gamma^{\frac{Q+2+2p}{p}} \hat{N}(u, f; \gamma r) + C_2 |c_{\alpha r} - c_{\gamma r}| \left(\int_{\mathcal{Q}^-_1} |P_*|^p \right)^{\frac{1}{p}} \qquad (3.3.13)
\end{aligned}$$

where in the last line we used case (i) of Lemma 3.3.1 and we introduced P_* as a solution to equation $\mathscr{L}_0 P_* = 1$. In particular, from (3.1.8), we obtain

$$\begin{aligned}
|c_{\alpha r} - c_{\gamma r}| &= \left(\frac{1}{|\mathcal{Q}^-_{\alpha r}|} \int_{\mathcal{Q}^-_{\alpha r}} |c_{\alpha r} - c_{\gamma r}|^p \right)^{\frac{1}{p}} \\
&\le \left(\frac{1}{|\mathcal{Q}^-_{\alpha r}|} \int_{\mathcal{Q}^-_{\alpha r}} |f - c_{\alpha r}|^p \right)^{\frac{1}{p}} + \gamma^{\frac{Q+2}{p}} \left(\frac{1}{|\mathcal{Q}^-_{\gamma r}|} \int_{\mathcal{Q}^-_{\gamma r}} |f - c_{\gamma r}|^p \right)^{\frac{1}{p}} \\
&\le \widetilde{\omega}(f; \alpha r) + \gamma^{\frac{Q+2}{p}} \widetilde{\omega}(f; \gamma r) \le 2\gamma^{\frac{Q+2}{p}} \widetilde{\omega}(f; \gamma r). \qquad (3.3.14)
\end{aligned}$$

Thus, combining (3.3.14) with (3.3.13) we infer that for any $\gamma > 1$ there esists a positive constant $C_\gamma = C_\gamma(p, Q, \gamma)$ s.t. for any $\alpha \in [1, \gamma]$ it holds

$$\begin{aligned}
\hat{N}(u, f; \alpha r) &\le C_\gamma \left(\hat{N}(u, f; \gamma r) + \widetilde{\omega}(f; \gamma r) \right), \\
\widetilde{\omega}(f; \alpha r) &\le C_\gamma \widetilde{\omega}(f; \gamma r). \qquad (3.3.15)
\end{aligned}$$

Eventually, putting together (3.3.12) and (3.3.15) and choosing $\gamma = 2$ we finally obtain

$$\begin{aligned}
\Big(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}^-_\rho} |u - P_1|^p \Big)^{\frac{1}{p}} &\le 3C \sum_{j=1}^{k} \left(\hat{N}(u, f; 2^{j+1} r) + \widetilde{\omega}(f; 2^{j+1} r) \right) \\
&\le 6C \sum_{j=1}^{k} \frac{\hat{N}(u, f; 2^{j+1} r) + \widetilde{\omega}(f; 2^{j+1} r)}{2^{j+1}} \left(2^{j+2} r - 2^{j+1} r \right) \\
&\le 6C \int_1^{4\rho} \frac{\hat{N}(u, f; s) + \widetilde{\omega}(f; s)}{s} ds.
\end{aligned}$$

□

As a consequence, we can prove the decay estimate below.

Proposition 3.3.4. *(Basic decay estimate)*
Given $p \in (1, +\infty)$, there exist constants $C_0 = C_0(p, Q) > 0$ and $\lambda = \lambda(p, Q), \mu = \mu(p, Q) \in$

$(0,1)$ *such that for every function* u *and* f *satisfying* (3.1.1), $\forall r \in (0,1]$, *the following estimates hold*

$$\hat{N}(u,f;\lambda^2 r,\lambda r) < \mu\ \hat{N}(u,f;\lambda r,r) \qquad \text{or} \qquad \hat{N}(u,f;\lambda^2 r,\lambda r) < C_0\ \widetilde{\omega}(f;\lambda^2 r,r). \tag{3.3.16}$$

Proof. The proof is carried out by contradiction. Namely, if (3.3.16) is not true, we can find the sequences $C_k \to \infty, r_k \in (0,1], \lambda_k \to 0$ and $\mu_k \to 1$ such that

$$\hat{N}(u_k,f_k;\lambda_k^2 r_k,\lambda_k r_k) \geq \mu_k \hat{N}(u_k,f_k;\lambda_k r_k,r_k) \tag{3.3.17}$$
$$\hat{N}(u_k,f_k;\lambda_k^2 r_k,\lambda_k r_k) \geq C_k \widetilde{\omega}(f_k;\lambda_k^2 r_k,r_k), \tag{3.3.18}$$

where $(f_k)_k$ and $(u_k)_k$ satisfy (3.1.1). Let us consider $\rho_k \in [\lambda_k^2 r_k, \lambda_k r_k]$ such that, according to (3.1.10)

$$\hat{N}(u_k,f_k;\lambda_k^2 r_k,\lambda_k r_k) = \hat{N}(u_k,f_k;\rho_k) =: \varepsilon_k. \tag{3.3.19}$$

Moreover, owing to (1.1.13), we define the rescaled functions

$$v_k(x,t) = \frac{u_k(\delta_{\rho_k}(x,t))}{\rho_k^2}$$

and

$$w_k(x,t) = \frac{u_k(\delta_{\rho_k}(x,t)) - P_k(\delta_{\rho_k}(x,t))}{\varepsilon_k \rho_k^2} \tag{3.3.20}$$

where $P_k \in \mathcal{P}_{c\rho_k}$ is the homogeneous polynomial realizing the infimum at the level ρ_k.

Now we want to control w_k in order to pass to the limit. We first notice that

$$\inf_{P\in\mathcal{P}} \Big(\int_{\mathcal{Q}_1^-} |w_k - P|^p \Big)^{\frac{1}{p}} = 1. \tag{3.3.21}$$

Indeed, first exploiting the definition of w_k in (3.3.20) and then using the change of variables $y = \delta^0_{\rho_k}(x),\ s = \rho_k^2 t$, owing to (1.1.14), we infer

$$\begin{aligned}
&\inf_{P\in\mathcal{P}} \Big(\int_{\mathcal{Q}_1^-} |w_k - P|^p \Big)^{\frac{1}{p}} \\
=& \inf_{P\in\mathcal{P}} \Big(\int_{\mathcal{Q}_1^-} \left| \frac{u_k(\delta^0_{\rho_k}(x),\rho_k^2 t) - P_k(\delta^0_{\rho_k}(x),\rho_k^2 t) - \varepsilon_k\rho_k^2 P(x,t)}{\varepsilon_k\rho_k^2} \right|^p dx\ dt \Big)^{1/p} \\
=& \frac{1}{\varepsilon_k} \inf_{P\in\mathcal{P}} \Big(\frac{1}{\rho_k^{Q+2+2p}} \int_{\mathcal{Q}_{\rho_k}^-} \left| u_k(y,s) - P_k(y,s) - \varepsilon_k\rho_k^2 P\left(\delta^0_{\frac{1}{\rho_k}}(y), \frac{1}{\rho_k^2} s\right) \right|^p dy\ ds \Big)^{1/p}.
\end{aligned}$$

Now, since $\mathscr{L}_0(P_k + \varepsilon_k\rho_k^2 P) = \mathscr{L}_0 P_k + \varepsilon_k\rho_k^2 \mathscr{L}_0 P = c_{\rho_k}$ by (3.1.4) and (3.1.5), the identity (3.3.21) follows from (3.3.19).

In addition it holds

$$\hat{N}(v_k, f_k; 1) = \Big(\frac{1}{\rho_k^{Q+2+2p}} \int_{\mathcal{Q}^-_{\rho_k}} |u_k - P_k|^p \Big)^{\frac{1}{p}}.$$

We now apply Lemma 3.3.3 to v_k, for $s \in \Big[1, \frac{r_k}{2\rho_k}\Big]$

$$\begin{aligned} \Big(\frac{1}{(s\rho_k)^{Q+2+2p}} \int_{\mathcal{Q}^-_{s\rho_k}} |u_k - P_k|^p \Big)^{\frac{1}{p}} &= \Big(\frac{1}{s^{Q+2+2p}} \int_{\mathcal{Q}^-_{s}} \Big| \frac{u_k(\delta_{\rho_k}(y,s)) - P_k(\delta_{\rho_k}(y,s))}{\rho_k^2} \Big|^p \Big)^{\frac{1}{p}} \\ &\leq C_1 \int_1^{4s} \frac{\hat{N}(v_k, g_k; \tau) + \widetilde{\omega}(g_k; \tau)}{\tau} d\tau \\ &\leq C_1 \int_1^{4s} \frac{\hat{N}(u_k, f_k; \tau\rho_k) + \widetilde{\omega}(f_k; \tau\rho_k)}{\tau} d\tau, \end{aligned} \qquad (3.3.22)$$

where in the second line we defined $g_k(x,t) = f_k(\delta_{\rho_k}(x,t))$ and in the third line we used the identities $\hat{N}(v_k, g_k; s) = \hat{N}(u_k, f_k; s\rho_k)$ and $\widetilde{\omega}(g_k; s) = \widetilde{\omega}(f_k; s\rho_k)$. As a consequence, for $s \in \Big[1, \frac{r_k}{2\rho_k}\Big]$, the following holds

$$\begin{aligned} \Big(\frac{1}{s^{Q+2+2p}} \int_{\mathcal{Q}^-_{s}} |w_k|^p \Big)^{\frac{1}{p}} &\leq \frac{C_1}{\varepsilon_k} \int_1^{4s} \frac{\hat{N}(u_k, f_k; \tau\rho_k) + \widetilde{\omega}(f_k; \tau\rho_k)}{\tau} d\tau \\ &\leq \frac{C_1}{\varepsilon_k} \int_1^{4s} \frac{\hat{N}(u_k, f_k; \lambda_k^2 r_k, r_k) + \widetilde{\omega}(f_k; \lambda_k^2 r_k, r_k)}{\tau} d\tau. \end{aligned} \qquad (3.3.23)$$

On the other hand, combining (3.3.17) with (3.3.19), we obtain

$$\hat{N}(u_k, f_k; \lambda_k^2 r_k, r_k) \leq \frac{\varepsilon_k}{\mu_k}$$

and

$$\widetilde{\omega}(f_k; \lambda_k^2 r_k, r_k) \leq \frac{\varepsilon_k}{C_k}. \qquad (3.3.24)$$

These two bounds together with (3.3.23), yield to

$$\Big(\frac{1}{s^{Q+2+2p}} \int_{\mathcal{Q}^-_{s}} |w_k|^p \Big)^{\frac{1}{p}} \leq C_2 \ln 4s \qquad (3.3.25)$$

where $s \in \Big[1, \frac{r_k}{2\rho_k}\Big]$ and C_2 is a positive constant depending on C_1, C_k and μ_k.

Now, according to the dilation invariance of $\mathscr{L}_0$ with respect to δ_r (see (1.1.11)) and (3.3.24), we find

$$\Big(\frac{1}{|\mathcal{Q}^-_s|} \int_{\mathcal{Q}^-_{s}} |\mathscr{L}_0 w_k|^p \Big)^{\frac{1}{p}} \leq \frac{1}{\varepsilon_k} \widetilde{\omega}(f_k; s\rho_k) \leq \frac{1}{\varepsilon_k} \widetilde{\omega}(f_k; \lambda_k^2 r_k, r_k) \leq \frac{1}{C_k} \to 0 \qquad (3.3.26)$$

The contradiction follows from passing to the limit. In order to do so, we need a compactness argument.

Applying Lemma 3.2.3 to $W_k = w_k |w_k|^{\frac{p}{2}-1}$, we obtain that for every $R \in \left[1, \frac{r_k}{\rho_k}\right)$

$$\frac{2(p-1)}{p^2} \int_{\mathcal{Q}_R^-} |D_m W_k|^2 \\ \leq \left(\frac{2}{(p-1)} \frac{c_2^2}{(r-\rho)^2} + \frac{2}{p} c_1 \frac{r^{2\kappa+1}}{r-\rho} \right) \int_{\mathcal{Q}_r^-} W_k^2 + \widetilde{\omega}(f;r) \, |\mathcal{Q}_r^-| \left(\frac{1}{|\mathcal{Q}_r^-|} \int_{\mathcal{Q}_r^-} \eta^{2p'} W_k^2 \right)^{\frac{1}{p'}},$$

for $r = \frac{r_k}{\rho_k}$. As a consequence, for every $R \in \left(0, \frac{r_k}{\rho_k}\right)$, we have

$$\| W_k \|_{S^2(\mathcal{Q}_R^-)} \leq C_R.$$

Thus, we can extract a non-relabelled subsequence W_k such that

$$W_k \rightharpoonup W_\infty = w_\infty |w_\infty|^{\frac{p}{2}-1} \qquad \text{weakly in } S^2_{loc}(\mathcal{Q}_R^-),$$

where we denoted by w_∞ the limit of the sequence w_k. Moreover, from the compact embedding provided by Theorem 3.2.2 it follows that

$$W_k \to W_\infty = w_\infty |w_\infty|^{\frac{p}{2}-1} \qquad \text{in } L^2_{loc}(\mathcal{Q}_R^-).$$

We now observe that

$$\| w_k \|^p_{L^p(\mathcal{Q}_R^-)} = \| W_k \|^2_{L^2(\mathcal{Q}_R^-)}, \qquad \| w_\infty \|^p_{L^p(\mathcal{Q}_R^-)} = \| W_\infty \|^2_{L^2(\mathcal{Q}_R^-)},$$

and therefore we have the following convergence result for every $R \in \left(0, \frac{r_k}{\rho_k}\right)$

$$w_k \to w_\infty \qquad \text{in } L^p_{loc}(\mathcal{Q}_r^-).$$

In particular, from (3.3.21), w_∞ satisfies

$$\inf_{P \in \mathcal{P}} \left(\int_{\mathcal{Q}_1^-} |w_\infty - P|^p \right)^{\frac{1}{p}} = 1. \tag{3.3.27}$$

Similarly, according to (3.3.25),

$$\left(\frac{1}{s^{Q+2+2p}} \int_{\mathcal{Q}_s^-} |w_\infty|^p \right)^{\frac{1}{p}} \leq C_2 \ln 4s.$$

Hence, w_∞ is a function that grows quadratically in space and linearly in time up to a logarithmic correction. Moreover, in virtue of (3.3.26), it follows that w_∞ a. e. belongs to $\mathcal{P}$. This contradicts (3.3.27) and therefore concludes the proof. □

We now establish a sort of monotonicity result for $\underline{N}$ and $\underline{\omega}$, defined in (3.1.12) and (3.1.13), respectively.

Proposition 3.3.5. *(Dini estimate)*
Let $N : (0,1] \to [0,+\infty)$, $\omega : (0,1] \to [0,+\infty)$ be two functions that satisfy

$$\forall r \in (0,1], \qquad N(\lambda r) < \mu\, N(r) \qquad \text{or} \qquad N(\lambda r) < \underline{C}\, \omega(r), \tag{3.3.28}$$

and

$$\forall r \in (0,1] \quad \forall \alpha \in [\lambda, 1], \qquad \begin{cases} N(\alpha r) \le \underline{C}\,(N(r) + \omega(r)), \\ \omega(\alpha r) \le \underline{C}\, \omega(r) \end{cases} \tag{3.3.29}$$

for some constants $\underline{C} > 0$ and for $\lambda, \mu \in (0,1)$. Moreover, we assume that ω is Dini. Then for every $\rho \in (0, \frac{\lambda}{4})$ and for $\beta = \frac{\ln \mu}{\ln \lambda}$ we have

$$\begin{aligned} &\int_0^{4\rho} \frac{N(r)}{r} dr \le \\ &\qquad \underline{C}\, \frac{1}{\beta} \left\{ \left(\frac{4\rho}{\lambda} \right)^{\beta} (N(1) + \omega(1)) + \underline{C}' \left(\int_0^{4\rho} \frac{\omega(r)}{r} dr + \rho^{\beta} \int_{4\rho}^{1} \frac{\omega(r)}{r^{1+\beta}} dr \right) \right\}. \end{aligned} \tag{3.3.30}$$

where $\underline{C}' = \underline{C}'(\lambda, \mu) = \frac{1}{\mu} \frac{1}{(1-\lambda)\lambda^{\beta}}$.

Proof. We first prove that for all $r \in (0, \lambda]$, we have

$$N(r) \le \max \left(\underline{C}_1 r^{\beta}, \underline{C}\, \frac{1}{\mu}\, r^{\beta} \sup_{\rho \in [r, \lambda]} \frac{\omega(\rho)}{\rho^{\beta}} \right), \tag{3.3.31}$$

where $\underline{C}_1$ is given by

$$\underline{C}_1 = \underline{C}\, \lambda^{-\beta} \, (N(1) + \omega(1)). \tag{3.3.32}$$

If $r \le \lambda$, we write it as $r = \lambda^k r_1$ with $k \ge 1$ and $r_1 \in (\lambda, 1]$. Then, taking advantage of (3.3.28), we infer

$$\begin{aligned} N(r) &\le \max \left(\underline{C}\, \omega \left(\frac{r}{\lambda} \right), \mu\, N \left(\frac{r}{\lambda} \right) \right) \\ &\le \max \left(\underline{C}\, \omega \left(\frac{r}{\lambda} \right), \underline{C}\, \mu\, \omega \left(\frac{r}{\lambda^2} \right), \mu^2\, N \left(\frac{r}{\lambda^2} \right) \right) \\ &\le \max \left(\underline{C}\, \omega \left(\frac{r}{\lambda} \right), \underline{C}\, \mu\, \omega \left(\frac{r}{\lambda^2} \right), \underline{C}\, \mu^2\, \omega \left(\frac{r}{\lambda^3} \right), \dots, \right. \\ &\qquad \left. \underline{C}\, \mu^{k-2}\, \omega \left(\frac{r}{\lambda^{k-1}} \right), \mu^k\, N \left(\frac{r}{\lambda^k} \right) \right). \end{aligned} \tag{3.3.33}$$

Now, if we set $\beta = \frac{\ln \mu}{\ln \lambda}$ and $\rho = \frac{r}{\lambda^{j+1}}$ for $j = 0, \dots, k-2$, we deduce

$$\mu^j\, \omega \left(\frac{r}{\lambda^{j+1}} \right) = e^{j \ln \mu} \omega(\rho) = \mu^{-1}\, e^{\ln(r/\rho)\beta}\, \omega(\rho) = \mu^{-1} \frac{\omega(\rho)}{\rho^{\beta}} r^{\beta}. \tag{3.3.34}$$

On the other hand, according to (3.3.29) and (3.3.32), we have

$$\mu^k N\left(\frac{r}{\lambda^k}\right) \leq \mu^k \underline{C}(N(1)+\omega(1)) = \underline{C}_1\,\mu^k\lambda^\beta \leq \underline{C}_1\,\mu^k r_1^\beta = \underline{C}_1\,\mu^k\left(\frac{r}{\lambda^k}\right)^\beta \leq \underline{C}_1\,r^\beta. \qquad (3.3.35)$$

Finally, using estimates (3.3.34) and (3.3.35) in (3.3.33), we get (3.3.31).

We now want to estimate $\sup_{\rho\in[r,\lambda]}\frac{\omega(\rho)}{\rho^\beta}$. To this end, for some $\rho_0 \in [r,\lambda]$, we write

$$\begin{aligned}\sup_{\rho\in[r,\lambda]}\frac{\omega(\rho)}{\rho^\beta} &= \frac{\omega(\rho_0)}{\rho_0^\beta}\\ &\leq \frac{1}{\rho_0^\beta}\frac{1}{t\rho_0}\int_{\rho_0}^{\rho_0+t\rho_0}\underline{C}\,\omega(\rho)d\rho\\ &\leq \frac{\underline{C}}{t\lambda^{1+\beta}}\int_{\rho_0}^{\rho_0/\lambda}\frac{\omega(\rho)}{\rho^{1+\beta}}d\rho\\ &\leq \underline{C}_2\int_r^1\frac{\omega(\rho)}{\rho^{1+\beta}}d\rho,\end{aligned}$$

where in the second line we have used the monotonicity of ω according to (3.3.29) and the constants t and $\underline{C}_2$ appearing in the second and forth line are equal to $(1-\lambda)/\lambda$ and $\underline{C}/((1-\lambda)\lambda^\beta)$ respectively.

Combining the previous inequality with (3.3.31) and setting $\underline{C}' := \frac{1}{\mu}\,\underline{C}_2$, we obtain for any $\rho \in \left(0,\frac{\lambda}{4}\right)$

$$\int_0^{4\rho}\frac{N(r)}{r}dr \leq \underline{C}_1\int_0^{4\rho}r^{\beta-1}dr + \underline{C}\,\underline{C}'\,J, \qquad (3.3.36)$$

with

$$\begin{aligned}J := &\int_0^{4\rho}r^{\beta-1}dr\left(\int_r^1\frac{\omega(\tau)}{\tau^{1+\beta}}d\tau\right)\\ \leq &\int_0^{4\rho}\frac{r^\beta}{\beta}\frac{\omega(r)}{r^{1+\beta}}dr + \left[\frac{r^\beta}{\beta}\left(\int_r^1\frac{\omega(\tau)}{\tau^{1+\beta}}d\tau\right)\right]_0^{4\rho}\\ = &\frac{1}{\beta}\int_0^{4\rho}\frac{\omega(r)}{r}dr + \frac{(4\rho)^\beta}{\beta}\left(\int_{4\rho}^1\frac{\omega(\tau)}{\tau^{1+\beta}}d\tau\right),\end{aligned} \qquad (3.3.37)$$

where in the second line we have integrated by parts and in the third we have applied the dominated convergence theorem.

Inequality (3.3.36), together with (3.3.37) and the definition of $\underline{C}_1$ in (3.3.32), yields

$$\begin{aligned}\int_0^{4\rho}\frac{N(r)}{r}dr \leq &\underline{C}\left(\frac{4\rho}{\lambda}\right)^\beta\frac{1}{\beta}(N(1)+\omega(1))\\ &+\underline{CC'}\left(\frac{1}{\beta}\int_0^{4\rho}\frac{\omega(r)}{r}dr + \frac{(4\rho)^\beta}{\beta}\left(\int_{4\rho}^1\frac{\omega(\tau)}{\tau^{1+\beta}}d\tau\right)\right),\end{aligned}$$

which concludes the proof. □

Remark 3.3.6. We observe that hypothesis (3.3.29) could be substituted by (3.3.15) and therefore owing to Proposition 3.3.4, the previous result Proposition 3.3.5 holds in parti-cular for $\hat{N}$ and $\widetilde{\omega}$.

Remark 3.3.7. We notice that:

1. the quantities $\underline{N}$ and $\underline{\omega}$ defined in (3.1.12) and (3.1.13) satisfy (3.3.28) in virtue of Proposition 3.3.4. Moreover, (3.3.15) with $\gamma = \frac{1}{\lambda}$ implies that $\underline{N}$ and $\underline{\omega}$ also satisfy (3.3.29);
2. we chose the limits of integration in order to combine effortlessly this result with the following Lemma 3.3.8.

We now focus on the following result, which differs from Lemma 3.3.3 in the choice of the polynomial and of ρ. More precisely, in Lemma 3.3.3, we derive an estimate on large cylinders, while we here consider smaller radii.

Lemma 3.3.8. *(Estimates on smaller cylinders)*
If u is defined in $\mathcal{Q}_1^-$, then there exist a unique polynomial $P_0 \in \widetilde{\mathcal{P}}$ such that for every $\rho \in (0, \frac{1}{4})$, we have

$$\left(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |u - P_0|^p \right)^{\frac{1}{p}} \leq C_1 \int_0^{4\rho} \frac{\hat{N}(u, f; r) + \widetilde{\omega}(f; r)}{r} dr. \tag{3.3.38}$$

Proof. We suppose that u is a solution to $\mathscr{L}_0 u = f$ in $\mathcal{Q}_1^-$. Applying Lemma 3.3.3 to a rescaled function

$$v(x, t) = \frac{u(\delta_r(x, t))}{r^2}$$

it follows that for $r \leq \frac{1}{4\gamma}$, with $\gamma \geq 1$

$$\left(\frac{1}{\gamma^{Q+2+2p}} \int_{\mathcal{Q}_\gamma^-} |v - P^v|^p \right)^{\frac{1}{p}} \leq C_1 \int_1^{4\gamma} \frac{\hat{N}(v, f; s) + \widetilde{\omega}(f; s)}{s} ds$$

where P^v realises the infimum in the definition of $\hat{N}(v, f; 1)$. Now performing a change of variables with $\rho = \gamma r$, we infer

$$\left(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |u - P_r|^p \right)^{\frac{1}{p}} \leq C_1 \int_r^{4\rho} \frac{\hat{N}(u, f; s) + \widetilde{\omega}(f; s)}{s} ds, \tag{3.3.39}$$

where we notice that $P^v(x, t) = \frac{P_r(\delta_r(x,t))}{r^2}$ and $\hat{N}(v, f; s) = \hat{N}(u, f; rs)$. Hence, fixing $\rho \in (0, 1/4)$, we may pass to the limit in (3.3.39) for $r \to 0$. Therefore up to extracting a subsequence, we can assume that P_r tends to a polynomial $P_0 \in \widetilde{\mathcal{P}}$, namely

$$\left(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |u - P_0|^p \right)^{\frac{1}{p}} \leq C_1 \int_0^{4\rho} \frac{\hat{N}(u, f; s) + \widetilde{\omega}(f; s)}{s} ds.$$

We now show that the Taylor-type polynomial P_0 is unique. In fact, if $P_1 \in \widetilde{\mathcal{P}}$ is another polynomial satisfying (3.3.38), then for every $\rho \in (0, \frac{1}{4})$ we have

$$\left(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |P_1 - P_0|^p \right)^{\frac{1}{p}} \leq 2C_1 \int_0^{4\rho} \frac{\hat{N}(u, f; r) + \widetilde{\omega}(f; r)}{r} dr.$$

On the other hand, as $P_1 - P_0 \in \widetilde{\mathcal{P}}$, assertion (ii) of Lemma 3.3.1 yields

$$\left(\int_{\mathcal{Q}_1^-} |P_1 - P_0|^p \right)^{\frac{1}{p}} \leq \widetilde{C}_2 \left(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |P|^p \right)^{\frac{1}{p}},$$

for any $\rho < 1$. Combining the previous two inequalities, infer

$$\|P_1 - P_0\|_{L^p(\mathcal{Q}_1^-)} \leq 2C_1 \int_0^{4\rho} \frac{\hat{N}(u, f; r) + \widetilde{\omega}(f; r)}{r} dr. \tag{3.3.40}$$

As the quantity on the right-hand side of (3.3.40) is finite and goes to 0 as $\rho \to 0$, we finally obtain $P_1 \equiv P_0$, which concludes the proof. □

We notice that, in Lemma 3.3.8 and the upcoming Proposition, we have P_0 belonging to the set $\widetilde{\mathcal{P}}$. Hence, in the proof of assertion (iii) of Theorem 3.1.2 it is only left to show that P_0 belongs in particular to $\mathcal{P}$ (i.e. $\mathscr{L}_0 P_0 = 0$) in order to prove (3.1.16).

Proposition 3.3.9. *(Modulus of continuity of the solution up to second order)*
Let us assume that $\widetilde{\omega}$ is Dini continuous and let us set $\beta = \ln \mu / \ln \lambda$. There exist a unique polynomial $P_0 \in \widetilde{\mathcal{P}}$ and a constant $C' = C'(\underline{C}, \lambda, \mu)$ such that for every $\rho \in (0, \frac{\rho^*}{4})$, we have

$$\begin{aligned} &\left(\frac{1}{\rho^{Q+2+2p}} \int_{\mathcal{Q}_\rho^-} |u - P_0|^p \right)^{\frac{1}{p}} \\ &\leq \underline{C} \frac{1}{\beta} \left\{ \left(\frac{4\rho}{\lambda} \right)^\beta \left(\hat{N}(u, f; 1) + \widetilde{\omega}(f; 1) \right) + \underline{C}' \left(\int_0^{4\rho} \frac{\widetilde{\omega}(f; r)}{r} dr + \rho^\beta \int_{4\rho}^1 \frac{\widetilde{\omega}(f; r)}{r^{1+\beta}} dr \right) \right\}. \end{aligned} \tag{3.3.41}$$

Proof. The proof simply follows from the combination of Proposition 3.3.5 with $N \equiv \hat{N}$ and $\omega \equiv \widetilde{\omega}$ and Lemma 3.3.8, where we remark that we reabsorbed the modulus of continuity in the right hand side of (3.3.30). We observe that the uniqueness of P_0 follows from Lemma 3.3.8. □

Remark 3.3.10. We observe that Proposition 3.3.9 holds, more generally, for two functions N and ω satisfying the assumptions of Proposition 3.3.5.

3.3.2 Proof of Theorem 3.1.2

(i) We first observe that from definitions (3.1.9) and (3.1.7), it holds that

$$\widetilde{N}(u;r) \leq \hat{N}(u,f;r). \tag{3.3.42}$$

The right hand side of (3.3.42) can be estimated combining (3.3.31) with (3.3.32), which yields for $r \in (0,\lambda]$

$$\underline{N}(r) \leq \underline{C}\left(\underline{N}(1) + \frac{1}{\mu}\sup_{\rho\in(0,1]}\underline{\omega}(\rho)\right),$$

where we recall that $\underline{N}$ and $\underline{\omega}$ were defined respectively in (3.1.12) and (3.1.13). Moreover owing to the motonicity-type estimate (3.3.15) for $\gamma = \frac{1}{\lambda}$ and $r \in (0,1]$

$$\hat{N}(u,f;r) \leq C\left(\hat{N}(u,f;1) + \sup_{\rho\in(0,1]}\widetilde{\omega}(f;\rho)\right),$$

with $C = C(\lambda,\mu,p,Q)$. On the other hand, from definition (3.1.9) we get

$$\hat{N}(u,f;1) \leq C\left(\|u\|_{L^p(\mathcal{Q}_1^-)} + \|f\|_{L^p(\mathcal{Q}_1^-)}\right).$$

Therefore, combining the estimates above, we conclude the proof of statement (i).

(ii) Assertion (ii) follows directly from estimate (3.3.31).

(iii) We observe that Proposition 3.3.5 with $N \equiv \hat{N}$ and $\omega \equiv \widetilde{\omega}$ yields statement (iii).

(iv) We recall that we have already proved estimate (3.1.16) in the case where $P_0 \in \widetilde{\mathcal{P}}$, according to Proposition 3.3.9 and namely to (3.3.41). Furthermore, we notice that the coefficients of P_0 are bounded by choosing $\rho = \frac{\lambda}{4}$ in (3.3.41).
Therefore it is only left to show that P_0 belongs in particular to $\mathcal{P}$, i.e. P_0 satisfies equation $\mathscr{L}_0 P_0 = 0$.
To this end, we define the function

$$u^\varepsilon(x,t) = \frac{u(\delta_\varepsilon(x,t)) - P_0(\delta_\varepsilon(x,t))}{\varepsilon^2}$$

which converges in L^p to a function $v \equiv 0$ by (3.3.41) for $\varepsilon \to 0$. Moreover from

$$\begin{aligned}\mathscr{L}_0(u^\varepsilon) &= \varepsilon^2\frac{\mathscr{L}_0 u(\delta_\varepsilon(x,t))}{\varepsilon^2} - \varepsilon^2\frac{\mathscr{L}_0 P_0(\delta_\varepsilon(x,t))}{\varepsilon^2}\\ &= f(\delta_\varepsilon(x,t)) - \mathscr{L}_0 P_0(\delta_\varepsilon(x,t))\end{aligned}$$

and according to (3.2.1), it follows that for $\varepsilon \to 0$,

$$0 = \mathscr{L}_0 v = f(0) - \mathscr{L}_0 P_0.$$

Since by assumption $f(0) = 0$, then we have showed that P_0 satisfies equation $\mathscr{L}_0 P_0 = 0$. This concludes the proof.

Part II

Rough coefficients

The second part of this dissertation is devoted to the study of the regularity of weak solutions to Kolmogorov-type equations with rough coefficients. The study of the regularity theory of degenerate Kolmogorov equations in divergence form with discontinuous coefficients has been an open problem for decades. Indeed, such an investigation started only at the end of the 1990s with [24,83,103,104] and is nowadays the main focus of the research community. Starting from the aforementioned papers, where vanishing-mean-oscillations (VMO) diffusion coefficients were considered, the weak regularity theory for Kolmogorov operators was developed in at least the following four directions.

Moser iteration. As far as rough coefficients a_{ij}'s are concerned, Pascucci and Polidoro proved in [99] that weak (sub)-solutions are locally bounded from above. Later on, Cinti, Pascucci and Polidoro extended this result to the non-dilation invariant case (see [32]). The non-dilation invariant case with lower order coefficients and positive divergence was eventually addressed by Anceschi, Polidoro and Ragusa in [9]).

Poincaré inequality and Hölder regularity. A weak Poincaré inequality and the Hölder continuity of weak solutions were proved by Wang and Zhang in [117] for the dilation-invariant case and in [116] for the non-dilation invariant one. Related results have been recently proved in a rather new functional setting by Armstrong and Mourrat (see [3]) for the kinetic Kolmogorov-Fokker-Planck equation

$$\Delta_p u(p, y, t) = \langle p, D_y u(p, y, t) \rangle + \partial_t u(p, y, t), \tag{II.1}$$

where $u : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$. Very recently, Litsgard and Nystrom in [78] took advantage of the same functional setting of [3] to prove existence and uniqueness of solutions to the Cauchy Dirichlet problem associated to equation (II.1) with rough coefficients.

Harnack inequality. Prior to this dissertation, the only results in this weak framework have been established in the particular case of the kinetic Fokker-Planck operator (II.1).

In particular, Golse, Imbert, Mouhot and Vasseur proved the Hölder continuity and a Harnack inequality for weak solutions to the kinetic Kolmogorov-Fokker-Planck equation (see [49]). The Harnack inequality established in [49] is quite a remarkable result as it comes more than sixty years after the analogous one for uniformly parabolic equations. The reason for this delay lies in the fact that we the classical regularity techniques cannot be applied to degenerate equations like the one in (II.1). To overcome this technical difficulty, the authors of [49] adopted an approach based on velocity averaging method. Based on the results of [49], Anceschi, Eleuteri and Polidoro established a geometric statement for the Harnack inequality (see [6]). More recently, a weak Harnack inequality for kinetic Fokker-Planck equations with essentially bounded coefficients was proved by Guerand and Imbert in [51]. Moreover, in [52], Guerand and Mouhot gave new proofs of weak Harnack and Harnack inequalities, as well as the De Giorgi intermediate-value lemma. Finally, it is worth mentioning the recent preprint [42], where a weak Harnack inequality for hypoelliptic equations including (II.1) was derived.

Gaussian bounds for the fundamental solution. An upper bound for the fundamental solution of Kolmogorov-type operators with measurable bounded coefficients is obtained by Pascucci and Polidoro in [98] and by Lanconelli and Pascucci [71] adapting Aronson's method. Moreover, starting from the invariant Harnack inequality in [49], Lanconelli, Pascucci and Polidoro proved Gaussian bounds for the fundamental solution of (II.1).

In the next chapters, we aim at developing the study of the weak regularity theory for Kolmogorov operators even further. The results we present here are part of a project which started in 2021. The project emerged from the scientific collaboration with Anceschi. The final aim of the project was to provide a complete characterization of the De Giorgi-Nash-Moser weak regularity theory in a suitable functional space for very general Kolmogorov equations of the form

$$\begin{aligned}\mathscr{L}u(x,t) := &\sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u(x,t)\right) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) \\ &+ \sum_{i=1}^{m_0} b_i(x,t)\partial_i u(x,t) + c(x,t)u(x,t) = f(x,t),\end{aligned} \tag{II.2}$$

where $z = (x,t) = (x_1,\ldots,x_N,t) \in \mathbb{R}^{N+1}$ and $1 \le m_0 \le N$.

More precisely, in Chapter 4, we prove a Harnack inequality and the Hölder continuity for weak solutions to the Kolmogorov equation (II.2) with measurable coefficients, integrable lower order terms and nonzero source term. We then introduce a functional space $\mathcal{W}$, suitable for the study of weak solutions to $\mathscr{L}u = f$, that allows us to prove a weak Poincaré inequality. Our analysis is based on a weak Harnack inequality, a weak Poincaré inequality combined with an $L^2 - L^\infty$ estimate and a classical covering argument (Ink-Spots Theorem). The results we present in Chapter 4 are contained in the paper [11].

As a second step, we then prove the existence of a fundamental solution associated to the Kolmogorov equation $\mathscr{L}u = f$, with bounded measurable coefficients. Finally, we prove Gaussian upper and lower bounds for the fundamental solution, and other related properties. These results will be presented in Chapter 5 and are the content of the recent paper [12].

Chapter 4

De Giorgi-Nash regularity theory

4.1 Motivation

This chapter is devoted to the study of the De Giorgi-Nash-Moser regularity theory for weak solutions to the second order partial differential equation of Kolmogorov-type of the form

$$\begin{aligned}\mathscr{L}u(x,t) : &= \sum_{i,j=1}^{m_0} \partial_{x_i} \left(a_{ij}(x,t) \partial_{x_j} u(x,t) \right) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t) \\ &+ \sum_{i=1}^{m_0} b_i(x,t) \partial_i u(x,t) + c(x,t) u(x,t) = f(x,t),\end{aligned} \tag{4.1.1}$$

where $z = (x,t) = (x_1, \dots, x_N, t) \in \mathbb{R}^{N+1}$ and $1 \le m_0 \le N$. In particular, following the lines of [11], we prove a Harnack inequality and the Hölder continuity for weak solutions to equation (4.1.1) under the hypotheses **(H1)-(H2)-(H3)** listed below.

First of all, we require that the matrices $A_0 = (a_{ij}(x,t))_{i,j=1,\dots,m_0}$ and $B = (b_{ij})_{i,j=1,\dots,N}$ satisfy the following structural assumption.

(H1) The matrix A_0 is symmetric with real measurable entries. Moreover, there exist two positive constants λ and Λ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(x,t) \xi_i \xi_j \le \Lambda |\xi|^2 \tag{4.1.2}$$

for every $(x,t) \in \mathbb{R}^{N+1}$ and $\xi \in \mathbb{R}^{m_0}$. The matrix B has constant entries.

In order to state assumption **(H2)**, let us consider the principal part operator

$$\begin{aligned}\mathscr{L}_0 u(x,t) :&= \sum_{i=1}^{m_0} \partial_{x_i}^2 u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t) \\ &= \Delta_{m_0} u(x,t) + Y u(x,t),\end{aligned} \tag{4.1.3}$$

where $(x,t) \in \mathbb{R}^{N+1}$ and, as usual, Yu denotes the Lie derivative introduced in (1.1.5). Operator $\mathscr{L}_0$ in (4.1.3) belongs to the class of constant-coefficients operators we introduced in Chapter 1, Section 1.1. In particular, it is known that, if the matrix B takes the block form (1.1.12), then $\mathscr{L}_0$ is hypoelliptic and invariant with respect to a certain family of dilations. In the sequel, we will therefore rely on the following assumption.

(H2) The *principal part operator* $\mathscr{L}_0$ *of* $\mathscr{L}$ is hypoelliptic and homogeneous of degree 2 with respect to the family of dilations $(\delta_r)_{r>0}$ introduced in (1.1.13).

In the following of this chapter, we will therefore assume that B has the canonical form (1.1.12) and we will also make use of the following notation, which allows us to introduce a compact formulation for operator $\mathscr{L}$. More precisely, here and in the sequel

$$D = (\partial_{x_1}, \ldots, \partial_{x_N}), \quad \langle \cdot, \cdot \rangle, \quad \text{div}$$

respectively denote the gradient, the inner product and the divergence in $\mathbb{R}^N$. In addition,

$$D_{m_0} = (\partial_{x_1}, \ldots, \partial_{x_{m_0}}), \quad \text{div}_{m_0}$$

denote as usual the partial gradient and the partial divergence in the first m_0 components, respectively. Moreover, we introduce the matrix

$$A(x,t) = (a_{ij}(x,t))_{1 \leq i,j \leq N},$$

where a_{ij}, for every $i, j = 1, \ldots, m_0$, are the coefficients appearing in (4.1.1), while $a_{ij} \equiv 0$ whenever $i > m_0$ or $j > m_0$, and we let

$$b(x,t) := (b_1(x,t), \ldots, b_{m_0}(x,t), 0, \ldots, 0). \tag{4.1.4}$$

Now, we are able to rewrite operator $\mathscr{L}$ in the following compact form

$$\mathscr{L}u = \text{div}(ADu) + Yu + \langle b, Du \rangle + cu. \tag{4.1.5}$$

We are now in a position to state our assumption on the integrability of b, c and of the source term f in terms of the homogeneous dimension defined in (1.1.17).

(H3) $c, f \in L^q_{loc}(\Omega)$, with $q > \frac{Q+2}{2}$, and $b \in (L^\infty_{loc}(\Omega))^{m_0}$.

4.1.1 Main results

In order to expose our main results, we first need to introduce some preliminary notation. From now on, we consider a set $\Omega = \Omega_{m_0} \times \Omega_{N-m_0+1}$ of $\mathbb{R}^{N+1}$, where Ω_{m_0} is a bounded Lipschitz domain of $\mathbb{R}^{m_0}$ and Ω_{N-m_0+1} is a bounded Lipschitz domain of $\mathbb{R}^{N-m_0+1}$. This is not restrictive since the cylinders $\mathcal{Q}$ that we consider in our local analysis (see (4.1.12)) satisfy the Lipschitz boundary assumption. Then we split the coordinate $x \in \mathbb{R}^N$ as

$$x = \big(x^{(0)}, x^{(1)}, \dots, x^{(\kappa)}\big), \qquad x^{(0)} \in \mathbb{R}^{m_0}, \quad x^{(j)} \in \mathbb{R}^{m_j}, \quad j \in \{1, \dots, \kappa\}, \tag{4.1.6}$$

where we have that in accordance with the scaling of the differential equation (see (1.1.13)) every m_j is a positive integer such that

$$\sum_{j=1}^{\kappa} m_j = N \qquad \text{and} \qquad N \geq m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1.$$

We denote by $\mathcal{D}(\Omega)$ the set of C^∞ functions compactly supported in Ω and by $\mathcal{D}'(\Omega)$ the set of distributions in Ω. From now on, $H^1_{x^{(0)}}$ denotes the Sobolev space of functions $u \in L^2(\Omega_{m_0})$ with distribution gradient $D_{m_0}u$ lying in $(L^2(\Omega_{m_0}))^{m_0}$, i.e.

$$H^1_{x^{(0)}} := \left\{ u \in L^2(\Omega_{m_0}) : \, D_{m_0}u \in (L^2(\Omega_{m_0}))^{m_0} \right\},$$

and we set

$$\|u\|^2_{H^1_{x^{(0)}}} := \|u\|^2_{L^2(\Omega_{m_0})} + \|D_{m_0}u\|^2_{L^2(\Omega_{m_0})}.$$

We let H^1_0 denote the closure of $C^\infty_c(\Omega_{m_0})$ in the norm of $H^1_{x^{(0)}}$ and we recall that $C^\infty_c(\overline{\Omega}_{m_0})$ is dense in $H^1_{x^{(0)}}$ since Ω_{m_0} is a bounded Lipschitz domain by assumption. Moreover, H^1_0 is a reflexive Hilbert space and thus we may consider its dual space

$$\big(H^1_0\big)^* = H^{-1}_{x^{(0)}} \quad \text{and} \quad \big(H^{-1}_{x^{(0)}}\big)^* = H^1_0,$$

where the notation we consider is the classical one. Hence, from now on we denote by $H^{-1}_{x^{(0)}}$ the dual of H^1_0 acting on functions in H^1_0 through the duality pairing $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{H^1_{x^{(0)}}, H^1_0}$. In a standard manner, see for instance [3, 78], we let $\mathcal{W}$ denote the closure of $C^\infty(\overline{\Omega})$ in the norm

$$\|u\|^2_{\mathcal{W}} = \|u\|^2_{L^2\left(\Omega_{N-m_0+1}; H^1_{x^{(0)}}\right)} + \|Yu\|^2_{L^2\left(\Omega_{N-m_0+1}; H^{-1}_{x^{(0)}}\right)}, \tag{4.1.7}$$

where the previous norm can be explicitly computed as follows:

$$\|u\|^2_{\mathcal{W}} = \int_{\Omega_{N-m_0+1}} \|u(\cdot, y, t)\|^2_{H^1_{x^{(0)}}} \, dy \, dt + \int_{\Omega_{N-m_0+1}} \|Yu(\cdot, y, t)\|^2_{H^{-1}_{x^{(0)}}} \, dy \, dt,$$

where $y = (x^{(1)}, \dots, x^{(\kappa)})$. In particular, $\mathcal{W}$ is a Banach space and we remark that the dual of $L^2(\Omega_{N-m_0+1}; H^1_0)$ satisfies

$$\big(L^2(\Omega_{N-m_0+1}; H^1_0)\big)^* = L^2(\Omega_{N-m_0+1}; H^{-1}_0) \quad \text{and}$$
$$\big(L^2(\Omega_{N-m_0+1}; H^{-1}_0)\big)^* = L^2(\Omega_{N-m_0+1}; H^1_0).$$

From now on, we consider the shorthand notation L^2H^{-1} to denote $L^2\left(\Omega_{N-m_0+1}; H_0^{-1}\right)$.

The space of functions $\mathcal{W}$ is the most natural framework for the study of the weak regularity theory for operator $\mathscr{L}$. Still, to the best of our knowledge it has never been considered in literature in the ultraparabolic setting. In particular, it is an extension of the functional setting firstly formally proposed by Armstrong and Mourrat in [3] for the study of the kinetic Kolmogorov-Fokker-Planck equation, that can be recovered from (4.1.1) by choosing $N = 2d$, $\kappa = 1$, $m_0 = m_1 = d$ and $c \equiv 0$. We refer the reader to Subsection 4.1.2 for an overview on the existing literature.

We refer to [78, Section 2] for some properties of the space $\mathcal{W}$. Lastly, we remark that the major issue when dealing with the space $\mathcal{W}$ is that it requires to handle the duality pairing between L^2H^1 and L^2H^{-1}. To this end, we take advantage of the following remark, see [65, Chapter 4].

Remark 4.1.1. For every open subset $A \subset \mathbb{R}^n$ and for every function $g \in H^{-1}(A)$ there exist two functions H_0, $H_1 \in L^2(A)$ such that

$$g = \operatorname{div}_{m_0} H_1 + H_0 \qquad \text{and} \qquad \|H_0\|_{L^2(A)} + \|H_1\|_{L^2(A)} \leq 2\|g\|_{H^{-1}(A)}.$$

Now, we introduce the definition of *weak solutions* we consider in this dissertation.

Definition 4.1.2. A function $u \in \mathcal{W}$ is a *weak solution* to (4.1.1) with source term $f \in L^2(\Omega)$ if for every non-negative test function $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega -\langle ADu, D\varphi\rangle - uY\varphi + \langle b, Du\rangle\varphi + cu\varphi = \int_\Omega f\varphi. \tag{4.1.8}$$

In the sequel, we will also consider weak sub-solutions to (4.1.1), namely functions $u \in \mathcal{W}$ that satisfy the following inequality

$$\int_\Omega -\langle ADu, D\varphi\rangle - uY\varphi + \langle b, Du\rangle\varphi + cu\varphi \overset{(\leq)}{\geq} \int_\Omega f\varphi, \tag{4.1.9}$$

for every non-negative test function $\varphi \in \mathcal{D}(\Omega)$. A function u is a super-solution to (4.1.1) if it satisfies (4.1.9) with $(\leq)$.

As mentioned above, the aim of this chapter is to prove the local Hölder continuity and a Harnack inequality for solutions to (4.1.1) in the sense of Definition 4.1.2. Our method is based on the combination of three fundamental ingredients - boundedness of weak solutions, weak Poincaré inequality and Log-transformation - in the same spirit of the recent paper [51] for the Fokker-Planck equation. First, we carry out a local study with $\mathcal{Q}^0$ at unit scale. For some reasons we expose below in Section 4.5, $\mathcal{Q}^0$ takes the form $B_{R_0} \times B_{R_0} \times \ldots \times B_{R_0} \times (-1, 0]$ for some large constant R_0 only depending on the dimension Q and on the ellipticity constants λ, Λ in **(H1)**.

As we rely on the Lie group structure associated to operator $\mathscr{L}_0$ in (4.1.3), the suitable geometry when dealing with operator $\mathscr{L}$ is given by the non-Euclidean structure defined in (1.1.6). Our results naturally reflect this non-Euclidean setting. Here and in the following of the chapter, we denote by $\mathcal{Q}_1$ and $\widetilde{\mathcal{Q}}_1$ the unit past cylinders

$$\begin{aligned}\mathcal{Q}_1 &:= B_1 \times B_1 \times \ldots \times B_1 \times (-1, 0), \\ \widetilde{\mathcal{Q}}_1 &:= B_1 \times B_1 \times \ldots \times B_1 \times (-1, 0],\end{aligned} \tag{4.1.10}$$

defined through the open balls

$$B_1 = \{x^{(j)} \in \mathbb{R}^{m_j} : |x| \leq 1\}, \tag{4.1.11}$$

where $j = 0, \ldots, \kappa$ and $|\cdot|$ denotes the euclidean norm in $\mathbb{R}^{m_j}$. Now, for every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$, we set

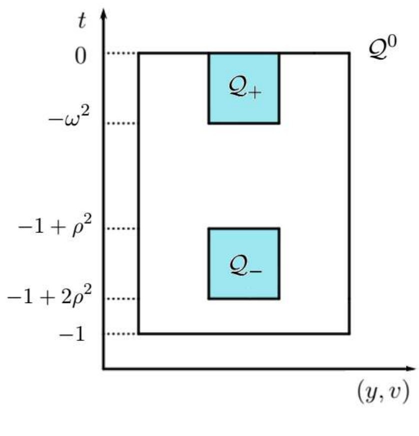
$$\mathcal{Q}_r(z_0) := z_0 \circ (\delta_r(\mathcal{Q}_1)) = \{z \in \mathbb{R}^{N+1} \, : \, z = z_0 \circ \delta_r(\zeta), \zeta \in \mathcal{Q}_1\}, \tag{4.1.12}$$

where "$\circ$" denote the composition law introduced in (1.1.6) and $(\delta_r)_{r>0}$ the family of dilations defined in (1.1.13). Moreover, we introduce

$$\begin{aligned}\mathcal{Q}_+ = \delta_\omega\left(\widetilde{\mathcal{Q}}_1\right) &= B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-\omega^2, 0] \quad \text{and} \\ \widetilde{Q}_- = (0, \ldots, 0, -1 + 2\rho^2) \circ \delta_\rho(\mathcal{Q}_1) &= B_\rho \times B_{\rho^3} \times \ldots \times B_{\rho^{2\kappa+1}} \times (-1 + \rho^2, -1 + 2\rho^2).\end{aligned}$$

We are now in a position to state one of our main results, namely the following Harnack inequality.

Figure 4.1: Geometric setting of the Harnack inequality for the degenerate kinetic Kolmogorov operator. The radius ω is small enough to ensure that, when stacking cylinders over a small one contained in $\mathcal{Q}_-$, the future cylinder $\mathcal{Q}_+$ is captured, see Lemma 4.B.1 and Figure 4.B. On the other hand, the radius R_0 of $\mathcal{Q}^0$ is large enough to allow us to apply Lemma 4.5.8 to every stacked cylinder.



Theorem 4.1.3 (Harnack inequality)**.** *Let u be a non-negative weak solution to $\mathscr{L}u = f$ in $\Omega \supset \widetilde{\mathcal{Q}}_1$ under assumptions **(H1)**-**(H2)**-**(H3)**. Then we have*

$$\sup_{\widetilde{\mathcal{Q}}_-} u \,\leq\, C \left(\inf_{\mathcal{Q}_+} u + \|f\|_{L^q(\mathcal{Q}^0)} \right), \tag{4.1.13}$$

where $0 < \omega < 1$ is given by Theorem 4.5.1 and $0 < \rho < \frac{\omega}{\sqrt{2}}$. Finally, the constants C, ω, ρ only depend on the homogeneous dimension Q defined in (1.1.17)*, on the ellipticity constants λ, Λ in* (4.1.2) *and on the norms $\|b\|_{L^\infty(\mathcal{Q}_{ext})}$ and $\|c\|_{L^q(\mathcal{Q}_{ext})}$.*

To the best of our knowledge, this is the first Harnack inequality available for weak solutions to (4.1.1), since in [118] the authors proved the local Hölder continuity for weak solutions only. It is obtained by combining Theorem 4.3.1 ($L^2 - L^\infty$ estimate) and Theorem 4.5.1 (weak Harnack inequality) and it is an extension of the analogous result for the particular case of the Fokker-Planck equation presented in [49].

Moreover, the weak Harnack inequality proved in Theorem 4.5.1 also implies the Hölder regularity of weak solutions in the sense of Definition 1.2.1. Specifically, the following result holds true.

Theorem 4.1.4 (Hölder regularity)**.** *There exists $\alpha \in (0,1)$ only depending on dimension Q, λ, Λ such that all weak solutions u to* (4.1.1) *under assumption **(H1)**-**(H2)**-**(H3)** in $\Omega \supset \mathcal{Q}_1$ satisfy*

$$[u]_{C^\alpha(Q_{\frac{1}{2}})} \leq C \left(\|u\|_{L^2(\mathcal{Q}_1)} + \|f\|_{L^q(\mathcal{Q}_1)} \right),$$

where the constant C only depends on the homogeneous dimension Q defined in (1.1.17)*, on the ellipticity constants λ, Λ in* (4.1.2) *and on the norms $\|b\|_{L^\infty(\mathcal{Q}_{ext})}$ and $\|c\|_{L^q(\mathcal{Q}_{ext})}$.*

The estimates presented in Theorem 4.1.3 and Theorem 4.1.4 can be stated and scaled in any arbitrary cylinder $\mathcal{Q}_r(z_0)$, thanks to the to translation and scaling invariance of the class of equations of the form (4.1.1), (see Chapter 1, Subsection 1.1.1).

4.1.2 Comparison with existing results

We now compare the main results of this chapter with the current literature concerning the weak regularity theory of solutions to (4.1.1).

The first results concerning weak solutions to (4.1.1) were obtained in the space of "strong" weak solutions (i.e. $Yu \in L^2$) proposed by Pascucci and Polidoro in [99] and, for some time, only concerned Moser's iterative scheme, see [32, 99]. Later on, Wang and Zhang proved the local Hölder continuity for the particular case of the Kolmogorov-Fokker-Planck equation [117]. Subsequently, in 2017 the same authors extended their procedure to the ultraparabolic case in the preprint [118]. Such procedure is based on the combination of Sobolev and Poincaré inequalities for "strong" weak solutions to (4.1.1), combined with the properties of a suitably chosen G function. The authors finally recover the local Hölder continuity of the "strong" weak solution by providing an estimate of the oscillations following Kruzkhov's level set method.

As in [118], we prove that weak solutions to equation (4.1.1) are Hölder continuous, but we rely on a different method that also allows to prove a Harnack inequality. Indeed, we extend the techniques employed by Guerand and Imbert in [51] for the study of the particular case of the Kolmogorov-Fokker-Planck equation in two different directions: we allow a higher number κ of commutators (this corresponds to a higher step property for the underlying Lie group structure) and we deal with possibly unbounded lower order coefficients and right-hand side. Our approach is based on the combination of a weak Poincaré inequality (Theorem 4.4.1) for functions belonging to the space $\mathcal{W}$ (so only depending on the geometrical structure of the underlying Lie group and not on the structure of the operator $\mathscr{L}$, i.e. on the lower order coefficients), with a $L^2 - L^\infty$ estimate (Theorem 4.3.1) for weak sub-solutions and a weak Harnack inequality (Theorem 4.5.1) for weak super-solutions, see Remark 4.5.3. This approach based on a weak Harnack inequality was considered for the first time by Moser [87] and Trudinger [112] in the setting of parabolic equations. Later on, Di Benedetto and Trudinger [39] extended it to non-negative functions in the elliptic De Giorgi's class, which correspond to super-solutions to elliptic equations. Eventually, Wang proved in [115] a weak Harnack inequality for the corresponding parabolic De Giorgi's class. Concerning De Giorgi Hölder regularity theory, we also mention the recent work [50].

The main motivation behind our studies is to provide the reader with a Harnack inequality for weak solutions to (4.1.1). To our knowledge, it is the first time such result is explicitly stated and proved for equation (4.1.1) in the framework $\mathcal{W}$. Indeed, prior to [51], a Harnack inequality for weak solutions for Kolmogorov-Fokker-Planck equations with rough coefficients was already proved in [49]. Still, it has never been extended to the more general framework of our interest since the argument is based on a priori fractional estimates only available for the particular case of the Fokker-Planck equation, see [23].

Another motivation behind our studies is the need to determine which are the lowest possible integrability assumptions for c, b and f that allow us to prove $L^2 - L^\infty$ estimates and a Harnack inequality for weak solutions. In particular, our attention is mainly focused on the behavior of the first order term b, which plays an important role in some applications, such as the Mean Field Games theory. Indeed, a Harnack inequality for weak solutions is the fundamental ingredient in the analysis of the maximal L^p regularity and well-posedness theory for Mean Field systems with degenerate diffusion, which were studied in the parabolic setting [34] and only very recently there has been a first attempt to consider the ultraparabolic setting [44].

As far as the $L^2 - L^\infty$ estimates are concerned, we were able to work under the following assumption

(M) $c, f \in L^q_{loc}(\Omega)$ and $b \in (L^q_{loc}(\Omega))^{m_0}$ for some $q > \frac{3}{4}(Q+2)$. Moreover, we assume $\operatorname{div} b \geq 0$ in Ω,

which is clearly less restrictive than **(H3)**. Moreover, we observe that our assumption **(M)** is less restrictive on the term b then the one proposed in [118], which reads as follows:

$$c, f \in L^q(\Omega) \text{ for some } q > \tfrac{Q+2}{2} \text{ and } b \in (L^q(\Omega))^{m_0} \text{ for some } q > Q+2 \text{ in } \Omega.$$

A scaling argument and the analogous parabolic case suggest that the optimal regularity for the coefficients is $b \in (L^q(\Omega))^{m_0}$, $c \in L^q(\Omega)$ for some $q > \frac{Q+2}{2}$.

Moreover, the physical interpretation of the sign of the divergence of b in assumption **(M)** can be understood by considering the Vlasov-Poisson-Fokker-Planck equation, e.g. [63], for which the lower order term b represents the electrostatic or gravitational forces. Equations whose term b satisfies the structural assumption $\operatorname{div} b \geq 0$ arise also in some other applications, like the ones presented in [66, 109]. Moreover, the sign assumption on the divergence of b is also quite relevant in the case of parabolic equations, since it has several applications. For instance, we recall the applications to incompressible flows and magnetostrophic turbulence models for the Earth's fluid core, e.g. [85]. In particular, nowadays it is known that the sign (or the divergence free, i.e. $\operatorname{div} b = 0$) assumption can be used to relax the regularity assumptions on b under which it is possible to prove a Harnack inequality and other results, see for instance [109]. Nevertheless, in our case as in the parabolic setting presented in [109], one still has to require that the divergence of b exists in the sense of distributions and that b is at least locally integrable up to a certain power (see also [9, 91]).

However, to prove our Harnack inequality Theorem 4.1.3, we had to require more restrictive assumption on the regularity of b, namely **(H3)**. This is due to a delicate step in the proof of Lemma 4.5.8 below.

Eventually, we point out that the main difficulties when dealing with weak solutions to (4.1.1) arise from the non-standard structure of the space $\mathcal{W}$. Indeed, it has been since the paper [99] that the classical Sobolev embedding and the Poincaré inequality required for the derivation of the Harnack inequality were replaced by specific *inequalities for (sub or super) solutions* to (4.1.1). Because of this need, it is not possible to lower the integrability requirements on the term b up to $\frac{Q+2}{2}$ (the hypoelliptic counterpart of the parabolic homogeneous dimension $\frac{N}{2}$) nor in our framework nor in the one of [118]. See [9, 91] and the references therein for further information on this fact in the parabolic and hypoelliptic setting, respectively. Hence, the proof of such classical results for functions simply belonging to space $\mathcal{W}$ would provide us with the necessary tools to carry out an elegant study of the weak regularity theory, that would also be independent of the structure of the operator $\mathscr{L}$ appearing in (4.1.1) and more specifically on the lower order coefficients b and c.

A first step towards this direction is represented by our extension of two interesting tools, which may be considered among the main novelties of this chapter: a weak Poincaré inequality (Theorem 4.4.1) for functions $u \in \mathcal{W}$; the Ink Spots Theorem (Theorem 4.A.1 in Appendix 4.A) on $\mathbb{R}^N$ equipped with the non-Euclidean geometry introduced in Section 4.2. The Poincaré inequality stated in Theorem 4.4.1 is called *weak* because it allows us to estimate the L^2 norm of the function with respect to a certain error, which replaces the role of the mean in our framework. Nevertheless, it provides us with enough information to conclude our argument and, differently from the one proposed in [118], it holds for functions belonging to $\mathcal{W}$. Hence, it is not subjected anymore to the structure of $\mathscr{L}$. As far as the Ink Spots Theorem is concerned, it allows us to recover a spreading of positivity result from a measure-to-pointwise estimate when used in combination with a suitable covering argument (see Appendix 4.B). The proof we propose in Appendix 4.A of this result is an extension of the one proposed by Imbert and Silvestre in [59] for a Lie group structure of step 1. We also complete our analysis by proving a Lebesgue differentiation theorem and introducing a family of cylinders suitable to carry out a covering argument in our setting (see Appendices 4.A and 4.B).

Since the structure of the newly introduced space $\mathcal{W}$ differs from the space of "strong" weak solutions considered in the existing literature, also the already established results (such as the Moser's iterative scheme) need to be analyzed again under this new light. Moreover, whenever the method allows it, we carry out a quantitative analysis explicitly computing the constants involved in our analysis. For these reasons, in the forthcoming sections all of the computations are explicitly stated providing the reader with a self-sufficient analysis.

4.1.3 Outline of the chapter

In Section 4.2 we recall some known facts about operators $\mathscr{L}$ and we state some preliminary results. The proofs of some intermediate theorems (a Sobolev-type and a Caccioppoli-type inequality), together with the Moser's iterative method, are presented in Section 4.3. Section 4.4 is devoted to the proof of a weak Poincaré inequality. In Section 4.5 we derive the weak Harnack inequality by combining the expansion of positivity and a covering argument known as Ink Spots theorem, whose proof is contained in Appendix 4.A. Moreover, in Section 4.6, we derive our main results Theorem 4.1.3 and Theorem 4.1.4 . Finally, in Appendix 4.B we state a technical lemma regarding stacked cylinders.

4.2 Preliminaries

Since $\mathscr{L}_0$ is dilation-invariant with respect to $(\delta_r)_{r>0}$, also its fundamental solution Γ is a homogeneous function of degree $-Q$, namely Γ satisfies (1.1.34). This property implies a L^p estimate for Newtonian potentials (see for instance [45]).

Theorem 4.2.1. *Let $\alpha \in (0, Q+2)$ and let $G \in C(\mathbb{R}^{N+1} \setminus \{0\})$ be a $\delta_\lambda-$homogeneous function of degree $\alpha - Q - 2$. If $f \in L^p(\mathbb{R}^{N+1})$ for some $p \in (1, +\infty)$, then the function*

$$G_f(z) := \int_{\mathbb{R}^{N+1}} G(\zeta^{-1} \circ z) f(\zeta) d\zeta,$$

is defined almost everywhere and there exists a constant $c = c(Q, p)$ such that

$$\| G_f \|_{L^q(\mathbb{R}^{N+1}} \leq c \max_{\|z\|_{\mathbb{K}}=1} |G(z)| \| f \|_{L^p(\mathbb{R}^{N+1})},$$

where q is defined by

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q+2}.$$

Now, we are able to define the $\Gamma-$*potential* of the function $f \in L^1(\mathbb{R}^{N+1})$ as follows

$$\Gamma(f)(z) = \int_{\mathbb{R}^{N+1}} \Gamma(z,\zeta)f(\zeta)d\zeta, \qquad z \in \mathbb{R}^{N+1}. \tag{4.2.1}$$

We also remark that the potential $\Gamma(D_{m_0}f) : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}^{m_0}$ is well-defined for any $f \in L^p(\mathbb{R}^{N+1})$, at least in the distributional sense, that is

$$\Gamma(D_{m_0}f)(z) := -\int_{\mathbb{R}^{N+1}} D^{(\xi)}_{m_0}\Gamma(z,\xi)\, f(\xi)\, d\xi,$$

where $D^{(\xi)}_{m_0}\Gamma(x,t,\xi,\tau)$ is the gradient with respect to $\xi_1,\dots,\xi_{m_0}$. Based on Theorem 4.2.1, we derive the following explicit potential estimates by substituting $\alpha = 1$ and $\alpha = 2$ when considering the Γ-potential for f and $D_0 f$, respectively. For the proof of this corollary we refer to [32, 99].

Corollary 4.2.2. *Let $f \in L^p(\mathcal{Q}_r)$. There exists a positive constant $c = c(T,B)$ such that*

$$\|\Gamma(f)\|_{L^{p**}(\mathcal{Q}_r)} \le c\|f\|_{L^p(\mathcal{Q}_r)}, \tag{4.2.2}$$

$$\|\Gamma(D_{m_0}f)\|_{L^{p*}(\mathcal{Q}_r)} \le c\|f\|_{L^p(\mathcal{Q}_r)}, \tag{4.2.3}$$

where $\frac{1}{p*} = \frac{1}{p} - \frac{1}{Q+2}$ and $\frac{1}{p**} = \frac{1}{p} - \frac{2}{Q+2}$.

Lastly, we show that it is possible to use the fundamental solution Γ as a test function in the definition of sub and super-solution. The following result extends [99, Lemma 2.5], [32, Lemma 3] and [9, Lemma 2.6] to the functional setting $\mathcal{W}$.

Lemma 4.2.3. *Let **(H1)**-**(H2)** hold. Let $c \in L^q(\Omega)$ and $b \in (L^q(\Omega))^{m_0}$ for some $q > \frac{Q+2}{2}$ and let $f \in L^2(\Omega)$. Moreover, let us assume that $\operatorname{div} b \ge 0$ in Ω. Let v be a non-negative weak sub-solution to $\mathscr{L}v = f$ in Ω. For every $\varphi \in \mathcal{D}(\Omega)$, $\varphi \ge 0$, and for almost every $z \in \mathbb{R}^{N+1}$, we have*

$$\int_\Omega -\langle ADv, D(\Gamma(z,\cdot)\varphi)\rangle + \Gamma(z,\cdot)\varphi Yv + \langle b, Dv\rangle\Gamma(z,\cdot)\varphi + cv\Gamma(z,\cdot)\varphi - \Gamma(z,\cdot)\varphi f \ge 0.$$

An analogous result holds for weak super-solutions to $\mathscr{L}u = f$.

Proof. For every $\varepsilon > 0$, we set

$$\psi_\varepsilon(z,\zeta) = 1 - \chi_{\varepsilon,2\varepsilon}\left(\|\zeta^{-1}\circ z\|\right) \tag{4.2.4}$$

where $\chi_{\rho,r} \in C^\infty([0,+\infty))$ is the cut-off function defined by

$$\chi_{\rho,r}(s) = \begin{cases} 0, & \text{if } s \geq r, \\ 1, & \text{if } 0 \leq s \leq \rho, \end{cases} \qquad |\chi'_{\rho,r}| \leq \frac{2}{r-\rho}, \tag{4.2.5}$$

with $\frac{1}{2} \leq \rho < r \leq 1$. As v is a weak-sub-solution, for every $\varepsilon > 0$ and $z \in \mathbb{R}^{N+1}$, we have

$$0 \leq -I_{1,\varepsilon}(z) + I_{2,\varepsilon}(z) - I_{3,\varepsilon}(z) + I_{4,\varepsilon}(z) + I_{5,\varepsilon}(z) + I_{6,\varepsilon}(z)$$

where

$$\begin{aligned}
I_{1,\varepsilon}(z) &= \int_\Omega [\langle ADv, D\Gamma(z,\cdot)\rangle \varphi\psi_\varepsilon(z,\cdot)](\zeta)d\zeta \\
I_{2,\varepsilon}(z) &= \int_\Omega [\Gamma(z,\cdot)\psi_\varepsilon(z,\cdot)(-\langle ADv, D\varphi\rangle + \varphi Yv)](\zeta)d\zeta \\
I_{3,\varepsilon}(z) &= \int_\Omega [\langle ADv, D\psi_\varepsilon(z,\cdot)\rangle \varphi\Gamma(z,\cdot)](\zeta)d\zeta \\
I_{4,\varepsilon}(z) &= \int_\Omega \langle b, Dv\rangle \Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)](\zeta)d\zeta \\
I_{5,\varepsilon}(z) &= \int_\Omega [cv\Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)](\zeta)d\zeta \\
I_{6,\varepsilon}(z) &= -\int_\Omega [\Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)f](\zeta)d\zeta.
\end{aligned}$$

Keeping in mind Corollary 4.2.2, it is clear that the integrals that define $I_{i,\varepsilon}(z)$, $i = 1,2,3$ are potentials and therefore convergent for almost every $z \in \mathbb{R}^{N+1}$. Thus, by a similar argument to the one used in [99] in the proof of Lemma 2.5, we infer that for almost every $z \in \mathbb{R}^{N+1}$

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} I_{1,\varepsilon}(z) &= \int_\Omega [\langle ADv, D\Gamma(z,\cdot)\rangle\varphi](\zeta)d\zeta \qquad\qquad \lim_{\varepsilon \to 0^+} I_{3,\varepsilon}(z) = 0 \\
\lim_{\varepsilon \to 0^+} I_{2,\varepsilon}(z) &= \int_\Omega [\Gamma(z,\cdot)(-\langle ADv, D\varphi\rangle + \varphi Yv)](\zeta)d\zeta,
\end{aligned}$$

where the passage to the limit for the term $I_{2,\varepsilon}$ is possible thanks to Remark 4.1.1 combined with the Lebesgue dominated convergence theorem.

We now take care of the term $I_{4,\varepsilon}$. Integrating by parts and taking advantage of the assumption $\text{div} b \geq 0$, we obtain

$$\begin{aligned}
I_{4,\varepsilon}(z) &= -\int_\Omega [\text{div} b\, \Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)v](\zeta)d\zeta - \int_\Omega [\langle b, D\left(\Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)\right)\rangle\, v](\zeta)d\zeta \\
&\leq -\int_\Omega [\langle b, D\left(\Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)\right)\rangle\, v](\zeta)d\zeta.
\end{aligned}$$

We are left with the estimate of a potential and therefore we exploit once again Corollary

4.2.2. Since we have $b \in (L^q(\Omega))^{m_0}$ and $v \in L^2(\Omega)$, we get

$$|\Gamma(z,\cdot)||\varphi||b||Dv| \in L^{2\alpha}(\Omega),$$

where
$$\alpha = \alpha(q) = \frac{q(Q+2)}{q(Q-2)+2(Q+2)} > 1 \qquad \text{if and only if} \qquad q > \frac{Q+2}{2}.$$

Hence, $|\langle b, D\left(\Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)\right)\rangle v| \leq |\langle b, D\left(\Gamma(z,\cdot)\varphi\right)\rangle v| \in L^1(\Omega)$. Thus, the Lebesgue convergence theorem yields

$$\begin{aligned} \lim_{\varepsilon\to 0^+} I_{4,\varepsilon}(z) &= \lim_{\varepsilon\to 0^+} -\int_\Omega [\langle b, D\left(\Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)\right)\rangle\, v](\zeta)d\zeta \\ &= \int_\Omega [\langle b, D\Gamma(z,\cdot)\varphi\rangle v](\zeta)d\zeta. \end{aligned}$$

Similarly, we can estimate the term $I_{5,\varepsilon}$ noting that $|c||v||\Gamma(z,\cdot)||\varphi| \in L^{2\alpha}(\Omega)$ with α as above. As a consequence, we have

$$|cv\Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)| \leq |cv\Gamma(z,\cdot)\varphi| \in L^1(\Omega), \quad \text{thus} \ \lim_{\varepsilon\to 0^+} I_{5,\varepsilon}(z) = \int_\Omega [cv\Gamma(z,\cdot)\varphi](\zeta)d\zeta.$$

Now, we are left with the estimate of term $I_{6,\varepsilon}$, which is again a Γ-potential such that

$$|\Gamma(z,\cdot)||\varphi||f| \in L^{2\kappa}(\Omega),$$

where $\kappa = \frac{Q+2}{Q-2}$. Thus, we infer $|\Gamma(z,\cdot)\varphi\psi_\varepsilon(z,\cdot)f| \leq |\Gamma(z,\cdot)\varphi f| \in L^1(\Omega)$. Therefore we conclude the proof by applying the dominated convergence theorem to $I_{6,\varepsilon}(z)$. □

We conclude this section by recalling the following lemma, for the proof of which we refer to [32, Lemma 6].

Lemma 4.2.4. *There exists a positive constant* $\overline{c} \in (0,1)$ *such that*

$$z \circ \mathcal{Q}_{\overline{c}(r-\rho)} \subseteq \mathcal{Q}_r, \qquad \textit{for every } 0 < \rho < r \leq 1 \quad \textit{and} \quad z \in \mathcal{Q}_\rho. \tag{4.2.6}$$

Remark 4.2.5. We recall that for every cylinder $\mathcal{Q}_r(z_0)$ defined in (4.1.12) there exists a positive constant $\overline{c}$ [117, equation (21)] such that

$$\begin{aligned} &B_{r_1}(x_0^{(0)}) \times B_{r_1^3}(x_0^{(1)}) \times \ldots \times B_{r_1^{2\kappa+1}}(x_0^{(\kappa)}) \times (t_0 - r_1^2, t_0] \\ &\qquad \subset \mathcal{Q}_r(z_0) \subset B_{r_2}(x_0^{(0)}) \times B_{r_2^3}(x_0^{(1)}) \times \ldots \times B_{r_2^{2\kappa+1}}(x_0^{(\kappa)}) \times (t_0 - r_2^2, t_0], \end{aligned}$$

where $r_1 = r/\overline{c}$ and $r_2 = \overline{c}r$. From now on, by abuse of notation we will sometimes consider the newly introduced ball representation instead of the definition of cylinder in (4.1.12).

4.3 Local boundedness for weak solutions to $\mathscr{L}u = f$

This section is devoted to the proof of the local boundedness of weak solutions to (4.1.1). As pointed out in Section 4.1, we provide the reader with the full computations because the functional framework considered here is weaker than the one proposed in the already existing literature, see [9, 32, 99, 118].

The procedure we follow here was first introduced by Moser in [88] and it is based on the iterative combination of a Caccioppoli and a Sobolev inequality. When dealing with the classical uniformly elliptic and parabolic settings, the Caccioppoli inequality provides us with an a priori estimate for the L^2 norm of the complete gradient of the solution in terms of the L^2 norm of the solution. This allows us to consider the classical Sobolev embedding to obtain a gain of integrability for the solution.

This is not the case when dealing with operator (4.1.1). Indeed, the degeneracy of the diffusion part allows us to estimate only the partial gradient $D_{m_0}u$ of the solution to $\mathscr{L}u = f$ (see Theorem 4.3.4). In addition, according to our definition of weak solution, u does not lie in a classical Sobolev space. In order to overcome these issues, we adopt a technique based on the representation of a solution u to $\mathscr{L}u = f$ in terms of the fundamental solution Γ (see (1.1.31)) of the principal part operator $\mathscr{L}_0$. Indeed, following the idea presented for the first time in [99] and later on applied in [9, 32, 118], we have that if u is a solution to $\mathscr{L}u = f$, then

$$\mathscr{L}_0 u = (\mathscr{L}_0 - \mathscr{L})\, u + f = \mathrm{div}_{m_0}\left((\mathbb{I}_{m_0} - A)\, D_{m_0} u\right) + f. \tag{4.3.1}$$

Hence, as pointed out at [99, p. 396], it seems quite natural to consider a representation formula in terms of the fundamental solution of the principal part operator $\mathscr{L}_0$ *"[...] since the classical Sobolev inequality can be proved by representing any function $u \in H^1$ as a convolution with the fundamental solution of the Laplace operator."*

Theorem 4.3.1. *Let $z_0 \in \Omega$ and $0 < \frac{r}{2} \le \rho < r \le 1$, be such that $\overline{\mathcal{Q}_r(z_0)} \subseteq \Omega$. Let u be a non-negative weak solution to $\mathscr{L}u = f$ in Ω under assumptions **(H1)**-**(H2)**-**(M)**. Then for every $p \ge 1$ there exists two positive constants $C = C\left(p, \lambda, \Lambda, Q, \| b \|_{L^q(\mathcal{Q}_r(z_0))}, \| c \|_{L^q(\mathcal{Q}_r(z_0))}\right)$, such that*

$$\sup_{\mathcal{Q}_\rho(z_0)} u_l^p \le \frac{C}{(r-\rho)^{\frac{Q+2}{\beta}}} \|u_l^p\|_{L^\beta(\mathcal{Q}_r(z_0))},$$

*where $\beta = \frac{q}{q-1}$, q introduced in **(M)** and $u_l := u + \|f\|_{L^q(\mathcal{Q}_R)}$. The same statement holds true if u is a non-negative weak sub-solution to (4.1.1) for $p \ge 1$; if u is a non-negative weak super-solution to (4.1.1) for $0 < p < \frac{1}{2}$. In particular, by choosing $p = 1$, for every sub-solution to (4.1.1) it holds*

$$\sup_{\mathcal{Q}_\rho(z_0)} u \le \frac{C}{(r-\rho)^{\frac{Q+2}{\beta}}} \left(\|u\|_{L^\beta(\mathcal{Q}_r(z_0))} + \|f\|_{L^q(\mathcal{Q}_r)}\right),$$

In literature, we find various proofs of the Moser's iterative scheme for a Kolmogorov operator of the type $\mathscr{L}$, see for instance [32, 99, 118]. Nevertheless, the functional framework proposed in those works is stronger than ours.

Remark 4.3.2. Theorem 4.3.1 holds true under the assumptions of [118], see Subsection 4.1.2. In particular, in this case, the constant α is replaced by $\alpha = 1 + \frac{2}{Q}$, that was firstly obtained in [99]. This is due the fact that the Sobolev inequality, Theorem 4.3.3, holds true with a greater exponent if we assume more integrability for the coefficient b. Thus, this allows us to obtain the local boundedness for weak solutions to $\mathscr{L}u = f$ with lower integrability for c and f, i.e. $c, f \in L^q(\Omega)$, with $q > \frac{Q+2}{2}$.

Finally, Theorem 4.3.1 holds true also for weak sub and super solutions, but not for the same values of the exponent p. This is due to the technique adopted for the proof of the Caccioppoli-type inequality (Theorem 4.3.4), see also [99, Remark 1.3], and it is a classical feature of all the local boundedness results appearing in the existing literature, see for instance [99, 118].

4.3.1 Sobolev-type Inequality

This subsection is devoted to the proof of a Sobolev-type inequality for weak solutions to $\mathscr{L}u = f$. Our approach is inspired by the paper [99] and allows us to construct an "ad hoc" Sobolev embedding for weak solutions to $\mathscr{L}u = f$ by overcoming the difficulties due to the degeneracy of the second order part of $\mathscr{L}$. However, the disadvantage of this method is that we are forced to lower the Sobolev exponent, that in our case depends on q and it is defined as

$$\alpha := \frac{q(Q+2)}{q(Q-2) + 2(Q+2)}. \tag{4.3.2}$$

We remark that the following statement holds true under lower integrability assumption than the one required in **(H3)**.

Theorem 4.3.3. *Let **(H1)**-**(H2)** hold. Let $c \in L^q(\Omega)$, $b \in (L^q(\Omega))^{m_0}$ for some $q > \frac{Q+2}{2}$ with $\operatorname{div} b \geq 0$ and let $f \in L^2(\Omega)$. Let v be a non-negative weak sub-solution of $\mathscr{L}v = f$ in $\mathcal{Q}_1$. Then there exists a constant $C = C(Q, \lambda, \Lambda) > 0$ such that $v \in L^{2\alpha}(\mathcal{Q}_1)$, and the following inequality holds*

$$\begin{aligned} \parallel v \parallel_{L^{2\alpha}(\mathcal{Q}_\rho(z_0))} \leq C \left(\parallel b \parallel_{L^q(\mathcal{Q}_r(z_0))} + \frac{r-\rho+1}{r-\rho} \right) \parallel D_{m_0} v \parallel_{L^2(\mathcal{Q}_r(z_0))} + \\ + C \left(\parallel c \parallel_{L^q(\mathcal{Q}_r(z_0))} + \frac{\rho+1}{\rho(r-\rho)} \right) \parallel v \parallel_{L^2(\mathcal{Q}_r(z_0))} + C \parallel f \parallel_{L^2(\mathcal{Q}_r(z_0))} \end{aligned}$$

for every ρ, r with $\frac{1}{2} \leq \rho < r \leq 1$ and for every $z_0 \in \Omega$, where $\alpha = \alpha(q)$ is defined as (4.3.2). *The same statement holds for non-negative super-solutions.*

Proof. Let v be a non-negative weak sub-solution to $\mathscr{L}v = f$. We represent v in terms of the fundamental solution Γ. To this end, we consider the cut-off function $\chi_{\rho,r}$ defined in (4.2.5) for $\frac{1}{2} \le \rho < r \le 1$. Then, if we consider the test function

$$\psi(x,t) = \chi_{\rho,r}(\| (x,t) \|), \tag{4.3.3}$$

the following estimates hold true

$$|Y\psi| \le \frac{c_0}{\rho(r-\rho)}, \qquad |\partial_{x_j}\psi| \le \frac{c_1}{r-\rho} \text{ for } j = 1, \dots, m_0 \tag{4.3.4}$$

where c_0, c_1 are dimensional constants. For every $z \in \mathcal{Q}_\rho$, we have

$$\begin{aligned} v(z) = v\psi(z) &= \int_{\mathcal{Q}_r} \left[\langle \mathbb{I}_{m_0} D(v\psi), D\Gamma(z,\cdot) \rangle - \Gamma(z,\cdot) Y(v\psi) \right](\zeta) d(\zeta) \\ &= I_0(z) + I_1(z) + I_2(z) + I_3(z) \end{aligned} \tag{4.3.5}$$

where

$$\begin{aligned} I_0(z) &= \int_{\mathcal{Q}_r} \left[\langle b, Dv \rangle \Gamma(z,\cdot)\psi \right](\zeta) d\zeta \; + \int_{\mathcal{Q}_r} \left[cv\Gamma(z,\cdot)\psi \right](\zeta) d\zeta - \int_{\mathcal{Q}_r} [\Gamma(z,\cdot)\psi f](\zeta) d\zeta \\ I_1(z) &= \int_{\mathcal{Q}_r} \left[\langle \mathbb{I}_{m_0} D\psi, D\Gamma(z,\cdot) \rangle v \right](\zeta) d\zeta \; - \int_{\mathcal{Q}_r} \left[\Gamma(z,\cdot) v Y\psi \right](\zeta) d\zeta = I_1^{'} + I_1^{''}, \\ I_2(z) &= \int_{\mathcal{Q}_r} \left[\langle (\mathbb{I}_{m_0} - A) Dv, D\Gamma(z,\cdot) \rangle \psi \right](\zeta) d\zeta \; - \int_{\mathcal{Q}_r} \left[\Gamma(z,\cdot) \langle ADv, D\psi \rangle \right](\zeta) d\zeta \\ I_3(z) &= \int_{\mathcal{Q}_r} \left[\langle ADv, D(\Gamma(z,\cdot)\psi) \rangle \right](\zeta) d\zeta \; - \int_{\mathcal{Q}_r} \left[(\Gamma(z,\cdot)\psi)\, Yv \right](\zeta) d\zeta \\ &\quad - \int_{\mathcal{Q}_r} \left[\langle b, Dv \rangle \Gamma(z,\cdot)\psi \right](\zeta) d\zeta - \int_{\mathcal{Q}_r} \left[cv\Gamma(z,\cdot)\psi \right](\zeta) d\zeta + \int_{\mathcal{Q}_r} [\Gamma(z,\cdot)\psi f](\zeta) d\zeta. \end{aligned}$$

Since v is a non-negative weak sub-solution to $\mathscr{L}v = f$, it follows from Lemma 4.2.3 that $I_3 \le 0$, then

$$0 \le v(z) \le I_0(z) + I_1(z) + I_2(z) \quad \text{for a.e. } z \in \mathcal{Q}_\rho.$$

To prove our claim is sufficient to estimate v by a sum of $\Gamma-$potentials. We start by estimating I_0. In order to do so, we recall that

$$\langle b, Dv \rangle,\, cv \, \in L^{2\frac{q}{q+2}} \quad \text{for } b, c \in L^q, \;\; q > \frac{Q+2}{2} \text{ and } v \in L^2.$$

Thus by Corollary 4.2.2 we get

$$\Gamma * \langle b, Dv \rangle, \Gamma * (cv) \in L^{2\alpha},$$

where $\alpha = \alpha(q)$ is defined in (4.3.2). In addition, for $f \in L^2$, we have

$$\Gamma * f \in L^{2\kappa}, \quad \kappa = \frac{Q+2}{Q-2}.$$

Observing that $\kappa > \alpha$, we obtain that $\Gamma * f \in L^{2\alpha}$ and therefore

$$\begin{aligned}\| I_0(\zeta) \|_{L^{2\alpha}(\mathcal{Q}_r)} &\leq \Gamma * (\langle b, D_{m_0} v\rangle \psi) + \Gamma * (cv\psi) \|_{L^{2\alpha}(\mathcal{Q}_r)} + \| \Gamma * f \|_{L^{2\alpha}(\mathcal{Q}_r)} \\ &\leq C \cdot \left(\| b \|_{L^q(\mathcal{Q}_r)} \| D_{m_0} v \|_{L^2(\mathcal{Q}_r)} + \| c \|_{L^q(\mathcal{Q}_r)} \| v \|_{L^2(\mathcal{Q}_r)} + \| f \|_{L^2(\mathcal{Q}_r)} \right).\end{aligned}$$

We now deal with the I_1. I_1' can be estimated by (4.2.3) of Corollary 4.2.2 as follows

$$\| I_1' \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \leq C \| I_1' \|_{L^{2^*}(\mathcal{Q}_\rho)} \leq C \| v D_{m_0} \psi \|_{L^2(\mathbb{R}^{N+1})} \leq \frac{C}{r-\rho} \| v \|_{L^2(\mathcal{Q}_r)},$$

where the last inequality follows from (4.3.4). To estimate I_1'' we use (4.2.2)

$$\begin{aligned}\| I_1'' \|_{L^{2\alpha}(\mathcal{Q}_\rho)} &\leq C \| I_1'' \|_{L^{2^*}(\mathcal{Q}_\rho)} \leq \text{meas}(\mathcal{Q}_\rho)^{2/Q} \| I_1'' \|_{L^{2^{**}}(\mathcal{Q}_r)} \\ &\leq C \| vY\psi \|_{L^2(\mathbb{R}^{N+1})} \leq \frac{C}{\rho(r-\rho)} \| v \|_{L^2(\mathcal{Q}_r)}.\end{aligned}$$

We can use the same technique to prove that

$$\| I_2 \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \leq C \left(1 + \frac{1}{r-\rho} \right) \| Dv \|_{L^2(\mathcal{Q}_\rho)},$$

for some constant $C = C(Q, \lambda, \Lambda)$. A similar argument proves the thesis when v is a super-solution to $\mathscr{L} v = f$. In this case we introduce the following auxiliary operator

$$\mathscr{L}_0 = \Delta_{m_0} + \widetilde{Y}, \qquad \widetilde{Y} \equiv -\langle x, BD\rangle - \partial_t. \tag{4.3.6}$$

Then we proceed analogously as in [99], Section 3, proof of Theorem 3.3. □

4.3.2 Caccioppoli-type inequality

As a second step, we prove a Caccioppoli-type inequality for powers of non-negative sub-solutions to $\mathscr{L} u = f$. In order to introduce the auxiliary function u_l that will play a relevant role in the proof of the Moser's iterative scheme, we hereby report the complete proof of this result, which can also be found with zero right-hand side in [9, 32, 99].

Theorem 4.3.4. *Let* ***(H1)****-****(H2)****-****(M)*** *hold and u be a non-negative weak sub-solution to $\mathscr{L} u = f$ in $\mathcal{Q}_r$, with $0 < \rho < r \leq 1$. For any $p \in (\frac{1}{2}, +\infty)$ such that $u^p \in L^2(\mathcal{Q}_r)$ the following estimate holds*

$$\begin{aligned}\frac{2p-1}{p} \lambda \| D_{m_0} u_l^p \|^2_{L^2(\mathcal{Q}_\rho)} &\leq \left(\frac{c_1}{(r-\rho)^2} \frac{p}{2p-1} \frac{\Lambda}{\lambda} + \frac{c_0}{\rho(r-\rho)} \right) \| u_l^p \|^2_{L^2(\mathcal{Q}_r)} \\ &\quad + \left(\frac{c_0}{r-\rho} \| b \|^2_{L^q(\mathcal{Q}_r)} + p \| c \|^2_{L^q(\mathcal{Q}_r)} + p \right) \| u_l^p \|^2_{L^{2\beta}(\mathcal{Q}_r)}\end{aligned}$$

where $u_l = u + \| f \|_{L^q(\mathcal{Q}_r)}$, $\beta = \beta(q) = \frac{q}{q-1}$, and c_0, c_1 are defined in (4.3.4).

Proof. Let us consider $0 < \rho < r \le 1$ and the test function ψ introduced in (4.3.3). The idea is to test the definition of weak sub-solution (4.1.9) against the test function

$$\varphi = p\psi^2 u_l^{2p-1}, \quad \text{with } p \ge \frac{1}{2}, \quad \text{and } u_l := u + l,$$

where $l > 0$ is chosen such that $l = \|f\|_{L^q(\mathcal{Q}_r)}$. For the sake of clarity, from now on we assume u is a bounded weak sub-solution to (4.1.1) such that $u \in C_c^\infty(\mathcal{Q}_r)$. This assumption is not restrictive thanks to the definition of the space $\mathcal{W}$ and if u where not bounded, then we would consider the test function

$$\varphi = p\psi^2 u_{l,M}^{2p-1}, \quad \text{where } u_{l,M} = \min\{u_l, M\} \quad \text{and let } M \text{ go to infinity.}$$

Hence, we test (4.1.9) against φ as defined above:

$$\underbrace{\boxed{\int_{\mathcal{Q}_r} \langle ADu, D\varphi\rangle}}_{A} \le -\underbrace{\boxed{\int_{\mathcal{Q}_r} uY\varphi}}_{Y} + \underbrace{\boxed{\int_{\mathcal{Q}_r} \langle b, Du\rangle\varphi}}_{B} + \underbrace{\boxed{\int_{\mathcal{Q}_r} cu\varphi}}_{C} - \underbrace{\boxed{\int_{\mathcal{Q}_r} f\varphi}}_{F}. \tag{4.3.7}$$

We begin estimating the boxed term A:

$$\begin{aligned}
\int_{\mathcal{Q}_r} \langle ADu, D\varphi\rangle &= p(2p-1)\int_{\mathcal{Q}_r} \langle ADu, Du_l\rangle u_l^{2p-2}\psi^2 + 2p\int_{\mathcal{Q}_r} \langle ADu, D\psi\rangle u_l^{2p-1}\psi \\
&= \frac{2p-1}{p}\int_{\mathcal{Q}_r} \langle ADu_l^p, Du_l^p\rangle\psi^2 + 2\int_{\mathcal{Q}_r} \langle ADu_l^p, D\psi\rangle u_l^p\psi \\
&\ge \frac{2p-1}{p}\lambda\int_{\mathcal{Q}_r} |D_{m_0}u_l^p|^2\psi^2 - 2\int_{\mathcal{Q}_r} |\langle ADu_l^p, D\psi\rangle| u_l^p\psi \\
&\ge \frac{2p-1}{p}\lambda\int_{\mathcal{Q}_r} |D_{m_0}u_l^p|^2\psi^2 - \Lambda\varepsilon_1\int_{\mathcal{Q}_r} |D_{m_0}u_l^p|^2\psi - \frac{1}{\varepsilon_1}\int_{\mathcal{Q}_r} |D_{m_0}\psi|^2|u_l^p|^2\psi
\end{aligned}$$

where in the first line we considered $Du_l = Du$; in the third line we applied **(H1)**; in the last line we employed Young inequality with $\varepsilon_1 > 0$ that will be chosen later on.

Next, we proceed by estimating the boxed term Y:

$$\begin{aligned}
-\underbrace{\boxed{\int_{\mathcal{Q}_r} uY\varphi}}_{Y} &= -p\int_{\mathcal{Q}_r} uY(u_l^{2p-1}\psi^2) = \frac{1}{2}\int_{\mathcal{Q}_r} Yu_l^{2p}\,\psi^2 \\
&= -\int_{\mathcal{Q}_r} u_l^{2p}\,Y\psi\,\psi \le \frac{c_0}{\rho(r-\rho)}\|u_l^p\|^2_{L^2(\mathcal{Q}_r)}
\end{aligned}$$

where we integrated by parts and we applied estimate (4.3.4) for the derivatives of the function ψ. As far as we are concerned with the term B:

$$\begin{aligned}
\underbrace{\boxed{\int_{\mathcal{Q}_r} \langle b, Du\rangle\varphi}}_{B} &= p\int_{\mathcal{Q}_r} \langle b, Du_l\rangle\psi^2 u_l^{2p-1} = \frac{1}{2}\int_{\mathcal{Q}_r} \langle b, Du_l^{2p}\rangle\psi^2 \\
&= -\frac{1}{2}\int_{\mathcal{Q}_r} \mathrm{div} b\, u_l^{2p}\psi^2 - \int_{\mathcal{Q}_r} \langle b, D\psi\rangle u_l^{2p}\psi
\end{aligned}$$

$$\leq - \int_{\mathcal{Q}_r} \langle b, D\psi \rangle u_l^{2p} \psi \leq \frac{c_0}{(r-\rho)} \|b\|_{L^q(\mathcal{Q}_r)} \|u_l^p\|^2_{L^{2\beta}(\mathcal{Q}_r)}$$

where we applied assumption **(M)**, Hölder's inequality and we defined $\beta := \frac{q}{q-1}$. Lastly, we are left with he estimates of the terms C and F:

$$\boxed{\int_{\mathcal{Q}_r} cu\varphi}_C \leq p \int_{\mathcal{Q}_r} |c||u|\psi^2 u_l^{2p-1} \leq p \int_{\mathcal{Q}_r} |c| u_l^{2p} \psi^2 \leq p \|c\|_{L^q(\mathcal{Q}_r)} \|u_l^p\|^2_{L^{2\beta}(\mathcal{Q}_r)}$$

$$-\boxed{\int_{\mathcal{Q}_r} f\varphi}_F \leq p \int_{\mathcal{Q}_r} |f| u_l^{2p-1} \psi^2 = p \int_{\mathcal{Q}_r} \mathcal{C}_f u_l^{2p} \psi^2 \leq p \|\mathcal{C}_f\|_{L^q(\mathcal{Q}_r)} \|u_l^p\|^2_{L^{2\beta}(\mathcal{Q}_r)} \leq p \|u_l^p\|^2_{L^{2\beta}(\mathcal{Q}_r)},$$

where we set $\mathcal{C}_f = \frac{|f|}{\|f\|_{L^q(\mathcal{Q}_r)}}$ so that we have $\frac{|f|}{u_l} \leq \mathcal{C}_f$ and $\|\mathcal{C}_f\|_{L^q(\mathcal{Q}_r)} \leq 1$. By choosing $\varepsilon_1 = \frac{2p-1}{2p^2} \frac{\lambda}{\Lambda}$, we conclude the proof. $\square$

4.3.3 Proof of Theorem 4.3.1

First of all, we consider a weak solution u to (4.1.1) and without loss of generality we assume $z_0 = 0$. As in Theorem 4.3.4, we set $u_l := u + l = u + \|f\|_{L^q(\mathcal{Q}_r)}$ and we remark from now on, l needs to be regarded as a constant. Our aim is to show there exists a positive constant c only depending on q, Q, λ and Λ such that

$$\sup_{\mathcal{Q}_\rho} u_l^p \leq \frac{C}{(r-\rho)^{\frac{Q+2}{\beta}}} \|u_l^p\|_{L^\beta(\mathcal{Q}_r)}. \tag{4.3.8}$$

To this end, we show that it is sufficient to prove that

$$\sup_{\mathcal{Q}_{\frac{1}{2}}} u_l^p \leq c_1 \|u_l^p\|_{L^\beta(\mathcal{Q}_1)}, \tag{4.3.9}$$

where c_1 is a constant not depending on r. We briefly explain how to infer (4.3.9) from (4.3.8) and we refer to [32] for further details. Indeed, by Lemma 4.2.4, if we set $\theta = \overline{C}(r-\rho)$ we obtain that (4.3.8) is equivalent to

$$\sup_{\mathcal{Q}_{\frac{\theta}{2}}} v^p \leq \frac{\overline{C}^{\frac{Q+2}{\beta}}}{\theta^{\frac{Q+2}{\beta}}} \|v^p\|_{L^\beta(\mathcal{Q}_\theta)}. \tag{4.3.10}$$

Then, we recall that if u_θ is a solution to $\mathscr{L} u_l = (f + cl)$ and $\theta > 0$, then we have that $v := u \circ \delta_\theta + l$ solves $\mathscr{L}^\theta v = f^\theta + c^\theta l$ in $\mathcal{Q}_1$, see [32], where

$$\mathscr{L}^\theta v := \mathscr{L}^\theta u(\delta_\theta(z)) = \left(\mathrm{div}(A^\theta Du) + \langle B^\theta x, Du \rangle - \partial_t u + \theta \langle b^\theta, Du \rangle + \theta^2 cu \right)(z),$$

with $A^\theta(z) = A(\delta_\theta(z))$, $B^\theta = \theta^2 D_\theta B D_{\frac{1}{\theta}}$, $b^\theta(z) = \theta b(\delta_\theta(z))$, $c^\theta(z) = \theta^2 c(\delta_\theta(z))$, $f^\theta(z) = \theta^2 f(\delta_\theta(z))$. Hence, by performing the change of variable $w(\zeta) = v(z \circ \delta_\theta(\zeta))$, with $\zeta \in \mathcal{Q}_1$, we imply (4.3.9).

We are now in a position to address the proof of (4.3.8). First of all, if we set $\frac{1}{2} = \rho$ and $r = 1$, the following estimates hold

$$\begin{gathered} \frac{1}{(\rho(r-\rho))^{\frac{1}{2}}(r-\rho)} \leq \frac{1}{(r-\rho)^2}, \quad \frac{1}{\rho(r-\rho)} \leq \frac{1}{(r-\rho)^2}, \\ \frac{1}{(\rho(r-\rho))^2} \leq \frac{1}{(r-\rho)^2}, \quad r^2 \leq \frac{1}{(r-\rho)^2}. \end{gathered} \tag{4.3.11}$$

Combining Theorems 4.3.3 and 4.3.4 for a non-negative sub-solution u, we obtain the following estimate. If $s > 1$, $\delta > 0$ verify the condition

$$\left|s - \frac{1}{2}\right| \geq \delta,$$

then we have

$$\| u_l^s \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \leq \widetilde{C}\left(s, \lambda, \Lambda, Q, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)}\right) \| u_l^s \|_{L^{2\beta}(\mathcal{Q}_r)}, \tag{4.3.12}$$

where $\widetilde{C}$ is a positive constant that we estimate as follows

$$\widetilde{C}(s, \lambda, \Lambda, Q, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)}) \leq \frac{K(\lambda, \Lambda, Q, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)})\sqrt{s}}{(r-\rho)^2}, \tag{4.3.13}$$

thanks to (4.3.11). Then we set $v = u_l^{\frac{p}{2}}$. Fixed a suitable $\delta > 0$, that we will specify later on, and a suitable $p \geq 1$ such that

$$\left|\frac{p}{2}\left(\frac{\alpha}{\beta}\right)^n - \frac{1}{2}\right| \geq 2\delta, \qquad \forall n \in \mathbb{N} \cup \{0\}, \tag{4.3.14}$$

we iterate inequality (4.3.12) by choosing

$$\rho_n = \frac{1}{2}\left(1 + \frac{1}{2^n}\right), \qquad p_n = \left(\frac{\alpha}{\beta}\right)^n \frac{p}{2}, \qquad n \in \mathbb{N} \cup \{0\}.$$

Thus, by combining (4.3.12) and (4.3.13), for every $n \in \mathbb{N} \cup \{0\}$ the following holds

$$\| v^{\left(\frac{\alpha}{\beta}\right)^n} \|_{L^{2\alpha}(\mathcal{Q}_{\rho_{n+1}})} \leq \frac{K(\lambda, \Lambda, Q, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)})\sqrt{p}}{(\rho_n - \rho_{n+1})^2} \| v^{\left(\frac{\alpha}{\beta}\right)^n} \|_{L^{2\beta}(\mathcal{Q}_{\rho_n})}.$$

From now on we denote $K = K(p, \lambda, \Lambda, Q, \| b \|_{L^q(\mathcal{Q}_r)}, \| c \|_{L^q(\mathcal{Q}_r)})$. Since

$$\| v^{\left(\frac{\alpha}{\beta}\right)^n} \|_{L^{2\kappa}} = \left(\| v \|_{L^{2\alpha\left(\frac{\alpha}{\beta}\right)^n}}\right)^{\left(\frac{\alpha}{\beta}\right)^n} \qquad \text{and} \qquad \| v^{\left(\frac{\alpha}{\beta}\right)^n} \|_{L^{2\beta}} = \left(\| v \|_{L^{2\beta\left(\frac{\alpha}{\beta}\right)^n}}\right)^{\left(\frac{\alpha}{\beta}\right)^n}$$

we are able to rewrite the previous estimate in the following form for every $n \in \mathbb{N} \cup \{0\}$

$$\| v \|_{L^{2\alpha\left(\frac{\alpha}{\beta}\right)^n}(\mathcal{Q}_{\rho_{n+1}})} \leq \left(\frac{K\sqrt{p}}{(\rho_n - \rho_{n+1})^2} \right)^{\left(\frac{\beta}{\alpha}\right)^n} \| v \|_{L^{2\beta\left(\frac{\alpha}{\beta}\right)^n}(\mathcal{Q}_{\rho_n})} .$$

Iterating this inequality and letting n go to infinity, we get

$$\sup_{\mathcal{Q}_\rho} v \leq \widetilde{K} \| v \|_{L^{2\beta}(\mathcal{Q}_r)}, \qquad \text{where} \quad \widetilde{K} = \prod_{j=0}^{+\infty} \left[K\sqrt{p} 2^{2(j+2)} \right]^{\left(\frac{\beta}{\alpha}\right)^j}$$

is a finite constant, since the product over j corresponds to a converging series, only depending on universal quantities Q, and q. This proves inequality (4.3.9) for p satisfying condition (4.3.14).

We now make a suitable choice of $\delta > 0$, only dependent on the homogeneous dimension Q, and on q in order to show that (4.3.14) holds for every $p \geq 1$. We notice that, if p is a number of the form

$$p_m = \frac{1}{2} \left(\frac{\alpha}{\beta} \right)^m \left(\frac{\alpha}{\beta} + 1 \right), \qquad m \in \mathbb{Z},$$

then (4.3.14) is satisfied with the following choice of δ for every $m \in \mathbb{Z}$

$$\delta = \frac{\alpha - \beta}{8\beta}.$$

Therefore (4.3.9) holds for such a choice of p, with $\widetilde{K}$ only dependent on Q, q, λ, Λ and $\| b \|_{L^q(\mathcal{Q}_r)}$, $\| c \|_{L^q(\mathcal{Q}_r)}$. On the other hand, if p is an arbitrary positive number, we consider $m \in \mathbb{Z}$ such that

$$p_m \leq p < p_{m+1}$$

and conclude the proof thanks to the monotonicity of the L^p means, see for instance [99]. Hence, the proof is complete. □

4.4 Weak Poincaré inequality

This section is devoted to the proof of a weak Poincaré inequality (see Theorem 4.4.1) for functions $u \in \mathcal{W}$. As one immediately understands, this Poincaré inequality is independent of the equation $\mathscr{L} u = f$ and only relies on the structure of the space $\mathcal{W}$. Its importance lies in the fact that it is a crucial tool in the proof of the Harnack inequality (see Theorem 4.1.3) and of the local Hölder continuity (see Theorem 4.1.4) of a solution u to (4.1.1). In order to state our result, we first need to introduce the following sets

$$\mathcal{Q}_{zero} = \{(x,t) : |x_j| \leq \eta^{\alpha_j}, j = 1, \ldots, N, -1 - \eta^2 < t \leq -1\}, \tag{4.4.1}$$
$$\mathcal{Q}_{ext} = \{(x,t) : |x_j| \leq 2^{\alpha_j} R, j = 1, \ldots, N, -1 - \eta^2 < t \leq 0\},$$

where $R > 1$, $\eta \in (0,1)$ and the exponents α_j, for $j = 1, \ldots, N$, are defined in (1.1.18). Thanks to Remark 4.2.5, $\mathcal{Q}_{zero}$ and $\mathcal{Q}_{ext}$ are completely equivalent to (4.1.12), but in this form they are more convenient for the construction of the cut-off function ψ_1 introduced in Lemma 4.4.3. Now, we state our weak Poincaré inequality.

Theorem 4.4.1 (Weak Poincaré inequality)**.** *Let $\eta \in (0,1)$ and let $\mathcal{Q}_{zero}$ and $\mathcal{Q}_{ext}$ be defined as in (4.4.1). Then there exist $R > 1$ and $\vartheta_0 \in (0,1)$, that only depend on Q and η, such that for any non-negative function $u \in \mathcal{W}$ such that $u \leq M$ in $\mathcal{Q}_1 = B_1 \times B_1 \times \ldots \times B_1 \times (-1,0)$ for a positive constant M and*

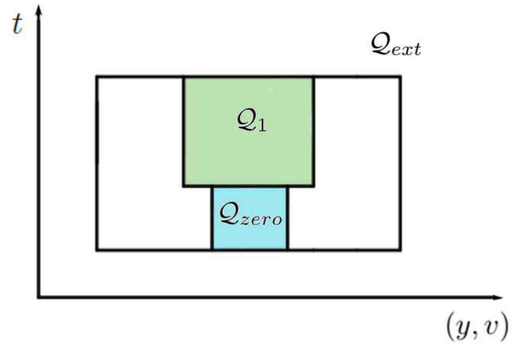
$$|\{u = 0\} \cap \mathcal{Q}_{zero}| \geq \frac{1}{4} |\mathcal{Q}_{zero}| , \qquad (4.4.2)$$

we have

$$\|(u - \vartheta_0 M)_+\|_{L^2(\mathcal{Q}_1)} \leq C_P \left(\|D_{m_0} u\|_{L^2(\mathcal{Q}_{ext})} + \|Yu\|_{L^2 H^{-1}(\mathcal{Q}_{ext})} \right) , \qquad (4.4.3)$$

where $C > 0$ is a constant only depending on Q.

Figure 4.2: Geometric setting of the Poincaré inequality (in the kinetic case).



The notation we consider here needs to be understood in the sense of (4.1.7). In particular, we have that $L^2 H^{-1}(\mathcal{Q}_{ext})$ is short for

$$L^2(B_{2^3 R} \times \ldots \times B_{2^{2\kappa+1} R} \times (-1 - \eta^2, 0], H^{-1}_{x^{(0)}}(B_{2R})), \qquad (4.4.4)$$

where we split $x = \left(x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}\right)$ according to (4.1.6).

A Poincaré inequality was already introduced by Wang and Zhang specifically for *strong* weak sub-solutions of ultraparabolic equations, i.e. $u \in L^2 H^1$ and $Yu \in L^2$, in the preprint [116, Lemmata 3.3 and Lemma 3.4] and the corresponding lemmata in [117]. This statement differs from our weak Poincaré inequality in three substantial aspects. First, the result we present here holds true for function u belonging to the space $\mathcal{W}$, and thus does not require the additional assumption for u to be a weak sub-solution to (4.1.1). Secondly, in our setting the transport operator Yu is merely assumed to be in $L^2 H^{-1}$. Finally, the proof we propose here differs from the ones in [116, 117], as we avoid using repeatedly the exact form of the fundamental solution of $\mathscr{L}$ and exploit arguments closer to the classical theory of parabolic equations (developed, for instance, in [76]).

In order to prove Theorem 4.4.1, the idea is to first derive a local Poincaré inequality in terms of an error function h defined as the solution to a suitable Cauchy problem. We then explicitly control the error function h through the L^∞ norm of the function u (see Lemma 4.4.4). This allows us to forget about the equation under study and to obtain a purely functional result. Since in the definition of our functional space $\mathcal{W}$ only the partial gradient D_{m_0} and the Lie derivative Y appear, we are allowed to work with the following Kolmogorov operator

$$\begin{aligned}\widetilde{\mathscr{L}_0}u(x,t) &:= -\sum_{i=1}^{m_0} \partial^2_{x_i} u(x,t) - \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) + \partial_t u(x,t), \qquad (x,t)\in\mathbb{R}^{N+1} \\ &:= -\Delta_{m_0}u(x,t) + \widetilde{Y}u(x,t),\end{aligned} \tag{4.4.5}$$

with zero first order terms and constant coefficients. Thus, following an approach similar to the one proposed in [51, 116], the error function h is defined as the solution to the Cauchy problem

$$\begin{cases} \widetilde{\mathscr{L}_0}h = u\widetilde{\mathscr{L}_0}\psi, & \textit{in } \mathbb{R}^N\times(-\rho^2,0) \\ h=0, & \textit{in } \mathbb{R}^N\times\{-\rho^2\}\end{cases} \tag{4.4.6}$$

where $\widetilde{\mathscr{L}_0}$ is the operator defined in (4.4.5) and ψ is a given cut-off function.

Lemma 4.4.2. *Let $\mathcal{Q}_{ext}$ be as defined in* (4.4.1) *and let $\psi:\mathbb{R}^{N+1}\to[0,1]$ be a C^∞ function, with support in $\mathcal{Q}_{ext}$ and such that $\psi=1$ in $\mathcal{Q}_1$. Then for any $u\in\mathcal{W}$, the following holds*

$$\|(u-h)_+\|_{L^2(\mathcal{Q}_1)} \le C\left(\|D_{m_0}u\|_{L^2(\mathcal{Q}_{ext})} + \|Yu\|_{L^2H^{-1}(\mathcal{Q}_{ext})}\right) \tag{4.4.7}$$

where h is the solution to (4.4.6)*, C is a constant only depending on $|\rho^2|$ and $\|D_{m_0}\psi\|_{L^\infty(\mathcal{Q}_{ext})}$, and the notation we consider needs to be intended in the sense of* (4.4.4)*.*

This local weak Poincaré inequality is an extension to operator $\mathscr{L}$ and to space $\mathcal{W}$ of the one proved in [51] and a simplification of the one proved in [116, 117]. Moreover, the result holds true for any cylinder of the form $\mathcal{Q}_{ext}=B_{R_0}\times\ldots\times B_{R_\kappa}\times(-\rho^2,0]$, provided that $\overline{\mathcal{Q}_1}\subset\mathcal{Q}_{ext}$. The proof of Lemma 4.4.2 is mainly based on the properties of the principal part operator $\mathscr{L}_0$ and of $\mathcal{W}$.

Proof of Lemma 4.4.2. As $u\in\mathcal{W}$, $\widetilde{Y}u\in L^2H^{-1}$ and therefore, in virtue of Remark 4.1.1, there exist $H_0,H_1\in L^2(\mathcal{Q}_{ext})$ such that

$$\widetilde{Y}u = \mathrm{div}_{m_0}H_1 + H_0, \tag{4.4.8}$$

with $\|H_0\|_{L^2(\mathcal{Q}_{ext})}+\|H_1\|_{L^2(\mathcal{Q}_{ext})}\le 2\|\widetilde{Y}u\|_{L^2H^{-1}}$. Thus, the function $g:=u\psi$ satisfies the following equation in the sense of distributions

$$\widetilde{\mathscr{L}}_0 g = u\widetilde{\mathscr{L}}_0\psi + \mathrm{div}_{m_0}\bar{H}_1 + \bar{H}_0, \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (-\rho^2, 0)),$$

where $\bar{H}_1 = (H_1 - D_{m_0}u)\psi$ and $\bar{H}_0 = H_0\psi - \langle H_1, D_{m_0}\psi\rangle - \langle D_{m_0}\psi, D_{m_0}u\rangle$. Thus, owing to (4.4.6), we obtain

$$\widetilde{\mathscr{L}}_0(g-h) = \mathrm{div}_{m_0}\bar{H}_1 + \bar{H}_0 =: \bar{H}, \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times (-\rho^2, 0)). \tag{4.4.9}$$

Now, choosing $2(g-h)_+\psi^2$ as a test function in (4.4.9) and integrating on the domain $D = \mathbb{R}^N \times [-\rho^2, 0]$, we get

$$\begin{aligned} 2\int_D |D_{m_0}((g-h)_+)|^2\psi^2 &+ \boxed{4\int_D \psi(g-h)_+\langle D_{m_0}((g-h)_+), D_{m_0}(\psi)\rangle}_A \\ &+ \boxed{2\int_D (g-h)_+\psi^2\widetilde{Y}((g-h)_+)}_B - \boxed{2\int_D (g-h)_+\psi^2 D_{m_0}\cdot\bar{H}_1}_C \\ &- \boxed{2\int_D (g-h)_+\psi^2\bar{H}_0}_D = 0. \end{aligned} \tag{4.4.10}$$

We estimate the boxed term A by applying Young's inequality and choosing $\varepsilon = 1$. As far as we are concerned with the boxed term B, we rewrite it as

$$\begin{aligned} \boxed{2\int_D (g-h)_+\psi^2\widetilde{Y}((g-h)_+)}_B &= -\int_D \psi^2\widetilde{Y}_0((g-h)_+^2) + \int_D \psi^2\partial_t((g-h)_+^2) \\ &= -\int_D [\psi^2\widetilde{Y}_0((g-h)_+^2) \\ &\quad + \int_D \partial_t((g-h)_+^2\psi^2) - (g-h)_+^2\partial_t(\psi^2)] \\ &= \int_D (g-h)_+^2\widetilde{Y}(\psi^2) + \int_D \partial_t((g-h)_+^2\psi^2), \end{aligned}$$

where in the first line we defined $\widetilde{Y}_0 = \langle Bx, D\rangle$, in the second line we used the equality $\partial_t((g-h)_+^2)\psi^2 = \partial_t((g-h)_+^2\psi^2) - \partial_t(\psi^2)(g-h)_+^2$ and in the third we integrated by parts the term involving Y_0. Finally, we take care of the boxed term C and D using Young's inequality as follows

$$\begin{aligned} -\boxed{2\int_D (g-h)_+\psi^2 D_{m_0}\cdot\bar{H}_1}_C &= 2\int_D \langle D_{m_0}((g-h)_+\psi^2), \bar{H}_1\rangle \\ &\leq \frac{1}{2}\|\psi D_{m_0}(g-h)_+\|_{L^2(D)}^2 + \frac{1}{2}\|(g-h)_+D_{m_0}\psi\|_{L^2(D)}^2 \\ &\qquad + 10\|\bar{H}_1\|_{L^2(D)}^2, \end{aligned}$$

$$-\boxed{2\int_D (g-h)_+\psi^2\bar{H}_0}_D \le 2\varepsilon\|(g-h)_+\|^2_{L^2(D)} + \frac{1}{2\varepsilon}\|\bar{H}_0\|^2_{L^2(D)}.$$

Combining the previous estimates, for every $T \in (-\rho^2, 0)$ and $\varepsilon > 0$ to be chosen later, we rewrite (4.4.10) as

$$\int_{-\rho^2}^{T}\int_{\mathbb{R}}^{N} \partial_t((g-h)_+^2\psi^2)dxdt + 2\int_D |D_{m_0}(g-h)_+|^2\,\psi^2 + \int_D (g-h)_+^2\widetilde{Y}(\psi^2)$$
$$\le \frac{1}{2}\|D_{m_0}(g-h)_+\psi\|^2_{L^2(D)} + 10\|\bar{H}_1\|^2_{L^2(D)} + 2\varepsilon\|(g-h)_+\|^2_{L^2(D)} + \frac{1}{2\varepsilon}\|\bar{H}_0\|^2_{L^2(D)}.$$

We now apply the fundamental theorem of calculus to the term involving the time derivative and we infer

$$\int_{\mathbb{R}}^{N} ((g-h)_+^2\psi^2)(x,T)dx + 2\int_D |D_{m_0}(g-h)_+|^2\,\psi^2 + \int_D (g-h)_+^2\widetilde{Y}(\psi^2) \tag{4.4.11}$$
$$\le \frac{1}{2}\|D_{m_0}(g-h)_+\psi\|^2_{L^2(D)} + 10\|\bar{H}_1\|^2_{L^2(D)} + 2\varepsilon\|(g-h)_+\|^2_{L^2(D)} + \frac{1}{2\varepsilon}\|\bar{H}_0\|^2_{L^2(D)}.$$

We then integrate in T from $-\rho^2$ to 0 and we obtain

$$\begin{aligned} &\|(g-h)_+\psi\|^2_{L^2(D)} \\ &\le -\frac{3}{2}\rho^2\|D_{m_0}(g-h)_+\psi\|^2_{L^2(D)} + 10\rho^2\|\bar{H}_1\|^2_{L^2(D)} + 2\rho^2\varepsilon\|(g-h)_+\|^2_{L^2(D)} + \frac{\rho^2}{2\varepsilon}\|\bar{H}_0\|^2_{L^2(D)} \\ &\le 10\rho^2\|\bar{H}_1\|^2_{L^2(D)} + 2\rho^2\varepsilon\|(g-h)_+\|^2_{L^2(D)} + \frac{\rho^2}{2\varepsilon}\|\bar{H}_0\|^2_{L^2(D)}, \end{aligned} \tag{4.4.12}$$

as $(g-h)_+\psi$ equals $(u-h)_+$ and $\widetilde{Y}(\psi^2)$ equals 0 in $\mathcal{Q}_1$. In addition, the following estimates hold

$$\begin{aligned} &\|\bar{H}_0\|_{L^2(D)} \le \|H_0\|_{L^2(\mathcal{Q}_{ext})} + \|D_{m_0}\psi\|_{L^\infty(\mathcal{Q}_{ext})}\left(\|D_{m_0}u\|_{L^2(\mathcal{Q}_{ext})} + \|H_1\|_{L^2(\mathcal{Q}_{ext})}\right), \\ &\|\bar{H}_1\|_{L^2(D)} \le \|H_1\|_{L^2(\mathcal{Q}_{ext})} + \|D_{m_0}u\|_{L^2(\mathcal{Q}_{ext})}. \end{aligned} \tag{4.4.13}$$

By combining (4.4.11), (4.4.12) and (4.4.13) and choosing $\varepsilon = \frac{1}{4\rho^2}$ the claim is proved. □

Given the local Poincaré inequality proved in Lemma 4.4.2, we just need to estimate the error function h defined in (4.4.6) in order to complete the proof of Theorem 4.4.1. In particular, our aim is to show that the error function h is bounded from above by $\vartheta_0 M$, where $\vartheta_0 \in (0,1)$ is a constant only depending on Q, λ and Λ. In order to prove this result, we first need to explicitly construct an appropriate cut-off function, that differs from the one considered in [51, Lemma 3.3] due to the more involved structure of our drift term Y. We observe that our construction of the suitable cut-off function is constructive, in contrast with the one proposed in [51]. More precisely, the cut-off function used in [51] is obtained as the product of three smooth functions that are not constructed explicitly but have the right dependency on velocity, space and time. However, in our case, it is necessary to explicitly contruct the smooth functions in the aforementioned product to make sure that the obtained cut-off function behaves nicely when applying the Lie derivative Y. This is the reason why we consider rather convoluted smooth functions in the proof of Lemma 4.4.3 (see (4.4.15) and (4.4.16) below).

Lemma 4.4.3. *Given $\eta \in (0,1]$ and $T \in (0,\eta^2)$, there exists a smooth function $\psi_1 : \mathbb{R}^N \times [-1-\eta^2, 0]$, supported in $\{(x,t) : |x_j| \leq 2^{\alpha_j}, j = 1, \ldots, N, t \in [-1-\eta^2, 0]\}$, equal to 1 in $\mathcal{Q}_1$, and such that the following conditions hold*

$$\begin{aligned} &\widetilde{Y}\psi_1 \geq 0 \quad \textit{everywhere} \\ &\widetilde{Y}\psi_1 \geq 1 \quad \textit{if } t \in (-1-\eta^2, -1-T]. \end{aligned} \tag{4.4.14}$$

Proof. Let us consider the cut-off function $\chi_1 \in C^\infty([0,+\infty))$ defined by

$$\chi_1(s) = \begin{cases} 0, & \text{if } s > \frac{2}{\sqrt{2}}, \\ 1, & \text{if } 0 \leq s \leq C+1, \end{cases} \qquad \chi_1' \leq 0, \tag{4.4.15}$$

where $C > 1$ is a constant we shall specify later on.

Now, we introduce a second cut-off function $\chi_2 \in C^\infty(\mathbb{R}^{N+1})$ defined as

$$\chi_2(x,t) = \chi_1\left(\sum_{j>m_0}^{N} \frac{2x_j^2}{2^{2\alpha_j}\sqrt{2}} - C\,t\right), \tag{4.4.16}$$

which is supported in $\mathcal{Q}_{ext}$ and equal to 1 in $\mathcal{Q}_1$.

Finally, we consider a smooth function $\Phi_t : [-1-\eta^2, 0] \to [0,1]$ equal to 1 in $[-1,0]$, with $\Phi_t(0) = 1$, $\Phi_t' \geq 0$ in $[-1-\eta^2, 0]$ and $\Phi_t' = 1$ in $[-1-\eta^2, -1-T]$. We are now in a position to define the cut-off function ψ_1 as follows

$$\psi_1(x,t) = \chi_1(\|(x_1, \ldots, x_{m_0})\|_{\mathbb{K}})\chi_2(x,t)\Phi_t(t).$$

We only have to check that conditions (4.4.14) hold, as the other desired properties immediately follow from the definition of ψ_1. To this end, we compute the following derivative

$$\widetilde{Y}\chi_2 = \chi_1'((\ldots))\left[-\sum_{i=1}^{N}\sum_{j>m_0} 2x_i b_{ij} x_j 2^{-2\alpha_j - 1/2} - C\right],$$

where (...) denotes $\left(\sum_{j=m_0}^{N} \frac{2x_j^2}{2^{2\alpha_j}\sqrt{2}} - C\,t\right)$. It can be shown (see [117, Lemma 3.2]) that there exists a constant $C > 1$ such that

$$C \geq \left|\sum_{i=1}^{N}\sum_{j>m_0} 2x_i b_{ij} x_j 2^{-2\alpha_j - 1/2}\right|,$$

for every $(x,t) \in \mathcal{Q}_{ext}$. Thus, with such a choice of C and keeping in mind that $\Phi_t' \geq 0$ in $[-1-\eta^2, 0]$ and $\Phi_t' = 1$ in $[-1-\eta^2, -1-T]$, we have

$$\begin{aligned} \widetilde{Y}\psi_1 &= \chi_1 \, \Phi_t \, \widetilde{Y}\chi_2 + \chi_1 \, \chi_2 \, \Phi_t' \geq 0 \qquad \text{everywhere}, \\ \widetilde{Y}\psi_1 &= \chi_1 \, \Phi_t \, \widetilde{Y}\chi_2 + \chi_1 \, \chi_2 \, \Phi_t' \geq 1 \quad \text{if } t \in (-1-\eta^2, -1-T]. \end{aligned}$$

□

Thus, we are now in a position to state and prove the following result regarding the control of the localization term h defined in (4.4.6).

Lemma 4.4.4. *Let $\eta \in (0,1]$ and let $\mathcal{Q}_{ext}$ be as defined in* (4.4.1). *Then there exist $R = R(Q,\eta) > 1$, $\vartheta_0 = \vartheta_0(Q,\eta) \in (0,1)$ and a C^∞ cut-off function $\psi : \mathbb{R}^{N+1} \to [0,1]$, with support in $\mathcal{Q}_{ext}$ and equal to 1 in $\mathcal{Q}_1$, such that for all $u \in \mathcal{W}$ non-negative bounded functions defined on $\mathcal{Q}_{ext}$ and satisfying*

$$|\{u = 0\} \cap \mathcal{Q}_{zero}| \geq \frac{1}{4} \, |\mathcal{Q}_{zero}| \, , \tag{4.4.17}$$

the function h solution to the Cauchy problem (4.4.6) *with $\rho^2 = 1 + \eta^2$ satisfies*

$$h \leq \vartheta_0 \|u\|_{L^\infty(\mathcal{Q}_{ext})}, \qquad \text{in } \mathcal{Q}_1. \tag{4.4.18}$$

Proof. We assume that u is not identically vanishing in $\mathcal{Q}_{ext}$. Indeed, if $u = 0$ in $\mathcal{Q}_{ext}$, then $h = 0$ and inequality (4.4.18) is trivially satisfied. Moreover, we can reduce to the case of a function u with L^∞- norm equal to 1 by taking $u/\|u\|_{L^\infty(\mathcal{Q}_{ext})}$.

We now fix $T = \eta^2/8$ and we introduce a time lap T between the top of the cylinder $\mathcal{Q}_{zero}$ and the bottom of the cylinder $\mathcal{Q}_1$. As $|\mathcal{Q}_{zero} \cap \{t \geq -1-T\}| = \frac{1}{8}|\mathcal{Q}_{zero}|$ by definition of T, inequality (4.4.17) yields the following inequality

$$|\mathcal{Q}_{zero} \cap \{t \leq -1-T\} \cap \{u = 0\}| \geq \frac{1}{8}|\mathcal{Q}_{zero}|. \tag{4.4.19}$$

We now consider the cut-off function

$$\psi(x,t) = \psi_1(x/R, t),$$

whew $R > 1$ is a constant we will specify later and ψ_1 is given by Lemma 4.4.3. We observe that, by definition of ψ_1, ψ is supported in $\mathcal{Q}_{ext}$ and equal to 1 in $\{(x,t) : |x_j| \leq R, j = 1, \ldots, N, t \in (-1, 0]\}$. In addition, it satifies

$$\widetilde{\mathscr{L}}_0 \psi(x,t) = -R^{-2} \Delta_{m_0} \psi_1(x/R, t) + \widetilde{Y}\psi_1(x/R, t).$$

We now consider the function h solution to the Cauchy problem (4.4.6). Thus, by definition of h, we have

$$\widetilde{\mathscr{L}_0}(h-\psi) = \frac{1-u}{R^2}\Delta_{m_0}\psi(x,t) + (u-1)\widetilde{Y}\psi(x,t).$$

We now rewrite the difference $h-\psi$ as

$$h - \psi = E_R - P_R, \tag{4.4.20}$$

where E_R and P_R stand respectively for error and positive term and are solutions in $\mathbb{R}^N \times (-1-\eta^2, 0)$ to the following Cauchy problems

$$\begin{cases} \widetilde{\mathscr{L}_0}E_R = \frac{1-u}{R^2}\Delta_{m_0}\psi(x,t), & \textit{in } \mathbb{R}^N \times (-1-\eta^2, 0) \\ E_R = 0, & \textit{in } \mathbb{R}^N \times \{-1-\eta^2\} \end{cases},$$

$$\begin{cases} \widetilde{\mathscr{L}_0}P_R = (1-u)\widetilde{Y}\psi(x,t), & \textit{in } \mathbb{R}^N \times (-1-\eta^2, 0) \\ P_R = 0, & \textit{in } \mathbb{R}^N \times \{-1-\eta^2\} \end{cases}.$$

Our aim is to show that there exist two positive constants $C' = C'(Q, \lambda, \Lambda, \eta)$ and $\delta_0 = \delta_0(Q, \lambda, \Lambda, \eta)$ such that

$$E_R \le \frac{C'}{R^2} \qquad \textit{and} \qquad P_R \ge \delta_0, \qquad \textit{in } \mathcal{Q}_1. \tag{4.4.21}$$

First, we focus on the term involving E_R, and we remark that

$$\widetilde{\mathscr{L}_0}E_R \le \frac{C'}{R^2}, \tag{4.4.22}$$

where $C' = \|\Delta_{m_0}\psi_1\|_{L^\infty}$ is a constant only depending on Q, λ, Λ and η. We then obtain the desidered estimate for E_R applying the maximum principle.

As far as the term involving P_R is concerned, we recall that, owing to the definition of ψ, there holds $\widetilde{Y}\psi \ge 1$ for $t \in (-1-\eta^2, -1-T)$. Moreover, as we restricted ourselves to the case $\|u\|_{L^\infty(\mathcal{Q}_{ext})} = 1$, we have $1 - u \ge 0$. Thus, the following inequality holds true

$$\widetilde{\mathscr{L}_0}P_R \ge \mathbb{I}_{\mathcal{Z}}, \quad \text{in } \mathbb{R}^N \times (-1-\eta^2, 0),$$

where $\mathbb{I}_{\mathcal{Z}}$ is the indicator function of the set $\mathcal{Z} := \mathcal{Q}_{zero} \cap \{t \le -1-T\} \cap \{u = 0\}$. Let us now consider P solution to the following Cauchy problem

$$\begin{cases} \widetilde{\mathscr{L}_0}P = \mathbb{I}_{\mathcal{Z}}, & \textit{in } \mathbb{R}^N \times (-1-\eta^2, 0) \\ P = 0, & \textit{in } \mathbb{R}^N \times \{-1-\eta^2\} \end{cases} \tag{4.4.23}$$

Then, the maximum principle [21] for the principle part operator $\widetilde{\mathscr{L}_0}$ yields

$$P \le P_R, \quad \text{in } \mathbb{R}^N \times (-1-\eta^2, 0).$$

As $\mathcal{Q}_1 \subset \mathbb{R}^N \times (-1-\eta^2, 0)$, the previous inequality implies in particular that $P \leq P_R$ in $\mathcal{Q}_1$. Owing to (4.4.23), we represent P using the fundamental solution $\widetilde{\Gamma}$ of $\widetilde{\mathscr{L}_0}$. More precisely, there exists $-1-\eta^2 \leq T_0 < 0$ such that

$$P(x,t) = \int_t^0 \int_{\mathbb{R}^N} \widetilde{\Gamma}(x,t;\xi,\tau)\mathbb{I}_{\mathcal{Z}}(\xi,\tau) d\xi d\tau,$$

for every $(x,t) \in \mathbb{R}^N \times (T_0, 0)$. In particular, for every $(x,t) \in \mathcal{Q}_1$, there holds

$$\begin{aligned} P(x,t) &= \int_t^0 \int_{\mathbb{R}^N} \widetilde{\Gamma}(x,t;\xi,\tau)\mathbb{I}_{\mathcal{Z}}(\xi,\tau) d\xi\, d\tau = \int_{\mathcal{Z}} \widetilde{\Gamma}(x,t;\xi,\tau) d\xi d\,\tau \\ &\geq \min_{\substack{\mathcal{Q}_1 \times \mathcal{Q}_{zero} \\ \cap\{t \leq -1-T\}}} \widetilde{\Gamma} \int_{\mathcal{Z}} d\xi\, d\tau \\ &= \min_{\substack{\mathcal{Q}_1 \times \mathcal{Q}_{zero} \\ \cap\{t \leq -1-T\}}} \widetilde{\Gamma}\, |\mathcal{Q}_{zero} \cap \{t \leq -1-T\} \cap \{u = 0\}| \\ &\geq \frac{1}{8} \min_{\substack{\mathcal{Q}_1 \times \mathcal{Q}_{zero} \\ \cap\{t \leq -1-T\}}} \widetilde{\Gamma}\, |\mathcal{Q}_{zero}| := \delta_0, \end{aligned}$$

where in the last line we have used inequality (4.4.19). We remark that the fundamental solution $\widetilde{\Gamma}$ is bounded from above and below by Gaussian bounds, see for instance [8, Theorem 1.6]. As a consequence,

$$P_R \geq P \geq \delta_0, \qquad in\ \mathcal{Q}_1 \tag{4.4.24}$$

and claim (4.4.21) is now proved. Using estimates (4.4.22) and (4.4.24) in (4.4.20), we finally obtain

$$h \leq 1 - \delta_0 + \frac{C'}{R^2}, \qquad in\ \mathcal{Q}_1.$$

We now observe that for R large enough we have $C'/R^2 \leq \delta_0/2$. Thus, setting $\vartheta_0 = 1-\delta_0/2 < 1$, we get the desired inequality (4.4.18). □

We remark that the proof (and therefore the statement) of Lemma 4.4.4 remains unchanged if we replace (4.4.17) with the less restrictive assumption $|\{u=0\} \cap \mathcal{Q}_{zero}| \geq \gamma_0 |\mathcal{Q}_{zero}|$ for some constant γ_0 such that $0 < \gamma_0 < 1$.

Proof of Theorem 4.4.1. The proof follows immediately combining Lemmata 4.4.2 and 4.4.4. □

4.5 Weak Harnack inequality

This section is devoted to the proof of a weak Harnack inequality, an intermediate result necessary to prove Theorem 4.1.3. The approach we present here is an extension of the classical method proposed in [68] for the elliptic and parabolic setting, and later on followed by Guerand and Imbert in [51] for the kinetic Kolmogorov-Fokker-Planck equation. This approach is very convenient for the study of the weak regularity theory, since it only relies on the functional structure of the space $\mathcal{W}$ and on the non-Euclidean geometrical setting presented in Section 4.2.

Theorem 4.5.1 (Weak Harnack inequality)**.** *Let R_0 and ω be two given positive constants. Let $\mathcal{Q}^0 = B_{R_0} \times B_{R_0} \times \ldots \times B_{R_0} \times (-1, 0]$ and let u be a non-negative weak solution to $\mathscr{L}u = f$ in $\Omega \supset \mathcal{Q}^0$ under assumptions **(H1)**-**(H2)**-**(H3)**. Then we have*

$$\left(\int_{\mathcal{Q}_-} u^p \right)^{\frac{1}{p}} \leq C \left(\inf_{\mathcal{Q}_+} u + \|f\|_{L^q(\mathcal{Q}^0)} \right), \tag{4.5.1}$$

where $\mathcal{Q}_+ = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-\omega^2, 0]$ and $\mathcal{Q}_- = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-1, -1+\omega^2]$. Moreover, the constants C, p, ω and R_0 only depend on the homogeneous dimension Q defined in (1.1.17)*, q and on the ellipticity constants λ and Λ in* (4.1.2)*. Additionally, if the term c is of positive sign, the statement holds true also for non-negative super-solutions to* (4.1.1)*.*

Remark 4.5.2. As in [51], the radius ω is small enough so that when "stacking cylinders" over a small initial one contained in $\mathcal{Q}^-$, the cylinder $\mathcal{Q}^+$ is captured, see Lemma 4.B.1. As far as we are concerned with R_0, it is large enough so that it is possible to apply the expansion of positivity lemma (see Lemma 4.5.8) to every stacked cylinder.

Finally, in order to carry out the proof of Theorem 4.5.1, we also require that the quantity R_0/ω is large enough. This condition is due to the fact that we want to apply the expansion of positivity Lemma 4.5.10 to cylinders which are contained in $\mathcal{Q}_-$ and therefore are of radius smaller than ω. To this end, we need to impose that $R_0 \geq C\omega$, where C is a constant that only depends on Q, λ and Λ and is explicitly computed in (4.5.16).

Remark 4.5.3. Differently from the weak Harnack inequality presented in [51], our statement holds true in general for weak solutions and not for weak super-solutions, hence the claim *"Such a weak Harnack inequality can be generalized to the ultraparabolic equations with rough coefficients"* stated in [51, Remark 4, p.2] is only partially true. This discrepancy is due to the presence of the lower order term cu in our analysis, which wasn't considered by the authors of [51]. Indeed, when $c \geq 0$ we exactly recover the statement proposed in [51], but if we only consider assumption **(H3)** we are forced to restrict ourselves to the case of weak solutions. This is mainly due to the methodology we follow to prove the expansion of positivity, Lemma 4.5.8, which is based on our local boundedness result Theorem 4.3.1, that holds true when the right-hand side is in L^q_{loc}, with $q > \frac{(Q+2)}{2}$. The extension of this result to weak super-solutions to (4.1.1) without any sign assumption on the term c will be the content of a forthcoming paper.

The proof of the weak Harnack inequality is obtained combining the fact that super-solutions to (4.1.1) expand positivity along times (Lemma 4.5.8) with the covering argument presented in Appendix 4.B.

We observe that, in contrast with parabolic equations, it is not possible to apply a classical Poincaré inequality in the spirit of [87]. Indeed, in our case there is a positive quantity replacing the average in the usual Poincaré inequality (see the statement of Theorem 4.4.1). Moreover, the lower order term involving c presents additional difficulties to the ones already addressed in [51], see Remark 4.5.3 above. We circumvent these difficulties by establishing a weak expansion of positivity result for solutions (respectively, for super-solutions when $c \geq 0$) to (4.1.1). More precisely, given a small cylider $\mathcal{Q}_{pos}$ lying in the past of $\mathcal{Q}_1$ (see Definition (4.5.6)), we show that the positivity of a non-negative (super-)solution u lying above 1 in a "big" part of $\mathcal{Q}_{pos}$ is spread to the whole $\mathcal{Q}_1$ (see Lemma 4.5.8). In other words, a positivity in measure in a smaller cylinder $\mathcal{Q}_{pos}$ is transformed into a pointwise positivity in a bigger cylinder $\mathcal{Q}_1$.

First of all, we introduce a convex function whose properties parallel the ones of the less regular function $\max(0, -\ln)$. Such a function was first constructed by Kruzhkov in [68] and later used in [51, 118]. For a proof of its existence we refer the reader to [118, Lemma 5.1].

Definition 4.5.4. Let $G : (0, +\infty) \to (0, +\infty)$ be a non-increasing and C^2 convex function such that

- $G'' \geq (G')^2$ and $G' \leq 0$ in $(0, +\infty)$,
- G is supported in $(0, 1]$,
- $G(t) \sim -\ln t$ as $t \to 0^+$,
- $-G'(t) \leq \frac{1}{t}$ for $t \in (0, \frac{1}{4}]$,
- $G(t) = 0$ for $t \geq 1$.

Our first aim is to show that, given a non-negative weak super-solution u to (4.1.1), the function G transforms it into a non-negative weak sub-solution to a suitably defined equation.

Lemma 4.5.5. *Let $\varepsilon \in \left(0, \frac{1}{4}\right]$ and u be a non-negative weak super-solution to (4.1.1) under the assumptions **(H1)-(H2)-(H3)** in a cylindrical shaped open set $\mathcal{Q}_{ext} = B_{R_0} \times B_{R_1} \times \ldots \times B_{R_\kappa} \times (t_0, T] \subset \Omega$, where $R_i > 0$ for $i = 0, \ldots, \kappa$. Then $g := G(u + \varepsilon^\gamma)$, for $\gamma > 0$, is a non-negative weak-sub-solution to the following equation:*

$$\operatorname{div}(ADg) + Yg + \langle b, Dg\rangle - \lambda |D_{m_0} g|^2 = -\varepsilon^{-\gamma}|cu - f|. \tag{4.5.2}$$

Proof. We consider $g = G(u+\varepsilon^\gamma)$, where G is the convex function defined above. In particular, we have that $|G'(u + \varepsilon^\gamma)| \leq |G'(\varepsilon^\gamma)| \leq \varepsilon^{-\gamma}$ as u is non-negative and $G'(u + \varepsilon^\gamma) \leq 0$ by assumption. For this reason, we test the definition of super-solution u against the function $\varphi = -G'(u + \varepsilon^\gamma)\, \psi$, where ψ is a non-negative test function belonging to $\mathcal{D}(\Omega)$:

$$\int\limits_{\mathcal{Q}_{ext}} \langle ADu, D(G'(u + \varepsilon^\gamma)\psi)\rangle - Yu(G'(u + \varepsilon^\gamma)\psi)$$

$$- \langle b, Du \rangle (G'(u+\varepsilon^\gamma)\psi) - cu(G'(u+\varepsilon^\gamma)\psi) \leq - \int\limits_{\mathcal{Q}_{ext}} f(G'(u+\varepsilon^\gamma)\psi).$$

Now, owing to $D_{m_0} g = G'(u+\varepsilon^\gamma) D_{m_0} u$ and $Yg = G'(u+\varepsilon^\gamma) Yu$, we can rewrite the previous inequality as follows:

$$\int\limits_{\mathcal{Q}_{ext}} \langle ADg, D\psi \rangle - Yg\ \psi - \langle b, Dg \rangle \psi + \int\limits_{\mathcal{Q}_{ext}} \langle ADu, Du \rangle G''(u+\varepsilon^\gamma)\psi \leq \int\limits_{\mathcal{Q}_{ext}} cuG'(u+\varepsilon^\gamma)\psi - \int\limits_{\mathcal{Q}_{ext}} fG'(u+\varepsilon^\gamma)\psi.$$

Hence, we are left with the estimate of the terms appearing on the right-hand side. Let us begin with the first one:

$$\int\limits_{\mathcal{Q}_{ext}} \langle ADu, Du \rangle G''(u+\varepsilon^\gamma)\psi \geq \lambda \int\limits_{\mathcal{Q}_{ext}} \langle D_{m_0} u, D_{m_0} u \rangle (G'(u+\varepsilon^\gamma))^2 \psi = \lambda \int\limits_{\mathcal{Q}_{ext}} |D_{m_0} g|^2 \psi,$$

where we employed assumption **(H1)** and $G''(u+\varepsilon^\gamma) \geq (G'(u+\varepsilon^\gamma))^2$. As far as we are concerned with the second and third term, we observe that

$$\int\limits_{\mathcal{Q}_{ext}} cuG'(u+\varepsilon^\gamma)\psi - \int\limits_{\mathcal{Q}_{ext}} fG'(u+\varepsilon^\gamma)\psi \leq \int\limits_{\mathcal{Q}_{ext}} |cu - f||G'(u+\varepsilon^\gamma)|\psi \leq \varepsilon^{-\gamma} \int\limits_{\mathcal{Q}_{ext}} |cu - f|\psi,$$

where the right-hand side is well-posed, since by definition of super-solution $u \in \mathcal{W}$ and by assumption **(H3)** the term $c \in L^q_{loc}$, with $q > \frac{(Q+2)}{2}$. Eventually, the function g satisfies the following inequality

$$\int\limits_{\mathcal{Q}_{ext}} -\langle ADg, D\psi \rangle + Yg\ \psi + \langle b, Dg \rangle \psi - \lambda |D_{m_0} g|^2 \psi \geq -\varepsilon^{-\gamma} \int\limits_{\mathcal{Q}_{ext}} |cu - f|\psi, \tag{4.5.3}$$

hence g is a non-negative weak-sub-solution to (4.5.2). □

We are now in a position to provide an estimate of the L^2-norm of the gradient $D_{m_0} g$.

Lemma 4.5.6. *Let $\mathcal{Q}_{ext} = B_{R_0} \times B_{R_1} \times \ldots \times B_{R_\kappa} \times (t_0, T_0] \subset \Omega$ and $\mathcal{Q}_{int} = B_{r_0} \times B_{r_1} \times \ldots \times B_{r_\kappa} \times (t_1, T_1]$, where $0 < r_i < R_i$ for every $i = 0, \ldots, \kappa$ and $t_0 < t_1 < T_1 < T_0$. Let g be a non-negative weak sub-solution to* (4.5.2) *under assumptions **(H1)-(H2)-(H3)**, then*

$$\frac{\lambda}{2} \int\limits_{\mathcal{Q}_{int}} |D_{m_0} g|^2 \leq C_G \int\limits_{\mathcal{Q}_{ext}} g + \varepsilon^{-\gamma} \left(\|c\|_{L^2(\mathcal{Q}_{ext})} \|u\|_{L^2(\mathcal{Q}_{ext})} + \|f\|_{L^1(\mathcal{Q}_{ext})} \right), \tag{4.5.4}$$

where $C_G = C_G(\|b\|_{L^\infty(\mathcal{Q}_{ext})}, \Lambda, Q, C_1, C_2)$ is a positive constant and C_1, C_2 are defined in

(4.5.5).

Proof. Let us consider a test function $\psi \in C_0^\infty$ valued in $[0,1]$, supported in $\mathcal{Q}_{ext}$ and equal to 1 in $\mathcal{Q}_{int}$. Moreover, we assume that the following estimates hold

$$|D_{m_0}\psi| \leq C_1, \qquad |Y\psi| \leq C_2 \tag{4.5.5}$$

for two non-negative constants C_1, C_2. Now, we may test (4.5.3) against ψ^2:

$$\int_{\mathcal{Q}_{ext}} \langle ADg, D\psi^2\rangle - Yg\ \psi^2 - \langle b, Dg\rangle\psi^2 + \lambda |D_{m_0}g|^2\psi^2 \leq \varepsilon^{-\gamma} \int_{\mathcal{Q}_{ext}} |cu - f|\psi^2.$$

Let us begin by estimating the term involving the matrix A through the Young's inequality:

$$\begin{aligned} -\int_{\mathcal{Q}_{ext}} 2\psi\langle ADg, D\psi\rangle &\leq \bar{\varepsilon} \int_{\mathcal{Q}_{ext}} |\langle ADg, Dg\rangle|\psi + \frac{1}{\bar{\varepsilon}} \int_{\mathcal{Q}_{ext}} |\langle AD\psi, D\psi\rangle|\psi \\ &\leq \bar{\varepsilon}\Lambda \int_{\mathcal{Q}_{ext}} |D_{m_0}g|^2\psi + \frac{\Lambda}{\bar{\varepsilon}} \int_{\mathcal{Q}_{ext}} |D_{m_0}\psi|^2\psi. \end{aligned}$$

Now, let us address the term involving c, u and f:

$$\begin{aligned} \varepsilon^{-\gamma} \int_{\mathcal{Q}_{ext}} |cu - f|\psi^2 &\leq \varepsilon^{-\gamma} \int_{\mathcal{Q}_{ext}} |c||u|\psi^2 + \varepsilon^{-\gamma} \int_{\mathcal{Q}_{ext}} |f|\psi^2 \\ &\leq \varepsilon^{-\gamma} \|c\|_{L^2(\mathcal{Q}_{ext})} \|u\|_{L^2(\mathcal{Q}_{ext})} + \varepsilon^{-\gamma} \|f\|_{L^1(\mathcal{Q}_{ext})}. \end{aligned}$$

As far as we are concerned with the term b, we integrate by parts and by assumption **(H3)** we have:

$$\begin{aligned} \int_{\mathcal{Q}_{ext}} \langle b, Dg\rangle\psi^2 &= -\int_{\mathcal{Q}_{ext}} \mathrm{div} b\ g\psi^2 - 2\int_{\mathcal{Q}_{ext}} bg\psi D\psi \leq 2\int_{\mathcal{Q}_{ext}} |b||g||D_{m_0}\psi|\psi \\ &\leq 2C_1 \|b\|_{L^\infty(\mathcal{Q}_{ext})} \int_{\mathcal{Q}_{ext}} g. \end{aligned}$$

Lastly, the term involving the transport operator can be treated as

$$\int_{\mathcal{Q}_{ext}} Yg\ \psi^2 \leq 2\int_{\mathcal{Q}_{ext}} g\,|Y\psi|\psi \leq 2C_2 \int_{\mathcal{Q}_{ext}} g.$$

Hence, by combining the above estimates we obtain

$$(\lambda - \bar{\varepsilon}\Lambda) \int_{\mathcal{Q}_{int}} |D_{m_0}g|^2 \leq \frac{\Lambda}{\bar{\varepsilon}} \int_{\mathcal{Q}_{ext}} |D_{m_0}\psi|^2\psi + \varepsilon^{-\gamma} \|c\|_{L^2(\mathcal{Q}_{ext})} \|u\|_{L^2(\mathcal{Q}_{ext})} + \varepsilon^{-\gamma} \|f\|_{L^1(\mathcal{Q}_{ext})}$$

$$+ C_1 \int\limits_{\mathcal{Q}_{ext}} g + 2C_2 \int\limits_{\mathcal{Q}_{ext}} g.$$

By choosing $\overline{\varepsilon} = \lambda/(2\Lambda)$ we get the desired estimate. □

We observe that assumption $b \in (L^\infty_{loc}(\Omega))^{m_0}$ was necessary to obtain the L^1-norm of g on the right-hand side of inequality (4.5.4). The presence of such a norm in the aforementioned inequality plays a crucial role in the proof of the upcoming Lemma 4.5.8.

Remark 4.5.7. A cut-off function satisfying the requirements stated in (4.5.5) exists. Indeed, we may for example consider a family of cut-off functions of the form

$$\chi^i_{r_i,R_i}(s) = \begin{cases} 0, & \text{if } s \geq R_i, \\ 1, & \text{if } 0 \leq s \leq r_i, \end{cases} \qquad |\chi'_{r_i,R_i}| \leq \frac{2}{R_i - r_i}, \quad i = 1, \ldots, \kappa,$$
$$\chi_{t_0,t_1}(s) = \begin{cases} 0, & \text{if } s \leq t_0, \text{ or } t \geq T_0, \\ 1, & \text{if } t_1 \leq s \leq T_1, \end{cases} \qquad |\chi'_{t_0,t_1}| \leq \max\left\{ \frac{2}{t_1 - t_0}, \frac{2}{T_0 - T_1} \right\},$$

with $0 < r_i < R_i$ for every $i = 1, \ldots, \kappa$ and $t_0 < t_1 < T_1 < T_0$. Then, we define

$$\psi(x,t) = \chi_{t_0,t_1}(t) \prod_{i=0}^{\kappa} \chi^i_{r_i,R_i}(\|x^{(i)}\|),$$

where the norm we consider is the Euclidean one. Eventually, the following estimates hold:

$$|\partial_{x_j}\psi| \leq \frac{c_0}{R_0 - r_0} \text{ for } j = 1, \ldots, m_0 \qquad |Y\psi| \leq \sum_{i=0}^{\kappa} \frac{c_i}{R_i - r_i} + \max\left\{ \frac{2}{t_1 - t_0}, \frac{2}{T_0 - T_1} \right\}.$$

Now, we are in a position to study how equation (4.1.1) spreads positivity of (super-)solutions. More precisely, we state the upcoming Lemma 4.5.8 in terms of the cylinders

$$\begin{aligned} \mathcal{Q}_{pos} &= B_\theta \times B_{\theta^3} \times \ldots \times B_{\theta^{2\kappa+1}} \times (-1-\theta^2, -1], \\ \widetilde{\mathcal{Q}}_{ext} &= B_{3R} \times B_{3^3 R} \times \ldots \times B_{3^{2\kappa+1}R} \times (-1-\theta^2, 0], \end{aligned} \tag{4.5.6}$$

where $R = R(\theta, Q, \lambda, \Lambda)$ is the constant given by Lemma 4.4.4 and $\theta \in (0,1]$ is a parameter we will choose later on. In particular, θ will be chosen such that the stacked cylinder $\overline{\mathcal{Q}}^m_{pos}$ (see definition (4.A.1)) is contained in $\mathcal{Q}_1$. To stress the dependence of R on θ, we will sometimes write R_θ instead of R.

Lemma 4.5.8. *Let $\theta \in (0,1]$ and $\mathcal{Q}_{pos}$, $\widetilde{\mathcal{Q}}_{ext}$ be the cylinders defined in* (4.5.6)*. Then there exist a small positive constant $\eta_0 = \eta_0(\theta, Q, \lambda, \Lambda) \in (0,1)$ such that for any non-negative weak solution u of* (4.1.1) *under assumptions **(H1)-(H2)-(H3)** in some cylindrical open set $\Omega \supset \widetilde{\mathcal{Q}}_{ext}$ such that*

$$|\{u \geq 1\} \cap \mathcal{Q}_{pos}| \geq \frac{1}{2} |\mathcal{Q}_{pos}|, \tag{4.5.7}$$

we have $u \geq \eta_0$ in $\mathcal{Q}_1$. If, additionally, $c \geq 0$ the statement holds true for non-negative weak super-solutions to (4.1.1).

Proof. Let us consider $g = G(u + \varepsilon^\gamma)$ for $\varepsilon \in \left]0, \frac{1}{4}\right]$ and $\gamma = \frac{1}{8}$. By Definition 4.5.4, g is non-negative and a sub-solution to (4.5.2). Since G is non-increasing and $\varepsilon \in \left]0, \frac{1}{4}\right]$ we also have $g \leq G(\varepsilon^{\frac{1}{8}}) \leq G(\varepsilon)$. Now, let us introduce two parameters $\eta \in (0, \frac{\theta}{2})$ and $\iota > 0$, both depending on a parameter θ we will choose later on.

The idea of the proof is to apply Theorem 4.3.1 to the function g (with $\mathcal{Q}_1$ as the small cylinder and $\mathcal{Q}_{1+\iota}$ as the big cylinder with an accurate choice of ι) combined with Theorem 4.4.1 scaled on the cylinder $\mathcal{Q}_{1+\iota} = \delta_{1+\iota}(\mathcal{Q}_{ext})$. Finally, we estimate the L^2-norm of the diffusive gradient of g via the square root of the square of its mass on a larger cylinder.

First of all, we need to show that we can choose ι small enough such that $\mathcal{Q}_{ext} \subset \mathcal{Q}_{1+\iota} \subset \delta_{(1+\iota)^2}\mathcal{Q}_{ext} \subset \widetilde{\mathcal{Q}}_{ext}$, where $\mathcal{Q}_{ext}$ is defined in (4.4.1). In particular, ι needs to be chosen such that

$$(1+\iota)^4(1+\eta^2) \leq 1+\theta^2 \quad \text{and} \quad 2^{2j+1}(1+\iota)^{2(2j+1)} \leq 3^{2j+1} \qquad \text{for } j = 1, \ldots, \kappa,$$

where the above inequalities take into consideration the scaling introduced in (1.1.13), (4.4.1) and (4.5.6). In particular, this holds true if we assume $\iota = \min\left\{\frac{4(1+\theta^2)}{4+\theta^2} - 1, \left(\frac{3}{2}\right)^{\frac{1}{2}} - 1\right\}$. Moreover, by recalling the definition of $\mathcal{Q}_{zero}$, see (4.4.1), we then pick $\eta \in (0,1)$ such that

$$|\mathcal{Q}_{pos} \setminus \delta_{(1+\iota)}(\mathcal{Q}_{zero})| \geq \frac{1}{4}|\delta_{(1+\iota)}(\mathcal{Q}_{zero})|.$$

Thanks to the geometry underlying operator $\mathscr{L}$, it is enough to pick $\eta = (2\kappa+1)^{-1/(2\kappa+1)}(1+\iota)^{-1}\theta$. Combining this fact with our assumption (4.5.7) we obtain

$$|\{u \geq 1\} \cap \delta_{(1+\iota)}(\mathcal{Q}_{zero})| \geq \frac{1}{4}|\delta_{(1+\iota)}(\mathcal{Q}_{zero})|. \tag{4.5.8}$$

Now, we want to check if we retrieve enough information on the function g. In particular, the set in which $g = 0$ needs to be at least of the same measure of the set where the function $u \geq 1$, i.e. we want to show that the inclusion

$$\{u \geq 1\} \subset \{g = 0\} \tag{4.5.9}$$

holds true. The function $g = G(u + \varepsilon^{\frac{1}{8}})$, by definition, is equal to zero if

$$u + \varepsilon^{\frac{1}{8}} > 1 \quad \Longleftrightarrow \quad u > 1 - \varepsilon^{\frac{1}{8}}.$$

Inclusion (4.5.9) follows immediately from the previous inequality observing that $1 - \varepsilon^{\frac{1}{8}} < 1$

for every $\varepsilon \in]0, \frac{1}{4}]$. Hence, equation (4.5.8) and inclusion (4.5.9) imply that

$$|\{g = 0\} \cap \delta_{(1+\iota)}(\mathcal{Q}_{zero})| \geq |\{u \geq 1\} \cap \delta_{(1+\iota)}(\mathcal{Q}_{zero})| \geq \frac{1}{4}|\delta_{(1+\iota)}(\mathcal{Q}_{zero})| \tag{4.5.10}$$

Thus, the function g satisfies inequality (4.4.3) with $\delta_{(1+\iota)}(\mathcal{Q}_{zero})$ taking the role of $\mathcal{Q}_{zero}$ and we are therefore allowed to apply Theorem 4.4.1.

Now, our aim is to estimate $g - \theta_0 G(\varepsilon)$, where $\theta_0 \in (0,1)$ is the constant given in Theorem 4.4.1 only depending on η and θ (since the dimension of the cylinders involved in our analysis depends on them). We now want to apply Theorem 4.3.1 to $(g - \theta_0 G(\varepsilon))_+$. To this end, we first observe $(g - \theta_0 G(\varepsilon))_+$ is still a non-negative sub-solution to (4.5.2) and

$$g - \theta_0 G(\varepsilon) \leq (g - \theta_0 G(\varepsilon))_+ \leq \sup_{\mathcal{Q}_1} (g - \theta_0 G(\varepsilon))_+.$$

We are not yet in a position to apply Theorem 4.3.1 to $(g-\theta_0 G(\varepsilon))_+$, as this result was derived for non-negative weak solutions to equation (4.1.1), where in particular the term $-\lambda |D_{m_0} u|^2$ does not appear. For this reason, we briefly explain how to take care of the additional term in the proof of the Sobolev-type and Caccioppoli-type inequalities presented in Section 4.3. In particular, in the proof of Theorem 4.3.3, we have additionally

$$\lambda \int_{\mathcal{Q}_r} [\langle D_{m_0} v, D_{m_0} v \rangle \Gamma(z, \cdot)\psi](\zeta) d\zeta,$$

which clearly satisfies

$$\lambda \int_{\mathcal{Q}_r} [\langle D_{m_0} v, D_{m_0} v \rangle \Gamma(z, \cdot)\psi](\zeta) d\zeta \leq C \| D_{m_0} v \|^2_{L^2(\mathcal{Q}_r)},$$

and can be treated like I_2. On the other hand, in the proof of Theorem 4.3.4, we get an additionally term of the form

$$\frac{\lambda}{p} \int_{\mathcal{Q}_r} |D_{m_0} u_l^p|^2 \psi^2$$

on the left-hand side of (4.3.7). By choosing $\varepsilon_1 = 2\lambda/\Lambda$, we get exactly the inequality stated in Theorem 4.3.4.

Eventually, we are able to provide the following chain of estimates

$$\begin{aligned}
\sup_{\mathcal{Q}_1} (g - \theta_0 G(\varepsilon))_+ &\leq C_\iota C_M \parallel (g - \theta_0 G(\varepsilon))_+ \parallel_{L^\beta(\mathcal{Q}_{1+\iota})} \\
&\quad + C_\iota C_M \varepsilon^{-\frac{1}{8}} \parallel cu - f \|_{L^q(\mathcal{Q}_{1+\iota})} && \text{Theorem 4.3.1} \\
&\leq C_\iota C_M \parallel (g - \theta_0 G(\varepsilon))_+ \parallel_{L^2(\mathcal{Q}_{1+\iota})} \\
&\quad + C_\iota C_M \varepsilon^{-\frac{1}{8}} \parallel cu - f \|_{L^q(\mathcal{Q}_{1+\iota})} && \beta := \frac{q}{q-1} < 2 \\
&\leq C_P C_\iota C_M \| D_{m_0} (g - \theta_0 G(\varepsilon))_+ \|_{L^2(\mathcal{Q}_{ext})} && \text{Theorem 4.4.1} \\
&\quad + C_P C_\iota C_M \| Y (g - \theta_0 G(\varepsilon))_+ \|_{L^2 H^{-1}(\mathcal{Q}_{ext})}
\end{aligned}$$

$$
\begin{aligned}
&\quad + C_\iota C_M \varepsilon^{-\frac{1}{8}} \, \| \, cu - f\|_{L^q(\mathcal{Q}_{1+\iota})} \\
&\leq C \left(C_P C_\iota C_M C_G\right)^{\frac{1}{2}} \left(\int_{\mathcal{Q}_{ext}} g \right)^{\frac{1}{2}} && \text{Lemma 4.5.6} \\
&\quad + C \left(C_P C_\iota C_M C_G\right)^{\frac{1}{2}} \varepsilon^{-\frac{1}{16}} \left(\|c\|_{L^2(\mathcal{Q}_{ext})} \|u\|_{L^2(\mathcal{Q}_{ext})} + \|f\|_{L^1(\mathcal{Q}_{ext})} \right)^{\frac{1}{2}} \\
&\quad + C_\iota C_M \varepsilon^{-\frac{1}{8}} \, \| \, cu - f\|_{L^q(\mathcal{Q}_{1+\iota})} \\
&\leq C_1 \sqrt{G(\varepsilon)} + C_2 \varepsilon^{-\frac{1}{8}} && G \text{ is non-increasing}
\end{aligned}
$$

where C_ι is a constant due to the scaling on the cylinder $\mathcal{Q}_{(1+\iota)}$ and, as C_1, C_2, only depends on Q, Λ, λ, $\|b\|_{L^\infty(\mathcal{Q}_{ext})}$, $\|c\|^2_{L^2(\mathcal{Q}_{ext})}$, $\|f\|_{L^q(\mathcal{Q}_{ext})}$ and ι and we used inequality $\varepsilon^{-\frac{1}{16}} \leq \varepsilon^{-\frac{1}{8}}$ for ε sufficiently small.

Thus, we obtain the following inequality

$$
G(u + \varepsilon^{\frac{1}{8}}) - \theta_0 G(\varepsilon) \leq C_1 \sqrt{G(\varepsilon)} + C_2 \varepsilon^{-\frac{1}{8}} \qquad \text{in } \mathcal{Q}_1.
$$

As $G(\varepsilon) \sim -\ln(\varepsilon)$ as $\varepsilon \to 0^+$ and $\varepsilon^{-\frac{1}{8}} \leq \sqrt{-\ln(\varepsilon)}$ for every ε sufficiently small (for instance, $\varepsilon \in]0, \frac{1}{5}]$), we have

$$
G(u + \varepsilon^{\frac{1}{8}}) \leq \theta_0 G(\varepsilon) + C \sqrt{G(\varepsilon)}, \tag{4.5.11}
$$

where $C = C(C_1, C_2)$ is a constant that does not depend on ε.
Since $G(\varepsilon) \to +\infty$ as $\varepsilon \to 0^+$, there exists $\bar{\varepsilon} \in]0, \frac{1}{5}]$ such that for every $\varepsilon \in]0, \bar{\varepsilon}]$ we have

$$
\theta_0 G(\varepsilon) + C \sqrt{G(\varepsilon)} \leq \frac{1 + 7\theta_0}{8} \, G(\varepsilon). \tag{4.5.12}
$$

Indeed, we can rewrite (4.5.12) as follows

$$
C \sqrt{G(\varepsilon)} \leq \frac{1 - \theta_0}{8} \, G(\varepsilon). \tag{4.5.13}
$$

As both sides of (4.5.13) are positive, we can raise the square and get

$$
C^2 G(\varepsilon) \leq \left(\frac{1 - \theta_0}{8} \right)^2 (G(\varepsilon))^2.
$$

Hence, dividing by $G(\varepsilon)$, we infer

$$
C^2 \leq \left(\frac{1 - \theta_0}{8} \right)^2 G(\varepsilon).
$$

In order to obtain (4.5.12), it is therefore sufficient to choose ε small enough such that

$$G(\varepsilon) \geq \left(\frac{8C}{1-\theta_0}\right)^2.$$

Combining (4.5.11) and (4.5.12), we finally get

$$g = G(u + \varepsilon^{\frac{1}{8}}) \leq \frac{1+7\theta_0}{8} G(\varepsilon).$$

Taking advantage again of $G(\varepsilon) \sim -\ln(\varepsilon)$ for $\varepsilon \to 0^+$, for every ε sufficiently small the previous inequality implies

$$\ln(u + \varepsilon^{\frac{1}{8}}) \geq \frac{1+7\theta_0}{8} \ln(\varepsilon).$$

Hence, we for every ε sufficiently small we infer

$$u + \varepsilon^{\frac{1}{8}} \geq \varepsilon^{\frac{1+7\theta_0}{8}} \qquad \Longrightarrow \qquad u \geq \varepsilon^{\frac{1+7\theta_0}{8}} - \varepsilon^{\frac{1}{8}} =: \eta_0.$$

We remark that $\eta_0 > 0$ for every ε sufficiently small, since $\frac{1+7\theta_0}{8} < \frac{1}{8}$ for $\theta_0 \in (0,1)$. This concludes the proof. □

Remark 4.5.9. We observe that our function $g = G(u + \varepsilon^{\frac{1}{8}})$ differs from the one considered in [51, 118]. This is due to the fact that in the proof of the weak Harnack inequality below we cannot reduce to the case where the right-hand side is equal to zero. This condition is imposed by our weaker integrability assumption on the right-hand side and lower order coefficients, namely $c, f \in L^q$ with q possibly less than $+\infty$.

As a straightforward consequence of Lemma 4.5.8 we have the following result, which is the extension of [51, Lemma 4.2] to our case.

Lemma 4.5.10. *Let $m \geq 3$ and let R be the constant given in Lemma 4.5.8 for $\theta = m^{-1/2}$. Then there exists a constant $M = M(m, Q, \lambda, \Lambda) > 1$ such that for any non-negative solution u to (4.1.1) under assumptions **(H1)-(H2)-(H3)** satisfying $|\{u \geq M\} \cap \mathcal{Q}_1| \geq \frac{1}{2}|\mathcal{Q}_1|$, we have $u \geq 1$ in $\overline{\mathcal{Q}}_1^m$ (see (4.A.1)). If, additionally, $c \geq 0$ the statement holds true for non-negative weak super-solutions to (4.1.1).*

Proof. Since $\theta = m^{-1/2}$, we have that $\overline{\mathcal{Q}}_1^m \subset B_{\theta^{-1}} \times B_{\theta^{-3}} \times \ldots \times B_{\theta^{-2\kappa-1}} \times (0, \theta^{-2}]$, see (4.A.1). We observe that u/M is a non-negative weak solution to (4.1.1) with right-hand side equal to f/M and satisfying assumption (4.5.7). Therefore we are in a position to apply Lemma 4.5.8 to this function. In this case $\mathcal{Q}_1$ and $B_{\theta^{-1}} \times B_{\theta^{-3}} \times \ldots \times B_{\theta^{-2\kappa-1}} \times (0, \theta^{-2}]$ take the roles of the cylinders $\mathcal{Q}_{pos}$ and $\mathcal{Q}_1$, thanks to a rescaling argument. Hence, we obtain inequality $u \geq \eta_0 M$ in $B_{\theta^{-1}} \times B_{\theta^{-3}} \times \ldots \times B_{\theta^{-2\kappa-1}} \times (0, \theta^{-2}]$ and therefore in $\overline{\mathcal{Q}}_1^m$. Thus, to conclude the proof it is sufficient to choose $M = 1/\eta_0 > 1$. □

Before proving the weak Harnack inequality, we need to show that we can spread positivity along "suitable" cylinders. More precisely, recalling the definition of the open ball in (4.1.11), we set

$$\mathcal{Q}_+ = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-\omega^2, 0], \quad \mathcal{Q}_- = B_\omega \times B_{\omega^3} \times \ldots \times B_{\omega^{2\kappa+1}} \times (-1, -1+\omega^2], \tag{4.5.14}$$

where ω is a small positive constant. In particular, we will choose ω small enough so that, when expanding positivity from a given cylinder $\mathcal{Q}_r(z_0)$ in the past, the union of the stacked cylinders where the positivity is spread includes $\mathcal{Q}_+$. Moreover, we will choose the radius R_0 in the statement of Theorem 4.5.1 so that Lemma 4.5.8 can be applied to every stacked cylinder. The two previous statements are specified in Lemma 4.B.1. The staking cylinders Lemma 4.B.1, combined with Lemma 4.5.8, implies the following result regarding the expansion of positivity for large times.

In the sequel, we will largely use the cylinders $\mathcal{Q}_r[k]$, for $k = 1, \ldots, N$ and $\mathcal{Q}_{R_{N+1}}[N+1]$, whose definition and properties are presented in Appendix 4.B.

Lemma 4.5.11. *Let $R_{1/2}$ be the constant given by Lemma 4.5.8 for $\theta = 1/2$ and let u be any non-negative solution to* (4.1.1) *in $\Omega \supset \mathcal{Q}$ under assumptions **(H1)-(H2)-(H3)** such that $|\{u \geq M\} \cap \mathcal{Q}_r(z_0)| \geq \frac{1}{2} |\mathcal{Q}_r(z_0)|$ for some $M > 0$ and for some cylinder $\mathcal{Q}_r(z_0) \subset \mathcal{Q}_-$. Then there exists a positive constant p_0, only depending on Q, λ, Λ, such that*

$$u \geq M \left(\frac{r^2}{4} \right)^{p_0}, \quad \text{in } \mathcal{Q}_+. \tag{4.5.15}$$

If, additionally, $c \geq 0$ the statement holds true for non-negative weak super-solutions to (4.1.1)*.*

Proof. We apply Lemma 4.5.8 for $\theta = \frac{1}{2}$ to the function u/M, with $\mathcal{Q}_r(z_0)$ and $\mathcal{Q}_r[1]$ taking the role of $\mathcal{Q}_{pos}$ and $\mathcal{Q}_1$ (this is achieved through a rescaling argument) and obtain $u/M \geq \eta_0$ in $\mathcal{Q}_r[1]$. We then apply it to $u/(M\eta_0)$ and get $u \geq M\eta_0^2$ in $\mathcal{Q}_r[2]$. Reasoning by induction on $k = 1, \ldots, N$ we infer $u \geq M\eta_0^k$ in $\mathcal{Q}[k]$.

By exploiting Lemma 4.5.8 again, we get $u \geq M\eta_0^{N+1}$ in $\mathcal{Q}_{R_{N+1}}[N+1]$, which implies that the same inequality holds true in $\mathcal{Q}_+$. As $T_N \leq -t_0 < 1$, we have in particular $4^N r^2 \leq 1$. Picking $p_0 > 0$ so that $\eta_0 = \left(\left(\frac{1}{4}\right)^{\frac{N}{N+1}} \right)^{p_0}$, we finally obtain

$$u \geq M \left(\left((1/4)^{\frac{N}{N+1}} \right)^{N+1} \right)^{p_0} = M \left((1/4)^N \right)^{p_0} \geq M \left(r^2/4 \right)^{p_0},$$

which concludes the proof. □

From now on we will assume $\omega < 1/10^2$, where κ is defined in (4.1.6). We are in a position to prove the main result of this Section, Theorem 4.5.1.

Proof of Theorem 4.5.1. We start the proof by fixing the parameters ω and R_0 in order to select the appropriate geometric setting. More precisely, we choose ω so that we capture

$\mathcal{Q}_+$ when applying Lemma 4.B.1, namely we fix $\omega < \frac{1}{10^2}$. In addition, we choose the radius R_0 so that the stacked cylinders do not leak out of $\mathcal{Q}^0$, i.e. $R_0 \geq 6\,(2\kappa+1)\,R_{1/2}$, where $R_{1/2}$ is the constant given by Lemma 4.5.8 when $\theta = 1/2$. As we want to apply Lemma 4.5.10 to cylinders contained in $\mathcal{Q}_-$, we also assume

$$R_0 \geq 3(2\kappa+1) R_{m^{-1/2}} m^{(2\kappa+1)/2} \omega^{2\kappa+1}, \tag{4.5.16}$$

where $R_{m^{-1/2}}$ is the constant given by Lemma 4.5.8 for $\theta = m^{-1/2}$.

We first restrict ourselves to the case where

$$\inf_{\mathcal{Q}_+} u \leq 1. \tag{4.5.17}$$

Indeed, if $\inf_{\mathcal{Q}_+} u > 1$ we can simply consider $\bar{u} = u / \left(\inf_{\mathcal{Q}_+} u + 1\right)$ and reduce to the case where $\inf_{\mathcal{Q}_+} u \leq 1$.

We now want to prove that for all $k \in \mathbb{N}$, the following inequality holds

$$\left|\{u > M^k\} \cap \mathcal{Q}_1\right| \leq \widetilde{C}(1-\widetilde{\mu})^k, \tag{4.5.18}$$

for some constants $\widetilde{\mu} \in (0,1)$ $M > 1$ and $\widetilde{C} > 0$ that only depend on Q, λ and Λ. The proof of this fact is carried out by induction. For $k = 1$ it is sufficient to choose $\widetilde{\mu} \leq \frac{1}{2}$ and $\widetilde{C}$ such that $|\mathcal{Q}_-| \leq \frac{1}{2}\widetilde{C}$. We now assume that (4.5.18) holds true for $k \geq 1$ and we prove it for $k+1$. To this end, we consider the sets

$$E := \{u > M^{k+1}\} \cap \mathcal{Q}_-, \qquad F := \{u > M^k\} \cap \mathcal{Q}_1. \tag{4.5.19}$$

We observe that E and F satisfy the assumptions of Corollary 4.A.3 with $\mathcal{Q}_1$ replaced by $\mathcal{Q}_-$ and $\mu = 1/2$. Indeed, by definition E and F are bounded measurable sets such that $E \subset \mathcal{Q}_- \cap F$. We now consider a cylinder $\mathcal{Q} = \mathcal{Q}_r(z) \subset \mathcal{Q}_-$ such that $|\mathcal{Q} \cap E| > \frac{1}{2}|\mathcal{Q}|$, i.e.

$$|\{u > M^{k+1}\} \cap \mathcal{Q}| > \frac{1}{2}|\mathcal{Q}|.$$

We show that r needs to be small, that is to say r is less than some parameter $r_0 = r_0(Q, \lambda, \Lambda, k)$. Indeed, applying Lemma 4.5.11 to u, we obtain $u \geq M^{k+1}(r^2/4)^{p_0}$ in $\mathcal{Q}_+$. Thus, owing to $\inf_{\mathcal{Q}_+} u \leq 1$, we infer $1 \geq M^{k+1}(r^2/4)^{p_0}$ and therefore it is sufficient to choose $r_0 \leq 2M^{-k-1/2p_0}$. In order to apply Corollary 4.A.3, we are left with proving that $\overline{\mathcal{Q}}^m \subset F$, which holds true if $\overline{\mathcal{Q}}^m \subset \{u > M^k\}$. To this end, we apply Lemma 4.5.10 to u/M^k after rescaling the cylinder $\mathcal{Q}$ in $\mathcal{Q}_1$.

In virtue of Corollary 4.A.3, there exist $c_{\mathrm{is}} \in (0,1)$ and $C_{\mathrm{is}} > 0$ such that

$$\begin{aligned}
|E| = \left|\{u > M^{k+1}\} \cap \mathcal{Q}_-\right| &\leq \frac{m+1}{m}\left(1 - \frac{c_{\mathrm{is}}}{2}\right)\left(\left|\{u > M^k\} \cap \mathcal{Q}_1\right| + C_{\mathrm{is}} m r_0^2\right) \\
&\leq \left(1 - \frac{c_{\mathrm{is}}}{4}\right)\left(\left|\{u > M^k\} \cap \mathcal{Q}_1\right| + C_{\mathrm{is}} m r_0^2\right),
\end{aligned}$$

provided that we chose $m \in \mathbb{N}$ so that $\frac{m+1}{m}\left(1 - \frac{c_{\text{is}}}{2}\right) \leq 1 - \frac{c_{\text{is}}}{4}$. Thanks to the induction assumption and our choice of r_0 we get

$$\begin{aligned} |E| &\leq \left(1 - \frac{c_{\text{is}}}{4}\right)\left(\widetilde{C}(1-\widetilde{\mu})^k + C_{\text{is}} m r_0^2\right) \\ &\leq \left(1 - \frac{c_{\text{is}}}{4}\right)\left(\widetilde{C}(1-\widetilde{\mu})^k + C_{\text{is}} m M^{-\frac{k+1}{p_0}}\right). \end{aligned}$$

Picking then $\widetilde{\mu}$ small enough so that $M^{-1/p_0} \leq (1-\widetilde{\mu})$ and $\widetilde{\mu} \leq \frac{c_{\text{is}}}{4}$, we obtain

$$\begin{aligned} |E| &\leq \widetilde{C}\left(1 - \frac{c_{\text{is}}}{4}\right)(1-\widetilde{\mu})^k\left(1 + \widetilde{C}^{-1} m M^{-\frac{1}{p_0}}\right) \\ &\leq \widetilde{C}(1-\widetilde{\mu})^{k+1}\left(1 + \widetilde{C}^{-1} m M^{-\frac{1}{p_0}}\right). \end{aligned}$$

Picking $\widetilde{C}$ large enough so that $\left(1 + \widetilde{C}^{-1} m M^{-\frac{1}{p_0}}\right) \leq 2$ we conclude the proof of (4.5.18). By extending estimate (4.5.18) to the continuous case (i.e. $k \in \mathbb{R}$ and $k \geq 1$) and applying the layer cake formula to $\int_{\mathcal{Q}_-} u^p$ for some exponent p, we obtain that $\int_{\mathcal{Q}_-} u^p$ is bounded from above by a constant that only depends on Q, λ and Λ. □

4.6 Proof of main results

Proof of Theorem 4.1.3. The full Harnack inequality is a direct consequence of the combination of the local boundedness of weak sub-solutions proved in Theorem 4.3.1 and the weak Harnack inequality of Theorem 4.5.1. The only delicate point in this proof is given by the size of the cylinders we consider. Indeed, when applying Theorem 4.3.1, one needs to consider a cylinder $\mathcal{Q}_\rho$ that is fully contained in $\mathcal{Q}_-$ introduced in the statement of Theorem 4.5.1, which plays the role of $\mathcal{Q}_r$ in Theorem 4.3.1. This is the reason behind the peculiar choice of the cylinder $\widetilde{\mathcal{Q}}_-$ of the statement of the Harnack inequality. □

Proof of Theorem 4.1.4. The Hölder continuity of weak solutions is classically obtained by proving that the oscillation of the solution decays by a universal factor. This can be achieved in two different ways. Either by applying Lemma 4.5.8 with $\theta = 1$ in the same spirit of [51, Appendix B], or by directly applying the weak Harnack inequality, Theorem 4.5.1, following a standard argument, for further reference see [48]. □

Appendix

4.A The Ink-Spots Theorem

For the sake of completeness, we provide here the proof of the Ink Spots Theorem for the case of ultraparabolic equations. This theorem involves a covering argument in the spirit of Krylov and Safonov [70] growing Ink Spots theorem, or the Calderón-Zygmund decomposition, and it is a fundamental ingredient for the proof of the weak Harnack inequality contained in this Chapter (see Theorem 4.5.1). In order to give its statement in our setting, we introduce the delayed cylinder

$$\overline{\mathcal{Q}}_r^m(z_0) = \big((0,\dots,0,mr^2)\circ \mathcal{Q}_r(z_0)\big) \cap \big(\mathbb{R}^{N+1}\times(t_0,+\infty)\big) \tag{4.A.1}$$

where $z_0=(x_0,t_0)=(x_0^{(0)},\dots,x_0^{(\kappa)},t_0)\in\mathbb{R}^{N+1}$. We remark that $\overline{\mathcal{Q}}_r^m(z_0)$ starts immediately at the end of $\mathcal{Q}_r(z_0)$, with which it shares the same values for $x^{(0)}$, and its structure follows the non-Euclidean geometry presented in Subsection 1.1.1 associated to the principal part operator $\mathscr{L}_0$. The aim of this appendix is to prove the following statement.

Theorem 4.A.1. *Let $E\subset F$ be two bounded measurable sets. We assume there exists a constant $\mu\in(0,1)$ such that*

- *$E\subset\mathcal{Q}_1$ and $|E|<(1-\mu)|\mathcal{Q}_1|$;*
- *moreover, there exist an integer m such that for any cylinder $\mathcal{Q}\subset\mathcal{Q}_1$ such that $\overline{\mathcal{Q}}^m\subset\mathcal{Q}_1$ and $|\mathcal{Q}\cap E|\geq(1-\mu)|\mathcal{Q}|$, we have that $\overline{\mathcal{Q}}^m\subset F$.*

Then for some universal constant $c_{\text{is}}\in(0,1)$ only depending on N, there holds

$$|E|\leq\frac{m+1}{m}(1-c_{\text{is}}\mu)|F|.$$

Remark 4.A.2. Theorem 4.A.1 still holds true if we replace $\mathcal{Q}_1$ with $\mathcal{Q}_-$ defined in (4.5.14).

As it has already been pointed out by Imbert and Silvestre in [61], there is no chance to adapt the Calderón-Zygmund decomposition to this context, because it would require to split a larger piece into smaller ones of the same type and this is impossible due to the non Euclidean nature of our geometry. What we do is a generalization of the procedure proposed in [61], that is in fact an adaptation of the growing Ink Spots theorem, whose original construction in the parabolic case dates back to Krylov and Safonov [70, Appendix A].

Moreover, when we need to confine both E and F to stay within a fixed cylinder, the following corollary directly follows.

Corollary 4.A.3. *Let $E \subset F$ be two bounded measurable sets. We assume*

- $E \subset \mathcal{Q}_1$*;*
- *there exist two constants $\mu, r_0 \in (0,1)$ and an integer m such that for any cylinder $\mathcal{Q} \subset \mathcal{Q}_1$ of the form $Q_r(z_0)$ such that $|\mathcal{Q} \cap E| \geq (1-\mu)|\mathcal{Q}|$, we have $\overline{\mathcal{Q}}^m \subset F$ and also $r < r_0$.*

Then for some universal constants c_{is} and C_{is} only depending on N

$$|E| \leq \frac{m+1}{m}(1-c_{\text{is}}\mu)\left(|F \cap \mathcal{Q}_1| + C_{\text{is}} m r_0^2\right).$$

4.A.1 Stacked cylinders

First of all we recall some important properties of the following family of stacked cylinders

$$k\mathcal{Q}_r = \left(0,\dots,0,\frac{k^2-1}{2}r^2\right) \circ \mathcal{Q}_{kr} \quad \text{and} \quad k\mathcal{Q}_r(x_0,t_0) = \left(0,\dots,0,\frac{k^2-1}{2}r^2\right) \circ \mathcal{Q}_{kr}(x_0,t_0),$$

where $(x_0,t_0) \in \mathbb{R}^{N+1}$, that are defined starting from the unit cylinder (4.1.10) for a certain $k > 0$. By definition, it is clear that $|k\mathcal{Q}_r(x_0,t_0)| = k^{Q+2}|\mathcal{Q}_r(x_0,t_0)|$, and that the cylinders $\mathcal{Q}_r(x_0,t_0)$ are not the balls of any metric. Thus, the important properties of the cylinders are explicitly given by the following lemmata.

Lemma 4.A.4. *Let $\mathcal{Q}_{r_0}(x_0,t_0)$ and $\mathcal{Q}_{r_1}(x_1,t_1)$ be two cylinders with non empty intersection, with $(x_0,t_0),(x_1,t_1) \in \mathbb{R}^{N+1}$ and $2r_0 \geq r_1 > 0$. Then*

$$\mathcal{Q}_{r_1}(x_1,t_1) \subset k\mathcal{Q}_{r_0}(x_0,t_0)$$

for some universal constant k.

Proof. Without loss of generality, we may assume $(x_0,t_0) = (0,0)$. Then we need to choose the constant k in order to satisfy our statement. In particular, if we consider the ball associated to the first m_0 variables we get that $B_{r_1}(x_1^{(0)}) \subset B_{kr_0}$ if

$$kr_0 \geq r_0 + 2r_1 \quad \Longrightarrow \quad k \geq 5.$$

By repeating the same argument for all the κ blocks of variables, we get that k must satisfy the following conditions:

$$k^{2j+1} \geq 1 + 2 \cdot 2^{2j+1} \qquad \text{for } j = 0,\dots,\kappa.$$

As far as we are concerned with the condition regarding the time interval, we need k to be such that
$$-\frac{k^2+1}{2}r_0^2 \le -r_0 - 2r_1^2 \quad \Longrightarrow \quad k^2 \ge 9.$$
All of these inequalities are satisfied when the first one, i.e. the one corresponding to $j = 0$, is satisfied. We choose k to be the smallest parameter satisfying these inequalities. □

Lemma 4.A.5. *Let $\{\mathcal{Q}_j\}_{j\in J}$ be an arbitrary collection of slanted cylinders with bounded radius. Then there exists a disjoint countable subcollection $\{\mathcal{Q}_{j_i}\}_{i\in I}$ such that*

$$\bigcup_{j\in J} \mathcal{Q}_j = \bigcup_{i=1}^{\infty} k\mathcal{Q}_{j_i}.$$

The proof of Lemma 4.A.5 is the same as the classical proof of the Vitali covering lemma, where we employ Lemma 4.A.4 instead of the fact that in any metric space $B_{r_1}(x_1) \subset 5B_{r_0}$, if $B_{r_1}(x_1) \cap B_{r_0} \neq \emptyset$ and $r_1 \le 2r_0$.

4.A.2 A generalized Lebesgue differentiation theorem

For the reader's convenience, we also recall the definition of maximal function:

$$Mf(x,t) = \sup_{\mathcal{Q}:(x,t)\in\mathcal{Q}} \frac{1}{|\mathcal{Q}|} \int_{\mathcal{Q}\cap\Omega} |f(y,s)|\, dy\, ds,$$

where the supremum is taken over cylinders of the form $(y,s) + R\mathcal{Q}_1$.

Lemma 4.A.6. *For every $\lambda > 0$ and $f \in L^1(\Omega)$, we have*

$$|\{Mf < \lambda\} \cap \Omega| \le \frac{C}{\lambda}\|f\|_{L^1(\Omega)}.$$

Proof. Let us consider $(x,t) \in \{Mf < \lambda\} \cap \Omega$. Then there exists a cylinder $\mathcal{Q}$ such that $(x,t) \in \mathcal{Q}$ and
$$\int_{\mathcal{Q}\cap\Omega} |f(y,s)|\, dy\, ds \ge \frac{\lambda}{2}|\mathcal{Q}\cap\Omega|.$$

Then $\{Mf < \lambda\} \cap \Omega$ is covered with cylinders $\{\mathcal{Q}_j\}$ such that the previous inequality holds. From Lemma 4.A.5, there exists a disjoint countable subcollection $\{Q_{j_i}\}$ so that

$$\{Mf < \lambda\} \cap \Omega = \bigcup_{j=1}^{\infty} Q_j \subset \bigcup_{i=1}^{\infty} kQ_{j_i},$$

for some integer k. Thus, we get

$$\|f\|_{L^1(\Omega)} \geq \int\limits_{\Omega \cap \cup_i Q_{j_i}} |f| \geq \frac{\lambda}{2} \sum_{i=1}^{\infty} |\mathcal{Q}_{j_i} \cap \Omega| = \frac{\lambda}{2k^{Q+2}} \Big| \bigcup_{i=1}^{\infty} k\mathcal{Q}_{j_i} \cap \Omega \Big| \geq \frac{\lambda}{2k^{Q+2}} \Big| \{Mf < \lambda\} \cap \Omega \Big|.$$

Thus, the claim is proved when $C = 2k^{Q+2}$. □

The following generalized version of the Lebesgue differentiation theorem holds.

Theorem 4.A.7 (Genaralized Lebesgue Differentiation Theorem)**.** *Let $f \in L^1(\Omega, dx \otimes dt)$, where Ω is an open subset of $\mathbb{R}^{N+1}$. Then for a.e. $(x,t) \in \Omega$*

$$\lim_{r \to 0^+} \frac{1}{|\mathcal{Q}_r(x,t)|} \int\limits_{\mathcal{Q}_r(x,t)} |f(y,s) - f(x,t)| \, dy \, ds = 0.$$

Theorem 4.A.7 is obtained from the following Lemma 4.A.6 exactly as in [57, Theorem 2.5.1] by considering Lemma 4.A.6.

4.A.3 Ink Spots theorem without time delay

Lemma 4.A.8. *Let $E \subset F \subset \mathcal{Q}_1$ be two bounded measurable sets. We make the following assumptions for some constant $\mu \in (0,1)$:*

- *$E < (1-\mu)|\mathcal{Q}_1|$;*
- *if for any cylinder $\mathcal{Q} \subset \mathcal{Q}_1$ such that $|\mathcal{Q} \cap E| \geq (1-\mu)|\mathcal{Q}|$, then $\mathcal{Q} \subset F$.*

Then $|E| \leq (1 - c\mu)|F|$ for some universal constant c only depending on N.

Proof. Thanks to Theorem 4.A.7, for almost all points $z \in E$ there is some cylinder $\mathcal{Q}^z$ containing z such that $|\mathcal{Q}^z \cap E| \geq (1-\mu)|\mathcal{Q}^z|$. Thus, for all Lebesgue points $z \in E$ we choose a maximal cylinder $\mathcal{Q}^z \subset \mathcal{Q}_1$ that contains z and such that $|\mathcal{Q}^z \cap E| \geq (1-\mu)|\mathcal{Q}^z|$. Here $\mathcal{Q}^z = \mathcal{Q}_{\overline{r}}(\overline{x}, \overline{t})$ for some $\overline{r} > 0$ and $(\overline{x}, \overline{t}) \in \mathcal{Q}_1$. In particular, we have that $\mathcal{Q}^z$ differs from $\mathcal{Q}_1$ and $\mathcal{Q}^z \subset F$ by our assumption.

First of all we prove that $|\mathcal{Q}^z \cap E| = (1-\mu)|\mathcal{Q}^z|$. By contradiction, let us suppose that is not true. Then there exists $\delta > 0$ small enough and $\overline{\mathcal{Q}}$ such that $\mathcal{Q}^z \subset \overline{\mathcal{Q}} \subset (1+\delta)\mathcal{Q}^z$, $\overline{\mathcal{Q}} \subset \mathcal{Q}_1$ and $|\overline{\mathcal{Q}} \cap E| > (1-\mu)|\mathcal{Q}^z|$, and this contradicts the maximality of the choice of $\mathcal{Q}^z$.

Then we recall that the family of cylinders $\{\mathcal{Q}^z\}_{z \in E}$ covers the set E. Thanks to Lemma 4.A.5 and considering that E is a bounded set, we can extract a finite subfamily of non overlapping cylinders $\mathcal{Q}_j := \mathcal{Q}^{z_j}$ such that $E \subset \cup_{j=1}^n k\mathcal{Q}_j$. Since $\mathcal{Q}_j \subset F$ and $|\mathcal{Q}_j \cap E| = (1-\mu)|\mathcal{Q}_j|$, we have that $|\mathcal{Q}_j \cap F \setminus E| = \mu|\mathcal{Q}^z|$. Therefore,

$$|F \setminus E| \geq \sum_{j=1}^{n} |\mathcal{Q}_j \cap F \setminus E| \geq \sum_{j=1}^{n} \mu|\mathcal{Q}_j| = k^{-(Q+2)}\mu \sum_{j=1}^{n} |k\mathcal{Q}_j| \geq k^{-(Q+2)}\mu|E|.$$

Thus, we get that $|F| \geq (1+\overline{c}\mu)|E|$, with $\overline{c} = k^{-(Q+2)}$. Since $\overline{c}\mu \in (0,1)$, we complete the proof by choosing $c = \overline{c}/2$. □

4.A.4 Proof of Theorem 4.A.1 and Corollary 4.A.3

In order to proceed with the proof of the Ink Spots Theorem, we first need to recall two preliminary results.

Lemma 4.A.9. *Consider a (possibly infinite) sequence of intervals $(a_j - h_k, a_j]$. Then*

$$\Big|\bigcup_k (a_k, a_k + mh_k]\Big| \geq \frac{m}{m+1}\Big|\bigcup_k (a_k - h_k, a_k]\Big|.$$

The proof of Lemma 4.A.9 can be found in [59, Lemma 10.8]. Here, we only report the proof of the following lemma, that is an extension of Lemma 10.9 [59].

Lemma 4.A.10. *Let $\{\mathcal{Q}_j\}$ be a collection of slanted cylinders and let $\overline{\mathcal{Q}}_j^m$ be the corresponding versions as in* (4.A.1). *Then*

$$\Big|\bigcup_j \overline{\mathcal{Q}}_j^m\Big| \geq \frac{m}{m+1}\Big|\bigcup_j \mathcal{Q}_j\Big|.$$

Proof. Because of Fubini's Theorem we know that for any set $\Omega \subset \mathbb{R}^{N+1}$

$$|\Omega| = \int |\{(x^{(1)}, \dots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t) \in \Omega\}|\, dx^{(0)}.$$

Therefore, in order to prove our statement it is sufficient to show that for every $x^{(0)} \in \mathbb{R}^{m_0}$

$$\Big|\Big\{(x^{(1)}, \dots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t) \in \bigcup_j \overline{\mathcal{Q}}_j^m\Big\}\Big| \\ \geq \frac{m}{m+1}\Big|\Big\{(x^{(1)}, \dots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t) \in \bigcup_j \mathcal{Q}_j\Big\}\Big|.$$

From now on, let us consider a fixed $\overline{x} \in \mathbb{R}^{m_0}$. Any cylinder $\mathcal{Q}_j$ is a cylinder with center

$$(x_j^{(0)}, x_j^{(1)}, \dots, x_j^{(\kappa)}, t_j) \in \mathbb{R}^{N+1}$$

and radius $r_j > 0$. $\overline{\mathcal{Q}}_j^m$ is its delayed version (4.A.1), that thanks to Remark 4.2.5 can equivalently be represented as follows

$$\overline{\mathcal{Q}}_j^m = (t_0, t_0 + mr_j^2] \times B_r(x_j^{(0)}) \times B_{(m+2)r_j^3}(x_j^{(1)}) \times \ldots \times B_{(m^\kappa + 2\sum_{i=0}^{\kappa} m^i) r_j^{2\kappa+1}}(x_j^{(\kappa)}).$$

On one hand, when $|\overline{x} - x_j^{(0)}| \geq r_j$ the set

$$\Big\{(x^{(1)}, \dots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \dots, x^{(\kappa)}, t) \in \overline{\mathcal{Q}}_j^m\Big\} \quad \text{is empty.}$$

On the other hand, when $|\overline{x} - x_j^{(0)}| < r_j$ we have that

$$\Big\{(x^{(1)}, \ldots, x^{(\kappa)}, t) : (\overline{x}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \overline{\mathcal{Q}}_j^m\Big\} \supset \widetilde{\mathcal{Q}}_j := (t_j, t_j + m r_j^2] \times B_{2r_j^3}(x_j^{(1)}) \times \ldots \times B_{2\sum_{i=0}^{\kappa-1} m^i r_j^{2\kappa+1}}(x_j^{(\kappa)}).$$

Based on these last observations, we have that

$$\Big|\Big\{(x^{(1)}, \ldots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \bigcup_j \overline{\mathcal{Q}}_j^m\Big\}\Big| \geq \Big| \bigcup_{j : |\overline{x} - x_j^{(0)}| < r_j} \widetilde{\mathcal{Q}}_j \Big|.$$

Now, thanks to Fubini's Theorem and Lemma 4.A.9 we have

$$\begin{aligned}
&\Big|\Big\{(x^{(1)}, \ldots, x^{(\kappa)}, t) : (x^{(0)}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \bigcup_j \overline{\mathcal{Q}}_j^m\Big\}\Big| \\
&\geq \frac{m}{m+1} \Big| \bigcup_{j : |\overline{x} - x_j^{(0)}| < r_j} (t_j - r_j^2, 0] \times B_{2r_j^3}(x_j^{(1)}) \times \ldots \times B_{2\sum_{i=0}^{\kappa-1} m^i r_j^{2\kappa+1}}(x_j^{(\kappa)}) \Big| \\
&\geq \frac{m}{m+1} \Big| \bigcup_{j : |\overline{x} - x_j^{(0)}| < r_j} (t_j - r_j^2, 0] \times B_{r_j^3}(x_j^{(1)}) \times \ldots \times B_{r_j^{2\kappa+1}}(x_j^{(\kappa)}) \Big| \\
&= \frac{m}{m+1} \Big|\Big\{(x^{(1)}, \ldots, x^{(\kappa)}, t) : (\overline{x}, x^{(1)}, \ldots, x^{(\kappa)}, t) \in \bigcup_j \mathcal{Q}_j\Big\}\Big|.
\end{aligned}$$

Combining all of the above results, the proof is complete. □

Proof of Theorem 4.A.1. Let Q be the collection of all cylinders $\mathcal{Q} \subset \mathcal{Q}_1$ such that $|\mathcal{Q} \cap E| \geq (1 - \mu)|\mathcal{Q}|$. Let $G := \cup_{\mathcal{Q} \in Q} \mathcal{Q}$. By construction, the sets E and G satisfy the assumptions of Lemma 4.A.8. Therefore $(1 - c_{\text{is}}\mu)|G| \geq |E|$. Combining the assumptions of the theorem with Lemma 4.A.10 we conclude the proof. □

Proof of Corollary 4.A.3. The condition $|E| \leq (1 - \delta)|\mathcal{Q}_1|$ is implied by the second assumption when $r_0 < 1$. Moreover, the result is trivial when $r_0 \geq 1$ choosing C sufficiently large. Let Q be the collection of all cylinders $\mathcal{Q} \subset \mathcal{Q}_1$ such that $|\mathcal{Q} \cap E| \geq (1 - \mu)|\mathcal{Q}|$. Let $G := \cup_{\mathcal{Q} \in Q} \overline{\mathcal{Q}}^m$. From Theorem 4.A.1 we have that $|E| \leq \frac{m}{m+1}(1 - c\mu)|G|$. Moreover, our assumptions tell us $G \subset F$. In order to conclude the proof is sufficient to estimate the measure $G \setminus \mathcal{Q}_1$ by considering that each cylinder $\mathcal{Q} = \mathcal{Q}_r(x, t) \subset \mathcal{Q}_1$ has radius bounded by r_0 (see [59, Corollary 10.2]). □

4.B Stacked cylinders

In addition to the paper [59], a covering argument was already proposed in [1] for the case of the Fokker-Planck equation in trace form with Cordes-Landis assumptions. For the sake of completeness, we here state the stacking cylinders lemma for our operator $\mathscr{L}$, that is a generalization of the one proposed in [59]. Such a result is used when applying the Ink-Spots Theorem in the proof of the weak Harnack inequality, Theorem 4.5.1.

Lemma 4.B.1. *Let $\omega < \frac{1}{10^2}$ and $\rho = ((3\kappa+1)\omega)^{\frac{1}{2\kappa+1}}$. We consider any non-empty cylinder $\mathcal{Q}_r(z_0) \subset \mathcal{Q}_-$ and we set $T_k = \sum_{j=1}^{k}(2^k r)^2$. Let $N \geq 1$ such that $T_N \leq -t_0 < T_{N+1}$ and let*

$$\mathcal{Q}_r[k] := \mathcal{Q}_{2^k r}(z_k), \quad k = 1, \dots, N$$
$$\mathcal{Q}_{R_{N+1}}[N+1] := \mathcal{Q}_{R_{N+1}}(z_{N+1}),$$

where $z_k = z_0 \circ (0, \dots, 0, T_k)$ and $R = |t_0 + T_N|^{\frac{1}{2}}$, $R_{N+1} = \max(R, \rho)$, and

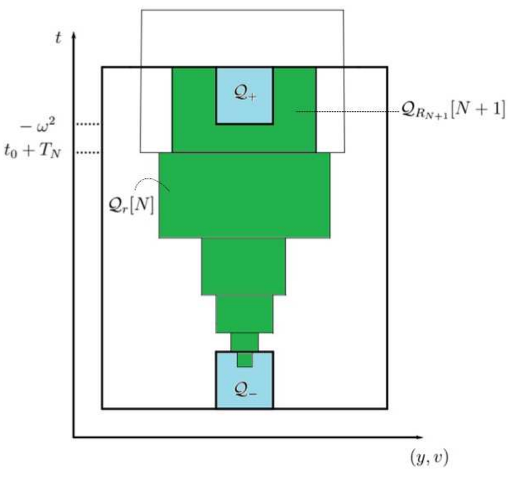
$$z_{N+1} = \begin{cases} z_N \circ (0, \dots, 0, R^2), & \text{if } R \geq \rho \\ (0,0), & \text{if } R < \rho \end{cases}$$

These cylinders satisfy

$$\mathcal{Q}_+ \subset \mathcal{Q}_{R_{N+1}}[N+1], \qquad \cup_{k=1}^{N+1} \mathcal{Q}_r[k] \subset (-1,0] \times B_2, \qquad \widetilde{\mathcal{Q}}[N] \subset \mathcal{Q}_r[N],$$

where $\widetilde{\mathcal{Q}}[N] = \mathcal{Q}_{R_{N+1}/2}\left(z_{N+1} \circ (0, \dots, 0, -R_{N+1}^2)\right)$.

Figure 4.3: Stacking cylinders above an initial one contained in $\mathcal{Q}_-$



Proof. As our derivation of the previous lemma follows very closely the one contained in [51, Appendix C], we here simply sketch the proof. Indeed, the proof of the result is merely geometric and the main difference with [51] lies in the fact that in our case we exploit the more general composition law and dilations defined in (1.1.6) and in (1.1.13), respectively. This explains why here the constants ω and ρ differ from the ones in [51].

We first check that the sequence of cylinders we consider is well-defined whenever $\omega < \frac{1}{10^2}$. As, by definition, $r \leq \omega$, we have that $t_0 + T_1 \leq -1 + \omega^2 + 4r^2 < 0$; moreover, there exists $N \geq 1$ such that $T_N < -t_0 \leq T_{N+1}$.

We now check that $\mathcal{Q}_+ \subset \mathcal{Q}_{R_{N+1}}[N+1]$. We first observe that, if $R < \rho$, then $R_{N+1} = \rho$ and therefore the desidered inclusion simply follows from the inequality $\omega \le \rho$. If, on the other hand, $R \ge \rho$, we have that $R_{N+1} = R$ and we can rewrite the desidered inclusion as $\mathcal{Q}_\omega(z_{N+1}^{-1}) \subset \mathcal{Q}_R$. In this case, in virtue of the definition of z_{N+1} and of the group law (1.1.6), we have $z_{N+1} = (E(R^2+T_N)x_0, t_0+T_N+R^2) = (E(R^2+T_N)x_0, 0)$, since $R^2 = |t_0+T_N| = -t_0 - T_N$. Hence, owing to (1.1.8), we obtain $z_{N+1}^{-1} = (E(R^2+T_N)x_0, 0)^{-1} = (-E(t_0)x_0, 0)$ and for every $z \in \mathcal{Q}_\omega$, we have

$$z_{N+1}^{-1} \circ z = (x - E(t+t_0)x_0, t) \in \mathcal{Q}_R$$

whenever $2\omega \le R$, $4\omega^3 \le R^3, \dots, (3\kappa+1)\omega^{2\kappa+1} \le R^{2\kappa+1}$ and $\omega^2 \le R^2$. The previous inequalities are clearly satisfied if $(3\kappa+1)\omega \le R^{2\kappa+1}$, i.e. if $\rho \le R$.

We now show that the inclusion $\mathcal{Q}_r[k] \subset B_2 \times \dots B_2 \times (-1,0]$ holds true for every $k \in \{1,\dots,N+1\}$. As far as $\mathcal{Q}_{R_{N+1}}[N+1]$ is concerned, i.e. as far as $k = N+1$, we take advantage of the inequalities $R = |t_0+T_N| \le 1$ and $\rho = ((3\kappa+1)\omega)^{\frac{1}{2\kappa+1}} \le 1$ to infer $R_{N+1} \le 1$. In addition, $z_{N+1} \in \mathcal{Q}_1$ and therefore $\mathcal{Q}_{R_{N+1}}[N+1] \subset B_2 \times \dots B_2 \times (-1,0]$. We now address the case $k \le N$. To this end, we observe that $T_N \le -t_0 \le 1$ as $t_0 \in (-1, -1+\omega^2]$ with $\omega \le 1$. As a consequence, by construction $(2^N r)^2 \le T_N \le 1$ and therefore $2^N r \le 1$. Moreover, if $\bar{z}_k = (x_k, t_k)$, then there exists $(x,t) \in \mathcal{Q}_1$ such that $\bar{z}_k = (x_0, t_0) \circ (0, T_k) \circ (\delta^0_{2^k r}(x), (2^k r)^2 t)$, where $(\delta^0_r)_{r>0}$ denotes the family of spatial dilations introduced in (1.1.14). In particular, this implies that $x_{k,i} = \sum_{j=1}^N (E((2^k r)^2 t + T_k))_{ij} x_{0,j} + (2^k r)^{2i+1} x_i$ and therefore

$$\begin{aligned}
\left|x_k^{(0)}\right| &\le \left|x_0^{(0)}\right| + \left|2^k r x^{(0)}\right| \le \omega + 1 \le 2, \\
\left|x_k^{(1)}\right| &\le \left|x_0^{(1)}\right| + 2\left|((2^k r)^2 t + T_k) x_0^{(0)}\right| + \left|(2^k r)^3 x^{(1)}\right| \le \omega^3 + 2\omega + 1 \le 3\omega + 1 \le 2 \\
&\vdots \\
\left|x_k^{(\kappa)}\right| &\le \left|x_0^{(\kappa)}\right| + 2\kappa \left|((2^k r)^2 t + T_k)\right| \omega + \left|(2^k r)^{2\kappa+1} x^{(\kappa)}\right| \le \omega^{2\kappa+1} + 2\kappa\omega + 1 \le 2
\end{aligned} \tag{4.B.1}$$

where we split $x = (x^{(0)}, \dots, x^{(\kappa)})$, $x_0 = (x_0^{(0)}, \dots, x_0^{(\kappa)})$, $x_k = (x_k^{(0)}, \dots, x_k^{(\kappa)})$ according to (4.1.6). We remark inequalities (4.B.1) hold true in virtue of the assumption $\rho^{2\kappa+1} = (3\kappa+1)\omega \le 1$, which in turn implies that $3\omega \le 1, \dots, (2\kappa+1)\omega \le 1$. In addition, in the above inequalities, we made use of $\left|(2^k r)^{2j+1} x^{(j)}\right| \le 1$ as $(2^k r)^{2j+1} \le 2^k r \le 2^N r \le 1$ for every $k \in \{1, \dots, N+1\}$ and $x^{(j)} \in B_1$. Finally, we observe that inequalities (4.B.1) implies $\mathcal{Q}_r[k] \subset B_2 \times \dots B_2 \times (-1,0]$ for every $k \le N$.

We are now left with proving $\widetilde{\mathcal{Q}}[N] \subset \mathcal{Q}_r[N]$, to conclude the proof of our lemma. If $R \ge \rho$, then the conclusion simply follows from $R/2 \le 2^N r$. On the other hand, if $R \le \rho$, we want to show that

$$\mathcal{Q}_{\rho/2}(\bar{z}) \subset \mathcal{Q}_{2^N r}, \tag{4.B.2}$$

where $\bar{z} = (0, -T_N) \circ (x_0, t_0)^{-1} \circ (0, -\rho^2)$. In order to prove inclusion (4.B.2), we first estimate the quantity $2^N r$ from below. From $t_0 + T_{N+1} > 0$ and $-t_0 \ge 1 - \omega^2$, we obtain that

$\frac{4}{3}\left(4^{N+1}-1\right)r^2 \geq 1-\omega^2$, which implies $4^N r^2 \geq \frac{1}{4}\left(\frac{3}{4}-\frac{7}{4}\omega^2\right) \geq \frac{1}{8}$. As a consequence, we infer that

$$2^N r \geq \frac{1}{2\sqrt{2}}. \tag{4.B.3}$$

We now observe that, by definition, $\bar{z} = (-E(-\rho^2 - t_0)x_0, -T_N - t_0 - \rho^2) = (-E(-\rho^2 - t_0)x_0, R^2 - \rho^2)$. Thus, for every $z \in \mathcal{Q}_{\rho/2}$, we compute $\bar{z} \circ z = (x - E(-\rho^2 - t_0 + t)x_0, R^2 - \rho^2 + t)$ and we observe that $\left|\left(x - E(-\rho^2 - t_0 + t)x_0\right)^{(i)}\right| \leq \omega^{2i+1} + (2i+1)\omega + (\rho/2)^{2i+1} \leq (3\kappa+1)\omega + (\rho/2)^{2i+1} = \rho^{2\kappa+1} + (\rho/2)^{2i+1} \leq \rho^{2i+1} + (\rho/2)^{2i+1} \leq 2\rho^{2i+1} \leq (2\rho)^{2i+1}$, for every $i = 0, \ldots, \kappa$. Moreover, we have $-(2\rho)^2 \leq -2\rho^2 < R^2 - \rho^2 - t_0$ and therefore $\bar{z} \circ z \in \mathcal{Q}_{2\rho}$. Hence, owing to (4.B.3), it is sufficient to choose ω such that $\rho \leq \frac{1}{2\sqrt{2}}$ (i.e. $\omega \leq \frac{1}{10^2}$) to get the desidered inclusion (4.B.2).

□

Chapter 5

Weak fundamental solution

5.1 Statement of the problem

In this chapter, we take advantage of the Schauder estimates contained in Chapter 2 to prove the existence of a fundamental solution associated to the Kolmogorov equation (4.1.1). Moreover, thanks to Moser's estimate and to the Harnack inequality established in the previous Chapter 4, we are able to prove Gaussian upper and lower bounds for the fundamental solution. We eventually remark that these results are published in [12].

In the following of this chapter, we therefore deal with weak solutions to equation (4.1.1) in the sense of Definition 4.1.2. Moreover, we always assume that hypothesis **(H1)-(H2)** of Chapter 4 hold true. Concerning the regularity of the coefficients a_{ij}, b_i and c, for $i, j = 1, \dots, N$, we mainly rely on assumption **(H3-ii)** below.

In order to expose the main results of this chapter, we first need to give the definition of *weak fundamental solution* for operator $\mathscr{L}$. To this end, we recall that the formal adjoint of operator $\mathscr{L}$ in (4.1.1) is defined as in (1.2.7).

Definition 5.1.1. A *weak fundamental solution* for $\mathscr{L}$ is a continuous positive function $\Gamma_L = \Gamma_L(x, t; \xi, \tau)$ defined for $t \in \mathbb{R}$, $0 \le T_0 < \tau < t < T_1$ and any $x, \xi \in \mathbb{R}^N$ such that:

1. $\Gamma_L = \Gamma_L(\cdot, \cdot; \xi, \tau)$ is a weak solution to $\mathscr{L}u = 0$ in $\mathbb{R}^N \times (\tau, T_1)$ and $\Gamma_L = \Gamma_L(x, t; \cdot, \cdot)$ is a weak solution of $\mathscr{L}^* v = 0$ in $\mathbb{R}^N \times (T_0, t)$;

2. for any bounded function $\varphi \in C(\mathbb{R}^N)$ and any $x, \xi \in \mathbb{R}^N$ we have

$$\begin{cases} \mathscr{L}u(x,t) = 0 & (x,t) \in \mathbb{R}^N \times (\tau, T_1), \\ \lim\limits_{\substack{(x,t)\to(\xi,\tau) \\ t>\tau}} u(x,t) = \varphi(\xi) & \xi \in \mathbb{R}^N, \end{cases} \tag{5.1.1}$$

$$\begin{cases} \mathscr{L}^* v(\xi,\tau) = 0 & (\xi,\tau) \in \mathbb{R}^N \times (T_0, t), \\ \lim\limits_{\substack{(\xi,\tau)\to(x,t) \\ t>\tau}} v(\xi,\tau) = \varphi(x) & x \in \mathbb{R}^N, \end{cases}$$

where the above equations need to be satisfied in the weak sense and

$$u(x,t) := \int_{\mathbb{R}^N} \Gamma_L(x,t;\xi,\tau)\,\varphi(\xi)\,d\xi, \quad v(\xi,\tau) := \int_{\mathbb{R}^N} \Gamma_L(x,t;\xi,\tau)\,\varphi(x)\,dx.$$

We remark that Definition 5.1.1 was first introduced by Lanconelli, Pascucci and Polidoro in [72, Definition 2.2].

Now, we are in a position to state our main results. First, we address and give answer to [72, Remark 2.3] by proving the existence of a weak fundamental solution for operator $\mathscr{L}$ in the sense of Definition 5.1.1 under the following assumptions on the coefficients of $\mathscr{L}$.

(H3-ii) The coefficients a_{ij}, b_i, $c \in L^1(S_{T_0T_1}) \cap L^\infty_{loc}(S_{T_0T_1})$ for $i,j,k = 1,\dots,N$, i.e. for any given compact subset K of $S_{T_0T_1}$ there exists a positive constant M such that

$$|a_{ij}(x,t)|, |b_i(x,t)|, |c(x,t)| \le M, \qquad \forall (x,t) \in K,\ \forall i,j = 1,\dots,N.$$

Moreover, the diffusion coefficients a_{ij} are such that

$$\operatorname{div} A_0^j = 0 \quad \forall j = 1,\dots,m_0 \text{ in the distributional sense,}$$

where A_0^j denotes the j^{th}-column of the matrix A_0 introduced in assumption **(H1)**.

Note that the divergence free assumption on the columns of the matrix A_0 is required to address technical issues arising in the proof of the forthcoming Theorem 5.1.2. Indeed, the existence of the weak fundamental solution is achieved by combining a regularization procedure with a diagonal argument. In particular, we perform a regularizing procedure on the coefficients a_{ij}, b_i, c and we then prove that the newly defined coefficients are Dini continuous. This allows us to apply Theorem 2.1.5 to the constructed regularized operator $\mathscr{L}_\varepsilon$ under the following assumption

(C') The matrix A_0 satisfies assumption **(H1)**, while the matrix B has constant entries. The principal part operator $\mathscr{L}_0$ satisfies assumption **(H2)**. Finally, the coefficients a_{ij}, b_i, c, and $\partial_{x_k} a_{ij}$, for $i,j,k = 1,\dots,N$, are bounded and Hölder continuous of exponent $\alpha \in (0,1]$ in the sense of Definition 1.2.1.

The proof of the existence of a fundamental solution associated to operator (4.1.1) finally follows from Theorem 1.2.4.

Hence, the reason why we work under assumption **(C')** is that we want to take advantage of Theorem 1.2.4, which, as discussed in Section 1.2, holds true under hypothesis **(C)**. The only difference between assumption **(C)** of Section 1.2 and our assumption **(C')** is that we here require the Hölder continuity of the derivatives of a_{ij}, $i,j = 1,\dots,N$. This additional requirement is simply due to the fact that the operator we consider in (4.1.1) is in divergence form, whereas Theorem 1.2.4 is proved for a trace form Kolmogorov operator, namely operator (1.2.6).

It is our belief that this additional assumption can be dropped by considering more refined analytical techniques, such as the ones recently proposed in the preprint [80] for the case of measurable in time and Hölder continuous in space diffusion coefficients. Finally, we point out that it is possible to replace the diverge free assumption on the diffusion coefficients with the following, more restrictive, one: the coefficients a_{ij} are measurable, doubly (weakly) differentiable with respect to the first m_0 components and such that $\partial^2_{lk} a_{ij} \in L^\infty_{loc}(S_{T_0T_1})$, for every $l, k = 2, \dots, m_0$. Indeed, this last assumption is enough to ensure that also the first order derivatives $\partial_k a_{ij}$ are Lipschitz continuous on $S_{T_0T_1}$, with a uniform modulus of continuity not depending on the set we are considering.

Theorem 5.1.2 (Existence of the weak fundamental solution)**.** *Let us consider operator $\mathscr{L}$ under assumptions **(H1)-(H2)-(H3-ii)**. Then there exists a fundamental solution Γ_L of $\mathscr{L}$ in the sense of Definition 5.1.1 and the following reproduction property holds. Indeed, for every $x, \xi \in \mathbb{R}^N$ and every $t, \tau \in \mathbb{R}$ with $\tau < s < t$ such that $\tau, t \in (T_0, T_1)$:*

$$\Gamma_L(x,t;\xi,\tau) = \int_{\mathbb{R}^N} \Gamma_L(x,t;y,s)\,\Gamma_L(y,s;\xi,\tau)\,dy.$$

Moreover, the function $\Gamma^*_L(x,t;\xi,\tau) = \Gamma_L(\xi,\tau;x,t)$ is the fundamental solution of $\mathscr{L}^*$ and verifies the dual properties of this statement.

The existence of a classical fundamental solution is a problem that has been thoroughly addressed over the years. In particular, we refer to the works by Hörmander [54] and Kolmogorov [67] for the analysis of the case with constant, or smooth coefficients. Among others, we recall the paper [102] for the proof of the existence of a classical fundamental solution through the Levi parametrix method (see [75]) and we refer to the last part of Section 1.2 for further reference.

To our knowledge, Theorem 5.1.2 is the first existence result available for the weak fundamental solution to (4.1.1) in the sense of Definition 5.1.1. The proof we propose here is based on a limiting procedure combined with Schauder type estimates and a diagonal argument. This procedure was first proposed in [7] to prove the existence of a classical fundamental solution when the coefficients of (4.1.1) are locally Hölder continuous. The main difficulties we encounter when adapting this argument to the weak case are given by the low regularity of the coefficients, hence a new regularizing procedure is introduced in Section 5.3.

We emphasize that the PDE approach adopted in this work improves the previously known results in that it allows us to consider differential operators with bounded measurable coefficients in *both time and space*, which is a milder assumption than the ones considered in the most recent literature. Indeed, on the one hand, in [25] the authors considered the case of bounded measurable time-depending coefficients, with a proof that is based on explicit computations involving the fundamental solution. On the other hand, in [80] the case of Hölder continuous in space and bounded measurable in time coefficients was addressed.

As a second issue, we extend [72, Theorem 1.3] providing Gaussian upper and lower bounds for the weak fundamental solution Γ of $\mathscr{L}$ under assumptions **(H1)-(H2)-(H3-ii)**.

As far as Gaussian upper bounds are concerned for the fundamental solution Γ associated to $\mathscr{L}$ with Hölder continuous coefficients, a first result dates back to [102], where the author proves Gaussian upper bounds depending on the Hölder norm of the coefficients a, b and c. Later on, Di Francesco and Pascucci [40], Di Francesco and Polidoro [41] proved upper and lower bounds for the classical fundamental solution, where also in this case the involved constants depend on the Hölder norm of the coefficients. A first result regarding Gaussian upper bounds independent of the Hölder norm of the coefficients is due to Pascucci and Polidoro, who studied operator (4.1.1) with $b = c = 0$ (see [98, Theorem 1.1]). Later on, Lanconelli, Pascucci [71] and Lanconelli, Pascucci, Polidoro [72] extended Nash upper bounds to non-homogeneous operators of the form (4.1.1) with bounded measurable coefficients.

On the other hand, if we consider Gaussian lower bounds independent of the Hölder norm of the coefficients, a first result is due to Lanconelli, Pascucci and Polidoro [72, Theorem 1.3] for the particular case of the kinetic Kolmogorov-Fokker-Planck equation. The proof of this result is based on the construction of a Harnack chain, alongside with the study of the control problem associated to the principal part operator $\mathscr{L}_0$. The authors of [72] already suggested this type of result could be extended to the general non-homogeneous Kolmogorov operator of step κ in (4.1.1), once a suitable Harnack inequality is established. The present work is a first step in this direction as it handles the homogeneous case, the only one for which a Harnack inequality is available, see Theorem 4.1.3.

Theorem 5.1.3 (Gaussian bounds). *Let $\mathscr{L}$ be an operator of the form (4.1.1) under assumptions **(H1)-(H2)-(H3-ii)**. Let $I = (T_0, T_1)$ be a bounded interval, then there exist four positive constants λ^+, λ^-, C^+, C^- such that*

$$C^- \, \Gamma_K^{\lambda^-}(x,t;\xi,\tau) \, \leq \, \Gamma_L(x,t;\xi,\tau) \, \leq \, C^+ \, \Gamma_K^{\lambda^+}(x,t;\xi,\tau) \tag{5.1.2}$$

for every $(x,t), (\xi,\tau) \in \mathbb{R}^N \times (T_0,T_1)$ with $\tau < t$. The constants λ^+, λ^-, C^+, C^- only depend on B, $(T_1 - T_0)$, M. Note that $\Gamma_K^{\lambda^-}$ and $\Gamma_K^{\lambda^+}$ denote the fundamental solution of $\mathscr{L}_0^{\lambda^-}$ and $\mathscr{L}_0^{\lambda^+}$, where

$$\mathscr{L}_0^{\lambda} u(x,t) := \frac{\lambda}{2} \sum_{i=1}^{m_0} \partial^2_{x_i} u(x,t) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t), \tag{5.1.3}$$

and the explicit expression of $\Gamma_K^{\lambda^\pm}$ is given by

$$\Gamma_K^{\lambda}((x,t);(\xi,\tau)) = \Gamma_K^{\lambda}((\xi,\tau)^{-1} \circ (x,t); 0, 0),$$

for every $(x,t),(\xi,\tau) \in \mathbb{R}^{N+1}$, *with* $(x,t) \neq (\xi,\tau)$ *and*

$$\Gamma_K^{\lambda}(x,t;0,0) = \begin{cases} \frac{(2\pi\lambda)^{-\frac{N}{2}}}{\sqrt{detC(t)}} \exp\left(-\frac{1}{2\lambda}\langle C^{-1}(t)x,x\rangle\right), & if\, t>0, \\ 0, & if\, t \leq 0. \end{cases} \tag{5.1.4}$$

Remark 5.1.4. As the Harnack inequality 4.1.3 holds true under the less restrictive assumption **(H3)** in Chapter 4, the lower bounds in (5.1.2) still holds if we assume $c, f \in L^q_{loc}(\Omega)$, with $q > \frac{(Q+2)}{2}$, and $b \in (L^{\infty}_{loc}(\Omega))^{m_0}$. In this case, the constants appearing on the left-hand side of inequality (5.1.2) will depend on B, $(T_1 - T_0)$, $\|b\|_q$, $\|c\|_q$.

Remark 5.1.5. Since the proof of the upper bound in (5.1.2) does not rely on the Harnack inequality stated in Theorem 4.1.3, the rightmost inequality of (5.1.2) holds true for the more general operator

$$\begin{aligned}\widetilde{\mathscr{L}}u(x,t) := &\sum_{i,j=1}^{m_0} \partial_{x_i}\left(a_{ij}(x,t)\partial_{x_j}u(x,t)\right) + \sum_{i=1}^{m_0} b_i(x,t)\partial_{x_i}u(x,t) + \\ &- \sum_{i=1}^{m_0} \partial_{x_i}\left(a_i(x,t)u(x,t)\right) + \sum_{i,j=1}^{N} b_{ij}x_j\partial_{x_i}u(x,t) - \partial_t u(x,t) + c(x,t)u(x,t),\end{aligned} \tag{5.1.5}$$

with $a \in (L^{\infty}_{\text{loc}}(S_T))^{m_0}$.

5.1.1 Outline of the chapter

This chapter is organized as follows. In Section 5.2 we prove Gaussian lower and upper bounds for the fundamental solution associated to operator $\mathscr{L}$. In Section 5.3 we prove the existence of a weak fundamental solution for operator $\mathscr{L}$ under assumptions **(H1)-(H2)-(H3-ii)**.

5.2 Proof of Theorem 5.1.3

This section is devoted to the proof of Gaussian bounds (Theorem 5.1.3) for the weak fundamental solution defined in Definition 5.1.1. All the results proved in Subsections 5.2.1-5.2.3 are obtained under assumptions **(H1)-(H2)-(H3)**, while the Gaussian upper bound in Subsection 5.2.2 still holds true when considering the more general operator $\widetilde{\mathscr{L}}$.

5.2.1 Harnack inequalities

The starting point in proving the Gaussian lower bound is an invariant Harnack inequality, which, prior to this work, was not avalaible in the ultraparabolic setting. For this reason, in this subsection we take advantage of Theorem 4.1.3. As Theorem 4.1.3 was established under assumptions **(H1)-(H2)-(H3)**, all the results in this subsection and in Subsection 5.2.3 hold true under these less restrictive assumptions.

Remark 5.2.1. When considering the more restrictive assumption **(H3-ii)** the constants appearing in the statement of Theorem 4.1.3 only depend on M, since we assume $|b(x,t)| \leq M$, $|c(x,t)| \leq M$ for every $(x,t) \in \mathbb{R}^{N+1}$.

We recall the following result, which will be useful in the proof of the upcoming Lemma 5.2.3.

Remark 5.2.2. Let u be a weak solution to $\mathscr{L}u = 0$ and $r > 0$. Then $v := u \circ \delta_r$ solves equation $\mathscr{L}^{(r)}v = 0$, where

$$\mathscr{L}^{(r)}v := \operatorname{div}(A^{(r)}Du) - \operatorname{div}(a^{(r)}v) + \langle b^{(r)}, Dv\rangle + c^{(r)}v + \langle Bx, Dv\rangle - \partial_t v$$

with $A^{(r)} = A \circ \delta_r$, $a^{(r)} = r(a \circ \delta_r)$, $b^{(r)} = r(b \circ \delta_r)$ and $c^{(r)} = r^2(c \circ \delta_r)$.

Moreover, if u is a solution to $\mathscr{L}u = 0$, then, for any $\zeta \in \mathbb{R}^{N+1}$, $v := u \circ \ell_z$ solves equation $(\mathscr{L} \circ \ell_z)v = 0$, where $\mathscr{L} \circ \ell_z$ is the operator obtained by $\mathscr{L}$ via a ℓ_z-translation of the coefficients.

For $\beta, r, R > 0$ and $z_0 \in \mathbb{R}^{N+1}$, we define the cones

$$P_{\beta,r,R} := \{z \in \mathbb{R}^{N+1} : z = \delta_\rho(\xi, \beta), |\xi|_{\mathbb{K}} < r, 0 < \rho \leq R\},$$

and we set $P_{\beta,r,R}(z_0) := z_0 \circ P_{\beta,r,R}$. We are now in a position to derive the following Lemma, which is a consequence of Theorem 4.1.3.

Lemma 5.2.3 (Local Harnack inequality)**.** *Let $z \in \mathbb{R}^{N+1}$ and $R \in (0,1]$. Moreover, let u be a continuous and non-negative weak solution to $\mathscr{L}u = 0$ in $\mathcal{Q}_R(z)$ under the assumptions **(H1)-(H2)-(H3)**. Then we have*

$$\sup_{P_{1,\omega,R/R_0}(z)} u \leq Cu(z),$$

where C, R_0 and ω are the constants appearing in Theorem 4.1.3 and they only depend on Q, λ, Λ and q.

Proof. Let $w \in P_{1,\omega,R}(z)$, i.e. $w = z \circ \delta_\sigma(\xi, 1)$ for some $\sigma \in (0, R]$ and $|\xi|_{\mathbb{K}} < \omega$. We now define the function $u_{z,\sigma} := u \circ \ell_z \circ \delta_\sigma$, which is a continuous and non-negative solution to $\mathscr{L}^{(\sigma)}u_{z,\sigma} = 0$ in $\mathcal{Q}_{R_0}(0,0) \subset \mathcal{Q}_{R/\sigma}(0,0)$ in virtue of Remark 5.2.2. Thus, we can apply the Harnack inequality (4.1.3) and infer

$$u(w) = u_{z,\sigma}(\xi, 1) \leq \sup_{\mathcal{Q}_-} u_{z,\sigma} \leq C \inf_{\mathcal{Q}_+} u_{z,\sigma} \leq Cu_{z,\sigma}(0,0) = Cu(z).$$

□

We next state a global version of the Harnack inequality, which is a crucial step in proving the Gaussian lower bound (see Theorem 5.2.14 below).

Theorem 5.2.4 (Global Harnack inequality)**.** *Let $t_0 \in \mathbb{R}$ and $\tau \in (0,1]$. If u is a continuous and non-negative weak solution to $\mathscr{L} u = 0$ in $\mathbb{R}^{N+1} \times (\tau - t_0, \tau + t_0)$ under the assumptions **(H1)-(H2)-(H3)**, then we have*

$$u(\xi, t) \leq c_0 e^{c_0 \langle C^{-1}(t-t_0)\left(\xi - e^{(t-t_0)B}x\right), \xi - e^{(t-t_0)B}x\rangle} u(x, t_0),$$

where $t \in (t_0, \tau + t_0)$, $x, \xi \in \mathbb{R}^N$, C is the matrix introduced in (1.1.32) and c_0 is a positive constant only depending on Q, λ, Λ and q.

The proof of Theorem 5.2.4 is based on a classical argument that makes use of the so-called Harnack chains, alongside with control theory. Moreover, the proof of this theorem follows the one of [72, Theorem 3.6], with the only difference that we here apply Theorem 4.1.3 and Lemma 5.2.3 instead of Theorem 3.1 and Lemma 3.5 of [72]. Indeed, the method we rely on has the advantage of highlighting the geometric structure of the operator $\mathscr{L}$ and can be therefore automatically extended to more general operators. Finally, we remark that a detailed contruction of Harnack chains through the repetition of an invariant Harnack inequality will be presented in Chapter 6 in a new relativistic setting.

Before proving Theorem 5.2.4, we recall that (see, for instance, [97, Section 9.5]) Hörmander's rank condition (1.0.6) is equivalent to the following property: for any two points $(y, t_0), (x, t) \in \mathbb{R}^{N+1}$ with $t > t_0$, we can find a control $\omega \in L^2(\mathbb{R}^{m_0}; [t_0, t])$ such that there exists a solution to the system

$$\begin{cases} \gamma'(s) = B\gamma(s) + \sigma\omega(s), \\ \gamma(t_0) = y, \quad \gamma(t) = x, \end{cases} \tag{5.2.1}$$

where B is the matrix defined in (1.1.12) and

$$\sigma = \begin{pmatrix} \mathbb{I}_{m_0} \\ \mathbb{O} \end{pmatrix}.$$

In the proof of Theorem 5.2.4 we will take advantage of the following result.

Lemma 5.2.5. *Let γ be the solution to*

$$\begin{cases} \gamma'(s) = B\gamma(s) + \sigma\omega(s), \quad s \in [t_0, t] \\ \gamma(t_0) = y, \end{cases} \tag{5.2.2}$$

with $t - t_0 \leq 1$, $x \in \mathbb{R}^N$ and $\omega \in L^2(\mathbb{R}^{m_0}; [t_0, t])$. Then we have

$$(\gamma(s), s) \in P_{1, c\|\omega\|_{L^2([t_0,t])}, \sqrt{t-t_0}}, \quad s \in [t_0, t], \tag{5.2.3}$$

where c is a constant that only depends on the matrix B.

We refer the reader to [72, Lemma 3.7] for the proof of the previous result. As the L^2-

norm of the control ω explicitly appears in (5.2.3), among the paths satisfying (5.2.2), we are interested in the one that minimizes the *total cost*

$$\|\omega\|^2_{L^2([t_0,t])} = \int_{t_0}^{t} |\omega(s)|^2 ds.$$

Classical control theory provides us with an explicit expression of the optimal control and of the associated the optimal cost (see, for instance, [97, Theorem 9.55]).

Lemma 5.2.6. *The optimal cost* $\Psi(y, t_0; x, t)$ *associated to problem* (5.2.1) *is given by*

$$\Psi(y, t_0; x, t) = \langle C^{-1}(t - t_0)\left(\xi - e^{(t-t_0)B}x\right), \xi - e^{(t-t_0)B}x\rangle.$$

Proof of Theorem 5.2.4. As we want to make use of the Harnack inequality Lemma 5.2.3, we first observe that, for every $(x, t) \in \mathbb{R}^N \times (t_0, t_0 + \tau)$, with $\tau > 0$, u is a continuous, non-negative solution to (4.1.1) (with $f \equiv 0$) in $\mathcal{Q}_{\sqrt{\tau}}(x, t)$. We now fix $x, y \in \mathbb{R}^N$, $t \in (t_0, t_0 + \tau)$ and we consider the solution γ to (5.2.1) corresponding to the optimal cost $\Psi(y, t_0; x, t)$. Additionally, we set $\widetilde{c} = \left(\frac{\omega}{c}\right)^2$, where ω and c are the constants given by Theorem 4.1.3 and Lemma 5.2.5, respectively.

If $\Psi(y, t_0; x, t) \le \widetilde{c}$, we can straightforwardly apply Lemma 5.2.5 and obtain

$$(x, t) \in P_{1, c\sqrt{\Psi(y,t_0;x,t)}, \sqrt{t-t_0}}(x, t) \subset P_{1,\omega,\sqrt{\tau}}(x, t).$$

Hence, we apply Lemma 5.2.3 and infer

$$u(x, t) \le C u(y, t_0),$$

where C is the constant given by Theorem 4.1.3. On the other hand, if the above inequality is not satisfied, we set

$$t_{j+1} = \inf\{s \in [t_j, t] : \Psi(\gamma(t_j), t_j; \gamma(s), s) \ge \widetilde{c}\}.$$

Clearly, for $j \ge \frac{\Psi(\gamma(t_j),t_j;\gamma(s),s)}{\widetilde{c}}$, we have that $t_j = t$, while, if $t_j < t$, there holds

$$(\gamma(t_{j+1}), t_{j+1}) \in P_{1,\omega,\sqrt{\tau}}(\gamma(t_j), t_j).$$

Thus, we are now in a position to apply Lemma 5.2.3, which yields

$$u(\gamma(t_{j+1}), t_{j+1}) \le C u(\gamma(t_j), t_j),$$

where, once again, C is the constant given by Theorem 4.1.3. As an immediate consequence of the previous inequalities, we get

$$u(x, t) \le C^{\frac{\Psi(y,t_0;x,t)}{\widetilde{c}}} u(y, t_0).$$

The thesis then follows thanks to the explicit expression of the optimal cost provided by Lemma 5.2.6.

□

5.2.2 Gaussian upper bound

As in the proof of the lower bound for the fundamental solution we make use of the upper bound provided by Theorem 5.2.12, we first focus our attention on proving this result.

The proof we propose in this subsection follows Aronson's method [14] and relies on the local boundedness of weak solutions to $\widetilde{\mathscr{L}}u = 0$ in (5.1.5). As the detailed derivation of the local boundedness of weak solutions was already provided in Chapter 4 through Moser's iterative method, we here just recall the main result, which in our setting reads as follows.

Theorem 5.2.7. *Let $z_0 \in \Omega$ and $0 < \frac{r}{2} \le \rho < r \le 1$, be such that $\overline{\mathcal{Q}_r(z_0)} \subseteq \Omega$. Let $\widetilde{\mathscr{L}}$ be an operator of the form* (5.1.5) *and let u be a non-negative weak solution to $\widetilde{\mathscr{L}}u = 0$ in Ω under assumptions **(H1)-(H2)-(H3-ii)**. Then for every $p \ge 1$ there exists two positive constants $C = C(p, \lambda, \Lambda, Q)$, such that*

$$\sup_{\mathcal{Q}_\rho(z_0)} u^p \le \frac{C}{(r-\rho)^{Q+2}} \int_{\mathcal{Q}_r(z_0)} u^p. \tag{5.2.4}$$

Proof. The argument relies on the combination of a Caccioppoli-type inequality and a Sobolev-type inequality (see Chapter 4, Section 4.3). The only difference with the proof presented in Chapter 4 is that here we handle the more general operator $\widetilde{\mathscr{L}}$ in (5.1.5) and we rely on the more restrictive assumption **(H3-ii)**. Thus, to carry out the proof in the present setting, we just need to show how to deal with the additional term $\sum_{i=1}^{m_0} \partial_{x_i}(a_i(x,t)u(x,t))$ in the Caccioppoli-type inequality and the Sobolev-type inequality proved in Chapter 4. More precisely, we consider the Caccioppoli inequality Theorem 4.3.4 and we focus on the new term involving the coefficient $a \in (L^\infty_{\text{loc}}(\Omega))^{m_0}$, which is handled as follows

$$\begin{aligned}
&(2p-1)\int_{\mathcal{Q}_r} \langle a, D_{m_0}u^p\rangle \psi^2 u^p + 2p\int_{\mathcal{Q}_r} \langle a u^{2p}, D_{m_0}\psi\rangle\psi \\
&\le |2p-1| \int_{\mathcal{Q}_r} |a|\,|D_{m_0}u^p|\psi^2 u^p + 2p\int_{\mathcal{Q}_r} u^{2p}\,|\langle a, D_{m_0}\psi\rangle|\,|\psi| \\
&\le \frac{|2p-1|}{\varepsilon_a}\int_{\mathcal{Q}_r} |a|^2\,\psi^2 u^{2p} + |2p-1|\varepsilon_a \int_{\mathcal{Q}_r} |D_{m_0}u^p|^2\psi^2 + 2p\int_{\mathcal{Q}_r} u^{2p}\,|\langle a, D_{m_0}\psi\rangle|\,|\psi| \\
&\le \frac{|2p-1|}{\varepsilon_a}\|a\|^2_{L^\infty(\mathcal{Q}_r)}\|u^p\|^2_{L^2(\mathcal{Q}_r)} + |2p-1|\varepsilon_a\int_{\mathcal{Q}_r}|D_{m_0}u^p|^2\psi^2 + \frac{|2p|\,c_1}{(r-\rho)}\|a\|_{L^\infty(\mathcal{Q}_r)}\|u^p\|^2_{L^2(\mathcal{Q}_r)}.
\end{aligned}$$

From this point, we obtain the Caccioppoli inequality reasoning as in the proof of Theorem 4.3.4.

As far as the Sobolev inequality is concerned, we find two extra terms in the representation formula of sub-solutions. More precisely, following the notation of Theorem 4.3.3, the term

$I_0(z)$ here becomes

$$I_0(z) = \int_{\mathcal{Q}_r} \left[-\langle a, D(\psi\Gamma(z,\cdot))\rangle v\right](\zeta)d\zeta + \\ + \int_{\mathcal{Q}_r} \left[\langle b, Dv\rangle\Gamma(z,\cdot)\psi\right](\zeta)d\zeta \ + \int_{\mathcal{Q}_r} \left[cv\Gamma(z,\cdot)\psi\right](\zeta)d\zeta.$$

Since

$$\langle a, Dv\rangle \in L^{2\frac{q}{q+2}} \quad \text{for } a \in L^q, \;\; q > \frac{Q+2}{2} \; \text{ and } v \in L^2,$$

reasoning as in Theorem 4.3.3 we infer

$$\begin{aligned}\| I_0(\zeta) \|_{L^{2\alpha}(\mathcal{Q}_\rho)} &\leq \| \Gamma * (\langle a, D_{m_0} v\rangle\psi) + \Gamma * (\langle b, D_{m_0} v\rangle\psi) + \Gamma * (cv\psi) \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \\ &\leq C \cdot (\| a \|_{L^q(\mathcal{Q}_\rho)} + \| b \|_{L^q(\mathcal{Q}_\rho)} \| D_{m_0} v \|_{L^2(\mathcal{Q}_\rho)} + \\ &\quad + \| c \|_{L^q(\mathcal{Q}_\rho)} \| v \|_{L^2(\mathcal{Q}_\rho)})),\end{aligned}$$

where

$$\alpha = \frac{q(Q+2)}{q(Q-2)+2(Q+2)}.$$

In addition, the term $I_3(z)$ here becomes

$$\begin{aligned}I_3(z) = &\int_{\mathcal{Q}_r} \left[\langle ADv, D(\Gamma(z,\cdot)\psi)\rangle\right](\zeta)d\zeta \ - \int_{\mathcal{Q}_r} \left[(\Gamma(z,\cdot)\psi)\, Yv\right](\zeta)d\zeta \\ &+ \int_{\mathcal{Q}_r} \left[\langle a, D(\Gamma(z,\cdot)\psi)\rangle v\right](\zeta)d\zeta - \int_{\mathcal{Q}_r} \left[\langle b, Dv\rangle\Gamma(z,\cdot)\psi\right](\zeta)d\zeta \\ &- \int_{\mathcal{Q}_r} \left[cv\Gamma(z,\cdot)\psi\right](\zeta)d\zeta\end{aligned}$$

and can be treated exactly as the analogous one in Theorem 4.3.3. The rest of the proof of the Sobolev inequality follows the one contained in Theorem 4.3.3. □

We now prove the following result, which is an important consequence of Moser's estimate (5.2.4).

Theorem 5.2.8 (Nash upper bound)**.** *Let $\widetilde{\mathscr{L}}$ be an operator of the form* (5.1.5) *satisfying assumptions* ***(H1)-(H2)-(H3-ii)****. Then there exists a positive constant C_0, only dependent on Q, λ and Λ, such that*

$$\Gamma(x,t;y,t_0) \leq \frac{C_0}{(t-t_0)^{\frac{Q}{2}}}, \tag{5.2.5}$$

for any $0 < t - t_0 \leq 1$ and $x, y \in \mathbb{R}^N$.

Proof. We apply Theorem 5.2.7, with $\rho = \frac{1}{2}\sqrt{t-t_0}$ and $r = \sqrt{2}\rho$, and we find

$$\Gamma(x,t;y,t_0) \leq \sup_{\mathcal{Q}_\rho(y,t_0)} \Gamma(x,t;\cdot,\cdot)$$

$$
\begin{aligned}
&\le \frac{C}{(t-t_0)^{\frac{Q+2}{2}}} \int_{\mathcal{Q}_r(y,t_0)} \Gamma(x,t;\xi,s)\, d\xi\, ds \\
&\le \frac{C}{(t-t_0)^{\frac{Q+2}{2}}} \int_{t_0-\frac{1}{2}(t-t_0)}^{t_0} \int_{\mathbb{R}^N} \Gamma(x,t;\xi,s)\, d\xi\, ds \\
&\le \frac{C_0}{(t-t_0)^{\frac{Q}{2}}},
\end{aligned}
$$

where in the last step we have used inequality $\int_{\mathbb{R}^N} \Gamma(x,t;\xi,s)\, d\xi, ds \le e^{(t-t_0)\|c\|_\infty}$. □

As a straightforward consequence of Theorem 5.2.8, we obtain the following corollary.

Corollary 5.2.9. *Under the assumptions of Theorem 5.2.8, there exists a positive constant C_0, only dependent on Q, λ and Λ, such that*

$$
\int_{\mathbb{R}^N} \Gamma^2(x,t;y,t_0)\, dx \le \frac{C_0}{(t-t_0)^{\frac{Q}{2}}},
$$

for any $0 < t - t_0 \le 1$ and $y \in \mathbb{R}^N$, and

$$
\int_{\mathbb{R}^N} \Gamma^2(x,t;y,t_0)\, dy \le \frac{C_0}{(t-t_0)^{\frac{Q}{2}}},
$$

for any $0 < t - t_0 \le 1$ and $x \in \mathbb{R}^N$.

Another crucial tool in proving the Gaussian upper bound is provided by the following theorem.

Theorem 5.2.10. *Let $y \in \mathbb{R}^N$, $\sigma > 0$ and let $u_0 \in L^2(\mathbb{R}^N)$ be such that $u_0(x) = 0$ whenever $|x-y|_{\mathbb{K}} < \sigma$. Let $\widetilde{\mathscr{L}}$ be an operator of the form (5.1.5) satisfying assumptions **(H1)**-**(H2)**-**(H3-ii)**. Moreover, we assume that u is a bounded solution to $\widetilde{\mathscr{L}} u = 0$ in $(\eta, \eta + \sigma^2]$ with initial value $u_0(x,\eta) = u_0(x)$. Then, there exist two positive constants k and C, only depending on M, such that for any τ satisfying $\eta \le \tau \le \eta + \frac{1\wedge\sigma^2}{k}$, we have*

$$
\left|u\left(\left(e^{-\eta B}y, 0\right) \circ (0,\tau)\right)\right| = |v(0,\tau)| \le \frac{C}{(\tau-\eta)^{\frac{Q}{4}}} \exp\left(-\frac{\sigma^2}{C(\tau-\eta)}\right) \|u_0\|^2_{L^2(\mathbb{R}^N)}.
$$

Proof. Let us first prove the theorem for $y = 0$. We fix s such that $0 \le s - \eta \le 1 \wedge \sigma^2$ and we define

$$
h(x,t) = -\frac{|x|^2_{\mathbb{K}}}{2(s-\eta) - k(t-\eta)} + \alpha(t-\eta), \quad \eta \le t \le \eta + \frac{s-\eta}{k}, \quad x \in \mathbb{R}^N, \tag{5.2.6}
$$

where α and k are positive constants we shall fix later on. In addition, for a radius $R \ge 2$, we consider a cut-off function $\gamma_R \in C_0^\infty(\mathbb{R}^N; [0,1])$ such that $\gamma_R(x) \equiv 1$ for $|x|_{\mathbb{K}} \le R - 1$, $\gamma_R(x) \equiv 0$ for $|x|_{\mathbb{K}} \ge R$ with $|D\gamma_R|$ bounded by a constant that does not depent on R. Now, multiplying both sides of equation $\widetilde{\mathscr{L}}u = 0$ by $\gamma_R^2 e^{2h} u$ and integrating over $\mathbb{R}^N \times [\eta, \tau]$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \gamma_R^2 e^{2h} u^2|_{t=\eta} - 2 \int\int_{\mathbb{R}^N \times [\eta,\tau]} \gamma_R^2 e^{2h} u^2 \ (3\langle A D_{m_0} h, D_{m_0} h\rangle - Yh - 2\langle a, D_{m_0} h\rangle + \mu\rangle) \ dx\, dt \\
&\leq \int_{\mathbb{R}^N} \gamma_R^2 e^{2h} u^2|_{t=\tau} dx + 2 \int\int_{\mathbb{R}^N \times [\eta,\tau]} e^{2h} u^2 \left(3\Lambda |D_{m_0} \gamma_R|^2 + |Y\gamma_R|^2 - 2\langle a, D_{m_0} \gamma_R\rangle \gamma_R\right) dx\, dt,
\end{aligned} \tag{5.2.7}$$

where $\mu = \frac{3}{2\lambda}\|a\|_\infty^2 + \frac{3}{2\lambda}\|b\|_\infty^2 + \|c\|_\infty$ is a positive constant only depending on M in assumption **(H3A)**. Since u is bounded by assumption and $e^{2h(x,t)} \leq e^{\frac{-|x|_{\mathbb{K}}^2}{s-\eta} + \alpha(s-\eta)}$, if we let R go to infinity in (5.2.7), the last integral tends to zero and we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} e^{2h} u^2|_{t=\eta} - 2 \int\int_{\mathbb{R}^N \times [\eta,\tau]} \gamma_R^2 e^{2h} u^2 \ (3\langle A D_{m_0} h, D_{m_0} h\rangle - Yh - 2\langle a, D_{m_0} h\rangle + \mu\rangle) \ dx\, dt \\
&\leq \int_{\mathbb{R}^N} e^{2h} u^2|_{t=\tau} dx.
\end{aligned} \tag{5.2.8}$$

We now make a suitable choice of k and α in (5.2.6), only dependent on M and B, we get

$$3\langle A D_{m_0} h, D_{m_0} h\rangle - Yh - 2\langle a, D_{m_0} h\rangle + \frac{3}{2\lambda}\|a\|_\infty^2 + \frac{3}{2\lambda}\|b\|_\infty^2 + \|c\|_\infty \leq 0, \tag{5.2.9}$$

where $\eta \leq t \leq \eta + \frac{s-\eta}{k}$, $x \in \mathbb{R}^N$. We now set $\delta = 2(s-\eta) - k(t-\eta)$, and we compute

$$\begin{aligned}
&3\langle A D_{m_0} h, D_{m_0} h\rangle - Yh - 2\langle a, D_{m_0} h\rangle + \mu \\
&\quad \leq \frac{12\Lambda |x|_{\mathbb{K}}^2}{\delta^2} + \frac{2\|B\| |x|_{\mathbb{K}}^2}{\delta} - \frac{k|x|_{\mathbb{K}}^2}{\delta^2} - \alpha + \frac{4\langle a, x\rangle}{\delta} + \mu \\
&\quad \leq \frac{|x|_{\mathbb{K}}^2}{\delta^2}\left(12\Lambda + 2\delta\|B\| - k + 2\right) - \alpha + 2\|a\|_\infty^2 + \mu \\
&\quad \leq \frac{|x|_{\mathbb{K}}^2}{\delta^2}\left(12\Lambda + 4\|B\| - k + 2\right) - \alpha 2\|a\|_\infty^2 + \mu,
\end{aligned}$$

and therefore inequality (5.2.9) holds true provided that we choose $\alpha = 2\|a\|_\infty^2 + \mu$. Combining inequalities (5.2.8) and (5.2.9), we obtain

$$\begin{aligned}
\max_{t \in \left]\eta, \eta + \frac{s-\eta}{k}\right[} \int_{\left|\delta^0_{2\sqrt{k}/\sqrt{s-\eta}}(x)\right|_{\mathbb{K}} \leq 1} e^{2h(x,t)} u^2(x,t)\, dx &\leq \max_{t \in \left]\eta, \eta + \frac{s-\eta}{k}\right[} \int_{\mathbb{R}^N} e^{2h(x,t)} u^2(x,t)\, dx \\
&\leq \int_{|x|_{\mathbb{K}} \geq \sigma} e^{2h(x,\eta)} u_0^2(x,t)\, dx,
\end{aligned} \tag{5.2.10}$$

We now want to bound the two exponents appearing on the left-hand side and on the right-hand side of (5.2.10). We first take care of the lower bound and we observe that, be definition

(5.2.6), for every $\eta \le t \le \eta + \frac{s-\eta}{k}$, we have

$$2h(x,t) \ge -2\frac{|x|_{\mathbb{K}}^2}{s-\eta} = -\frac{2\left|\delta_r^0\left(\delta_{r^{-1}}^0(x)\right)\right|^2}{s-\eta}. \tag{5.2.11}$$

We now set $r = \frac{\sqrt{s-\eta}}{2\sqrt{k}}$ and we observe that, if $\left|\delta^0_{\frac{2\sqrt{k}}{\sqrt{s-\eta}}}(x)\right|_{\mathbb{K}} = \left|\delta^0_{r^{-1}}(x)\right|_{\mathbb{K}} \le 1$, then we have

$$-\frac{2\left|\delta_r^0\left(\delta_{r^{-1}}^0(x)\right)\right|^2}{s-\eta} \ge -\frac{2\|\delta_r^0(x)\|^2}{s-\eta} \ge -\frac{2r^2}{s-\eta} = \frac{1}{2k}, \tag{5.2.12}$$

where in the first inequality we used the fact that $\delta < 1$ by assumption. Combining inequalities (5.2.11) and (5.2.12), we therefore obtain

$$2h(x,t) \ge \frac{1}{2k}, \quad \text{for every } \eta \le t \le \eta + \frac{s-\eta}{k}. \tag{5.2.13}$$

On the other hand, if $|x|_{\mathbb{K}} \ge \sigma$, we infer

$$-2h(x,\eta) = \frac{2|x|_{\mathbb{K}}^2}{2(s-\eta)} \ge \frac{\sigma^2}{s-\eta}. \tag{5.2.14}$$

Combining inequalities (5.2.10), (5.2.13) and (5.2.14), we get

$$\max_{t\in\left]\eta,\eta+\frac{s-\eta}{k}\right[} \int_{\left|\delta^0_{2\sqrt{k}/\sqrt{s-\eta}}(x)\right|_{\mathbb{K}}\le 1} e^{2h(x,t)}u^2(x,t)\,dx \le e^{\frac{1}{2k}} \exp\left(-\frac{\sigma^2}{s-\eta}\right)\|u_0\|^2_{L^2(\mathbb{R}^N)}. \tag{5.2.15}$$

To conclude the proof of Theorem 5.2.10, we rely once again on Theorem 5.2.7. More precisely, we let $\tau = \eta + \frac{s-\eta}{k}$ and therefore $\tau \in \left[\eta, \eta+\frac{1}{k}\right]$, $s-\eta = k(\tau-\eta)$. As a consequence, in virtue of Theorem 5.2.7, we have

$$\begin{aligned}
|u(0,\tau)|^2 \le \sup_{\mathcal{Q}_{\frac{\sqrt{s-\eta}}{4\sqrt{k}}}(0,\tau)} |u|^2 = \le \frac{C}{(s-\eta)^{\frac{Q+2}{2}}} \int\int_{\mathcal{Q}_{\frac{\sqrt{s-\eta}}{4\sqrt{k}}}(0,\tau)} u^2(x,t)\,dx\,dt \\
= \frac{C}{(s-\eta)^{\frac{Q+2}{2}}} \int_{\tau-\frac{s-\eta}{4k}}^{\tau} \int_{\left|\delta^0_{2\sqrt{k}/\sqrt{s-\eta}}(x)\right|_{\mathbb{K}}\le 1} u^2(x,t)\,dx\,dt.
\end{aligned}$$

We now apply (5.2.15) and we obtain

$$\begin{aligned}
\frac{C}{(s-\eta)^{\frac{Q+2}{2}}} \int_{\tau-\frac{s-\eta}{4k}}^{\tau} \int_{\left|\delta^0_{2\sqrt{k}/\sqrt{s-\eta}}(x)\right|_{\mathbb{K}}\le 1} u^2(x,t)\,dx\,dt &\le \frac{C}{(s-\eta)^{\frac{Q}{2}}} \exp\left(-\frac{\sigma^2}{C(s-\eta)}\right)\|u_0\|^2_{L^2(\mathbb{R}^N)} \\
&= \frac{C}{k^{\frac{Q}{2}}(\tau-\eta)^{\frac{Q}{2}}} \exp\left(-\frac{\sigma^2}{Ck(\tau-\eta)}\right)\|u_0\|^2_{L^2(\mathbb{R}^N)},
\end{aligned}$$

where C is a constant that only depends on M and k. Combining the two previous inequalities we obtain

$$|u(0,\tau)|^2 \le \frac{C}{k^{\frac{Q}{2}}(\tau-\eta)^{\frac{Q}{2}}} \exp\left(-\frac{\sigma^2}{Ck(\tau-\eta)}\right) \|u_0\|^2_{L^2(\mathbb{R}^N)}, \tag{5.2.16}$$

which concludes the proof in the case $y = 0$.

We now address the general case $y \in \mathbb{R}^N$. For any u, u_0 and (η, y) as in the statement, we set

$$v(x,\tau) := u\left(\left(e^{-\eta B}y, 0\right) \circ (x,\tau)\right), \quad \tau < \eta, x \in \mathbb{R}^N,$$

and we observe that be definition $v(x,\eta) = u(x+y,\eta) = u_0(x+y) = 0$ if $|x|_{\mathbb{K}} \le \sigma$. Moreover, as the vector field Y is invariant with respect to the left translation ℓ_z in (1.1.9), we have that $(\mathscr{L} \circ \ell_z)\, v = 0$, where $z = \left(e^{-\eta B}y, 0\right)$. Hence, taking advantage of (5.2.16), we infer

$$|u\,(z \circ (0,\tau))\,| = |v(0,\tau)| \le \frac{C}{(\tau-\eta)^{\frac{Q}{4}}} \exp\left(-\frac{\sigma^2}{C(\tau-\eta)}\right) \|u_0\|^2_{L^2(\mathbb{R}^N)}$$

and we conclude the proof of the general statement. □

As a simply consequence of Theorem 5.2.10, we obtain the following corollary.

Corollary 5.2.11. *Under the same assumptions of Theorem 5.2.10, there exist two constants k and C, only depending M, such that for every $\sigma > 0$ and $\eta \in \mathbb{R}$, we have*

$$\int_{\left|\xi - e^{(\eta-t)B}x\right|_{\mathbb{K}} \ge \sigma} \Gamma^2(x,t;\xi,\eta)d\xi \le \frac{Ce^{-\frac{\sigma^2}{C(t-\eta)}}}{(t-\eta)^{\frac{Q}{2}}}, \qquad (x,t) \in \mathbb{R}^N \times \left(\eta, \eta + \frac{1\wedge\sigma^2}{k}\right), \tag{5.2.17}$$

and

$$\int_{\left|x - e^{(\eta-t)B}\xi\right|_{\mathbb{K}} \ge \sigma} \Gamma^2(x,t;\xi,\eta)dx \le \frac{Ce^{-\frac{\sigma^2}{C(t-\eta)}}}{(t-\eta)^{\frac{Q}{2}}}, \qquad (x,t) \in \mathbb{R}^N \times \left(\eta, \eta + \frac{1\wedge\sigma^2}{k}\right). \tag{5.2.18}$$

Proof. We only show how to prove inequality (5.2.17), as (5.2.18) is proved similarly. As a first step, we observe that

$$\begin{aligned}\int_{\left|\xi - e^{(\eta-t)B}x\right|_{\mathbb{K}} \ge \sigma} \Gamma^2(x,t;\xi,\eta)d\xi &= \int_{|\xi-y|_{\mathbb{K}} \ge \sigma} \Gamma^2\left(e^{(\eta-t)B}y, t; \xi, \eta\right) d\xi \\ &= \int_{|\xi-y|_{\mathbb{K}} \ge \sigma} \Gamma^2\left(\left(e^{-tB}y, 0\right) \circ (0,\eta); \xi, \eta\right) d\xi.\end{aligned}$$

We now define the function

$$v(w,s) := \int_{|\xi-y|_{\mathbb{K}} \ge \sigma} \Gamma(w,s;\xi,\eta)\Gamma\left(\left(e^{-tB}y,0\right) \circ (0,\eta); \xi, \eta\right) d\xi,$$

which is a non-negative solution to (4.1.1) for $s > \eta$, with initial condition $v(w,\eta) = 0$ if

$|w-y|_{\mathbb{K}} < \sigma$ and $v(w,\eta) = \Gamma\left(\left(e^{-tB}y,0\right)\circ(0,\eta);w,\eta\right)$ if $|w-y|_{\mathbb{K}} \geq \sigma$. Setting $(w,s) = \left(e^{-tB}y,0\right)\circ(0,\eta)$ and applying Theorem 5.2.10, we get

$$\int_{|\xi-y|_{\mathbb{K}}\geq\sigma} \Gamma^2\left(\left(e^{-tB}y,0\right)\circ(0,\eta);\xi,\eta\right)d\xi = v\left(\left(e^{-tB}y,0\right)\circ(0,\eta)\right)$$
$$\leq \frac{Ce^{-\frac{\sigma^2}{C(t-\eta)}}}{(t-\eta)^{\frac{Q}{4}}}\|\Gamma\left(\left(e^{-tB}y,0\right)\circ(0,\eta);\cdot,\eta\right)\|_{L^2(\mathbb{R}^N)},$$

and the proof of inequality (5.2.17) follows from Corollary 5.2.9. □

We are now in a position to state and prove the following result concerning the upper bound for the fundamental solution. We remark that the exponent $\frac{Q}{2}$ appearing in estimate (5.2.19) is optimal, as we can easily see in the case of constant coefficients Kolmogorov operators, whose fundamental solution is explicit and given by formula (1.1.31).

Theorem 5.2.12 (Gaussian upper bound)**.** *Let $\widetilde{\mathscr{L}}$ be an operator of the form* (5.1.5) *satisfying assumptions **(H1)**-**(H2)**-**(H3-ii)**. Then there exists a positive constant C, only dependent on Q, λ, Λ and q, such that*

$$\Gamma(x,t;y,t_0) \leq \frac{C}{(t-t_0)^{\frac{Q}{2}}}\exp\left(-\frac{1}{C}\left|\delta^0_{1/\sqrt{t-t_0}}\left(x-e^{(t-t_0)B}y\right)\right|^2_{\mathbb{K}}\right), \qquad (5.2.19)$$

for any $0 < t-t_0 \leq 1$ and $x,y \in \mathbb{R}^N$.

Proof. We split the proof into four steps.

Step 1. We first prove the statement for $y=0$ and $t-t_0 = \frac{1}{k}$, where k is the constant given by Theorem 5.2.10. For a fixed $x \in \mathbb{R}^N$, we set

$$\sigma(x) = \frac{|x|_{\mathbb{K}}}{2\left\|e^{\frac{(t-t_0)}{2}B}\right\|}.$$

If $\sigma(x) \leq 1$ and thus $|x|_{\mathbb{K}} \leq 2\left\|e^{\frac{(t-t_0)}{2}B}\right\|$, then the thesis simply follows from Theorem 5.2.8, as $t-t_0 = \frac{1}{k}$, with k only depending on M.

On the other hand, if $\sigma(x) > 1$, then we set $\eta = t + \frac{t_0-t}{2}$ and we apply the reproduction property

$$\Gamma(x,t;0,t_0) = \int_{\mathbb{R}^N}\Gamma(x,t;\xi,\eta)\Gamma(\xi,\eta;0,t_0)d\xi = J_1 + J_2,$$

where

$$J_1 := \int_{\left|\xi-e^{\frac{(t-t_0)B}{2}}x\right|_{\mathbb{K}}\geq\sigma(x)}\Gamma(x,t;\xi,\eta)\Gamma(\xi,\eta;0,t_0)d\xi,$$
$$J_2 := \int_{\left|\xi-e^{\frac{(t-t_0)B}{2}}x\right|_{\mathbb{K}}<\sigma(x)}\Gamma(x,t;\xi,\eta)\Gamma(\xi,\eta;0,t_0)d\xi.$$

By the Cauchy-Schwarz inequality, we have

$$(J_1)^2 \le \int_{\left|\xi - e^{\frac{(t-t_0)B}{2}}x\right|_{\mathbb{K}} \ge \sigma(x)} \Gamma^2(x,t;\xi,\eta)d\xi \int_{\left|\xi - e^{\frac{(t-t_0)B}{2}}x\right|_{\mathbb{K}} \ge \sigma(x)} \Gamma^2(\xi,\eta;0,t_0)d\xi. \tag{5.2.20}$$

Applying inequality (5.2.17) and Corollary 5.2.9 to the right-hand side of (5.2.20), we infer

$$(J_1)^2 \le \frac{Ce^{-\frac{\sigma^2(x)}{C(t-t_0)}}}{(t-t_0)^Q} = Ck^Q \exp\left(-\frac{k|x|^2_{\mathbb{K}}}{4C\|e^{\frac{1}{2k}B}\|^2}\right).$$

As far as J_2 is concerned, we observe that, if $\left|\xi - e^{\frac{(t-t_0)}{2}B}x\right|_{\mathbb{K}} < \sigma(x)$, in virtue of the definition of $\sigma(x)$, we have

$$|\xi|_{\mathbb{K}} \ge \left|e^{-\frac{(t-t_0)}{2}B}x\right|_{\mathbb{K}} - \left|\xi - e^{-\frac{(t-t_0)}{2}B}x\right|_{\mathbb{K}} \ge \frac{|x|_{\mathbb{K}}}{\|e^{\frac{(t-t_0)}{2}B}\|} - \sigma(x) = \sigma(x). \tag{5.2.21}$$

Owing to (5.2.21) and applying once again the Cauchy-Schwarz inequality, inequality (5.2.18) and Corollary 5.2.9, we obtain

$$\begin{aligned}(J_2)^2 &\le \int_{|\xi|_{\mathbb{K}} \ge \sigma(x)} \Gamma^2(x,t;\xi,\eta)d\xi \int_{|\xi|_{\mathbb{K}} \ge \sigma(x)} \Gamma^2(\xi,\eta;0,t_0)d\xi \\ &\le \frac{Ce^{-\frac{\sigma^2(x)}{C(t-t_0)}}}{(t-t_0)^{\frac{Q}{2}}} \int_{\mathbb{R}^N} \Gamma^2(x,t;\xi,\eta)d\xi \\ &\le \frac{C}{(t-t_0)^Q} e^{-\frac{\sigma^2(x)}{C(t-t_0)}} = Ck^Q \exp\left(-\frac{-k|x|^2_{\mathbb{K}}}{4C\|e^{\frac{1}{2k}B}\|^2}\right).\end{aligned}$$

This concludes the proof of the statement in the case where $\sigma(x) \ge 1$, that is

$$\Gamma(x,t;0,t_0) \le Ce^{-\frac{|x|^2}{C}}, \qquad t - t_0 = \frac{1}{k}, x \in \mathbb{R}^N, \tag{5.2.22}$$

where the constant C only depends on M and B.

Step 2. In order to generalize estimate (5.2.22) to the case $0 < t - t_0 \le \frac{1}{k}$, we use a scaling argument. For $r \in [0,1]$, we set $\Gamma^r(x,t;0,t_0) = r^Q\Gamma\left(\delta_r(x,t);\delta_r(0,t_0)\right)$ and we observe that it is a fundamental solutions of operator $\mathscr{L}^{(r)}$ in Remark 5.2.2. We now fix a time t such that $0 < t - t_0 \le \frac{1}{k}$ and set $r = k(t-t_0)$. Then we have

$$\Gamma(x,t;0,t_0) = r^{-\frac{Q}{2}}\Gamma^{(\sqrt{r})}\left(\delta^0_{1/\sqrt{r}}(x), \frac{t}{r}; 0, \frac{t_0}{r}\right) \le Cr^{-\frac{Q}{2}}e^{-\frac{1}{C}\left|\delta^0_{1/\sqrt{r}}(x)\right|^2_{\mathbb{K}}}$$

where in the last inequality we took advantage of (5.2.22). To summarize, we have proved that

$$\Gamma(x,t;0,t_0) \le \frac{C}{(t-t_0)^{\frac{Q}{2}}} e^{-\frac{|x|^2_{\mathbb{K}}}{C(t-t_0)}}, \qquad 0 < t - t_0 \le \frac{1}{k}, x \in \mathbb{R}^N. \tag{5.2.23}$$

Step 3. We now remove the restriction $y = 0$. We set $z = (e^{-t_0 B}y, 0)$ and we denote by $\Gamma^{(z)}$ the fundamental solution of operator $\mathscr{L}^{(z)} := \mathscr{L} \circ \ell_z$, where ℓ_z is the left translation defined in (1.1.9). $\Gamma^{(z)}$ satisfies inequality (5.2.23) and therefore we have

$$\Gamma(x,t;y,t_0) = \Gamma^{(z)}\left(z^{-1} \circ (x,t); 0, t_0\right) \le \frac{C}{(t-t_0)^{\frac{Q}{2}}} \exp\left(-\frac{1}{C}|\delta^0_{1/\sqrt{t-t_0}}\left(x - e^{(t-t_0)B}y\right)|^2\right)$$

for every $x, y \in \mathbb{R}^N$ and $0 < t - t_0 \le \frac{1}{k}$.

Step 4. To prove the general statement, we are only left with relaxing the condition on the length of the time interval. As a first step, we suppose that $0 < t - t_0 \le \frac{2}{k}$ and we set $\tau = \frac{t-t_0}{2}$. We now apply the reproduction property and infer

$$\begin{aligned}
\Gamma(x,t;y,t_0) &= \int_{\mathbb{R}^N} \Gamma(x,t;\xi,t_0+\tau)\Gamma(\xi,t_0+\tau;y,t_0)d\xi \\
&\le \frac{C}{\tau^Q}\int_{\mathbb{R}^N} e^{-\frac{1}{C}\left|\delta_{1/\sqrt{\tau}}(x-e^{-\tau B}\xi)\right|^2_{\mathbb{K}}} e^{-\frac{1}{C}\left|\delta_{1/\sqrt{\tau}}(\xi-e^{-\tau B}y)\right|^2_{\mathbb{K}}} d\xi \\
&\le \frac{C}{\tau^Q}\int_{\mathbb{R}^N} e^{-\frac{1}{C}\left|\delta_{1/\sqrt{\tau}}(x-e^{-\tau B}\xi)\right|^2_{\mathbb{K}}} e^{-\frac{1}{C}\left|\delta_{1/\sqrt{\tau}}(e^{-\tau B}\xi-e^{-(t-t_0)B}y)\right|^2_{\mathbb{K}}} d\xi \\
&\le \frac{C}{(t-t_0)^{\frac{Q}{2}}} e^{-\frac{1}{C}\left|\delta_{1/\sqrt{\tau}}(x-e^{-(t-t_0)B}y)\right|^2_{\mathbb{K}}},
\end{aligned}$$

where in the last line we exploited the reproduction property for a standard Gaussian kernel. By iterating this procedure, we can extend estimate above to any bounded domain in time and therefore conclude the proof. □

5.2.3 Gaussian lower bound

Lemma 5.2.13. *Let $\mathscr{L}$ be an operator of the form* (4.1.1) *satisfying assumptions **(H1)**-**(H2)**-**(H3)**. Then there exist two positive constants R and c_2, only dependent on Q and B, such that*

$$\int_{\left|\delta^0_{(\sqrt{t-t_0})}\left(y-e^{(t-t_0)B}x\right)\right|_{\mathbb{K}} \le R} \Gamma(x,t;y,t_0)dx \ge c_2, \tag{5.2.24}$$

for any $0 < t - t_0 \le 1$ and $y \in \mathbb{R}^N$.

Proof. We first notice that for a small enough constant c_3, which depends only on Q and B, the function

$$v(y,t_0) := \int_{\mathbb{R}^N} \Gamma(x,t;y,t_0) - e^{-c_3(t-t_0)}, \quad t > t_0, \quad y \in \mathbb{R}^N,$$

is a weak super-solution of the Cauchy problem

$$\begin{cases} \mathscr{L}^* v(y,t_0) = -e^{-c(t-t_0)}(c - \text{Tr}(B) + c_3) \le 0, & t > t_0, \quad y \in \mathbb{R}^N, \\ v(y,t) = 0, & y \in \mathbb{R}^N, \end{cases}$$

where $\mathscr{L}^*$ is the adjoint operator defined in (1.2.7). Hence, in virtue of the maximum principle we infer $v \geq 0$, that is

$$\int_{\mathbb{R}^N} \Gamma(x,t;y,t_0) \geq e^{-c_3(t-t_0)}, \quad t > t_0, \quad y \in \mathbb{R}^N. \tag{5.2.25}$$

We now observe that

$$\begin{aligned}
&\int_{\left|\delta^0_{(\sqrt{t-t_0})}\left(y-e^{(t-t_0)B}x\right)\right|_{\mathbb{K}} \geq R} \Gamma(x,t;y,t_0)dx \qquad (5.2.26)\\
&\leq \frac{c_1}{(t-t_0)^{\frac{Q}{2}}} \int_{\left|\delta^0_{(\sqrt{t-t_0})}\left(y-e^{(t-t_0)B}x\right)\right|_{\mathbb{K}} \geq R} \exp\left(-\frac{1}{c_1}\left|\delta^0_{(t-t_0)^{-\frac{1}{2}}}\left(y-e^{(t-t_0)B}x\right)\right|^2_{\mathbb{K}}\right) dx\\
&= c_1 \int_{|z|_{\mathbb{K}} \geq R} \exp\left(-\frac{1}{c_1}|z|^2_{\mathbb{K}}\right) dz,
\end{aligned}$$

where in the second line we have used the upper bound (5.2.19) and in the third line we have performed the change of variables $z = \delta^0_{(\sqrt{t-t_0})}\left(y - e^{(t-t_0)B}x\right)$. Combining (5.2.25) and (5.2.26) and choosing c_3 small enough we obtain the thesis. □

We are now in a position to state and prove the following result concerning the Gaussian lower bound of the fundamental solution.

Theorem 5.2.14 (Gaussian lower bound)**.** *Let $\mathscr{L}$ be an operator of the form* (4.1.1) *satisfying assumptions **(H1)-(H2)-(H3)**. Then there exists a positive constant c_4, only dependent on Q, λ, Λ and q, such that*

$$\Gamma(x,t;y,t_0) \geq \frac{c_4}{(t-t_0)^{\frac{Q}{2}}} e^{-c_4\langle C^{-1}(t-t_0)\left(y-e^{(t-t_0)B}x\right),\, y-e^{(t-t_0)B}x\rangle} \tag{5.2.27}$$

for any $0 < t - t_0 \leq 1$ and $x, y \in \mathbb{R}^N$.

Proof. We restrict ourselves to the case where $x = 0$, as the general statement can be obtained from the dilation and translation-invariance of the operator $\mathscr{L}$. Then, for every $y \in \mathbb{R}^N$ and $R > 0$, we set

$$D_R := \left\{\xi \in \mathbb{R}^N : \left|\delta^0_{\sqrt{\tau}}(y - e^{\tau B}\xi)\right|_{\mathbb{K}} \leq R\right\}$$

and we compute

$$\begin{aligned}
\text{meas}(D_R) &= \int_{D_R} d\xi = R^Q \int_{\left|\delta^0_{\sqrt{\tau}}(y-e^{\tau B}\xi)\right|_{\mathbb{K}} \leq 1} d\xi = R^Q \tau^{Q/2} \int_{|(y-e^{\tau B}\xi)|_{\mathbb{K}} \leq 1} d\xi\\
&= R^Q \tau^{Q/2} \int_{|\xi|_{\mathbb{K}} \leq 1} d\xi = R^Q \tau^{Q/2} \text{meas}(B_1(0)) = c_5 \tau^{Q/2},
\end{aligned}$$

where the constant c_5 only depends on B and R. We also note that the function $\langle C^{-1}(t - t_0)\left(y - e^{(t-t_0)B}x\right), y - e^{(t-t_0)B}x\rangle$ is bounded by a constant M in D_R (see [73, Lemma 3.3]). Lastly, we now set $\tau = \frac{t-t_0}{2}$ and apply to Γ the global Harnack inequality stated in Theorem

5.2.4, which yields

$$\Gamma(y,t;y,t_0) \geq c_0 e^{-c_0\langle C^{-1}(\tau)\left(\xi-e^{\tau B}x\right),\xi-e^{\tau B}x\rangle}\Gamma(\xi,t+\tau;y,t_0), \quad y,\xi\in\mathbb{R}^N. \tag{5.2.28}$$

Hence, integrating inequality (5.2.28) over D_R, we infer

$$\begin{aligned}\Gamma(y,t;y,t_0) &= \frac{c_6}{\tau^{Q/2}}\int_{D_R}\Gamma(y,t;y,t_0)d\xi \\ &\geq \frac{c_6\,c_0}{\tau^{Q/2}}\int_{D_R} e^{-c_0\langle C^{-1}(\tau)\left(\xi-e^{\tau B}x\right),\xi-e^{\tau B}x\rangle}\Gamma(\xi,t+\tau;y,t_0)d\xi \\ &\geq \frac{c_6\,c_0}{\tau^{Q/2}}\int_{D_R} e^{-M}\Gamma(\xi,t+\tau;y,t_0)d\xi \\ &\geq \frac{c_7}{(t-t_0)^{Q/2}},\end{aligned}$$

where the last inequality is a direct consequence of Lemma 5.2.13 and the constant c_7 only depends on Q, λ, Λ, q and B.
Setting $\tau=\frac{3}{4}(t-t_0)$ and $x=0$, we apply once again Theorem 5.2.4 and we get

$$\begin{aligned}\Gamma(0,t;y,t_0) &\geq c_0 e^{-c_0\langle C^{-1}(\tau)y,y\rangle}\Gamma(y,t+\tau;y,t_0) \\ &\geq \frac{c_8}{(t-t_0)^{Q/2}}e^{-c_0\langle C^{-1}(\tau)y,y\rangle} \geq \frac{c_9}{(t-t_0)^{Q/2}}e^{-c_9\langle C^{-1}(t-t_0)y,y\rangle},\end{aligned}$$

where the last inequality is a consequence of a property of the covariance matrix C (see [72, Remark 4.5]). This concludes the proof. □

5.3 Proof of Theorem 5.1.2

This section is devoted to the proof of our existence result, Theorem 5.1.2, under assumptions **(H1)-(H2)-(H3-ii)**. Our idea is to adapt the limiting procedure proposed in [7] to the case of our interest.

Let us consider the operator $\mathscr{L}$ in (4.1.1) under the assumptions **(H1)-(H2)-(H3-ii)**. Our first aim is to build a sequence of operators $(\mathscr{L}_\varepsilon)_\varepsilon$ satisfying assumptions **(C)** of Theorem 1.2.4. Without loss of generality we restrict ourselves to the case of $(T_0,T_1)=(0,T)$, with $T>1$ and hence we denote $S_T := S_{0T}$. Thus, we may consider $\rho\in C_0^\infty(\mathbb{R})$ and $\psi\in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned}&\int_{\mathbb{R}}\rho(t)\,dt=1, \quad \operatorname{supp}\rho\subset B\left(\frac{T}{2},\frac{T}{4}\right), \quad \text{and} \\ &\int_{\mathbb{R}^N}\psi(x)\,dx=1, \quad \operatorname{supp}\psi\subset B(0,1),\end{aligned}$$

where by abuse of notation $B\left(\frac{T}{2},\frac{T}{4}\right)$ denotes the Euclidean ball on $\mathbb{R}$ of radius $\frac{T}{4}$ and center

$\frac{T}{2}$ of suitable dimension and $B(0,1)$ denotes the Euclidean ball of $\mathbb{R}^N$ of radius 1 and center 0 of suitable dimension. Then, for every $\varepsilon \in (0,1]$ we classically construct two families of mollifiers

$$\rho_\varepsilon(t) = \frac{1}{\varepsilon}\,\rho\left(\frac{t-\frac{T}{2}}{\varepsilon}\right), \qquad \psi_\varepsilon(x) = \frac{1}{\varepsilon^N}\,\psi\left(\frac{x}{\varepsilon}\right).$$

Lastly, for every $\varepsilon \in (0,1]$, for every $t \in (0,T)$ and $x \in \mathbb{R}^N$ we define

$$\begin{aligned}
(a_{ij})_\varepsilon(x,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^N} a_{ij}(x-y,(1-\varepsilon)t+\tau)\psi_\varepsilon(y)\,\rho_\varepsilon(\tau)\,dy\,d\tau, \quad \forall i,j=1,\dots,N,\\
(b_i)_\varepsilon(x,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^N} b_i(x-y,(1-\varepsilon)t+\tau)\psi_\varepsilon(y)\,\rho_\varepsilon(\tau)\,dy\,d\tau, \qquad \forall i=1,\dots,N,\\
c_\varepsilon(x,t) &:= \int_{\mathbb{R}}\int_{\mathbb{R}^N} c(x-y,(1-\varepsilon)t+\tau)\psi_\varepsilon(y)\,\rho_\varepsilon(\tau)\,dy\,d\tau.
\end{aligned}$$

These newly defined coefficients are smooth and such that $(a_{ij})_\varepsilon \to a_{ij}$, $(b_i)_\varepsilon \to b_i$, $(c)_\varepsilon \to c$ in $L^1(S_T)$. Hence, the L^1 convergence implies the pointwise convergence a.e. Moreover, for every $\varepsilon \in (0,1]$ the coefficients $(a_{ij})_\varepsilon$, $(b_i)_\varepsilon$ and $(c)_\varepsilon$ are bounded from above by the same constant appearing in assumption **(H3-ii)**. Indeed, for every $(x,t) \in K$, with $K \subset \mathbb{R}^{N+1}$ compact,

$$|(a_{ij})_\varepsilon(x,t)| \le \sup_{(x,t)\in K} |a_{ij}(x,t)| \le M,$$

for every $i,j = 1,\dots,N$. The same statement holds true for the coefficients c_ε and $(b_i)_\varepsilon$, with $i = 1,\dots,N$ and $\varepsilon \in (0,1]$.

In addition, given assumption **(H3-ii)** and our definition of the family of mollifiers we have

$$\begin{aligned}
\left|\tfrac{\partial}{\partial x_k}(b_i)_\varepsilon(x,t)\right| &= \left|\int_{\mathbb{R}}\int_{\mathbb{R}^N} b_i(y,(1-\varepsilon)t+\tau)\tfrac{\partial\psi_\varepsilon}{\partial x_k}(x-y)\,\rho_\varepsilon(\tau)\,dy\,d\tau\right|\\
&\le M \int_{B(\frac{T}{2},\varepsilon)} |\rho_\varepsilon(\tau)|d\tau \int_{B(0,\varepsilon)} |\tfrac{\partial\psi_\varepsilon}{\partial x_k}(x-y)|\,dy \le MC_1
\end{aligned}$$

for every $i = 1,\dots,m_0$ and for every $k = 1,\dots,m_0$, where C_1 is a constant depending on ψ. Indeed, for every $y \in B(0,\varepsilon)$, we have

$$\left|\tfrac{\partial\psi_\varepsilon}{\partial x_k}(x-y)\right| \le \frac{1}{\varepsilon^{N+1}}\left\|\tfrac{\partial\psi}{\partial x_k}\right\|_{L^\infty(B(0,\varepsilon))} \le \frac{1}{\varepsilon^{N+1}}\left(\frac{2}{e}\right)^2 \sup_{[-\varepsilon,\varepsilon]} |2y| \le \frac{C_1}{\varepsilon^N},$$

where C_1 is a constant that does not dependent on ε.

The same statement holds true also for $\partial_{x_k}(a_{ij})_\varepsilon$ and $\partial_{x_k}c_\varepsilon$, with $k = 1, \dots, m_0$ and $\varepsilon \in (0, 1]$. Hence, thanks to the mean value theorem along the direction of the vector fields ∂_{x_k}, the coefficients $(a_{ij})_\varepsilon$,c_ε and $(b_i)_\varepsilon$, with $i = 1, \dots, N$ and $\varepsilon \in (0, 1]$, are uniformly Lipschitz continuous (i.e. Hölder continuous of exponent $\alpha = 1$), and therefore Dini continuous.

Hence, we can apply Theorem 2.1.5 to $(\Gamma_L^\varepsilon)_\varepsilon$ for every $\varepsilon \in (0, 1]$. Thus, there exists a sequence of equibounded fundamental solutions $(\Gamma_L^\varepsilon)_\varepsilon$, in the sense that each of them satisfies Theorem 5.1.3, i.e. for every $(x, t), (\xi, \tau) \in S_T$, with $0 < \tau < t < T$

$$C^- \, \Gamma_K^{\lambda^-}(x, t; y, \tau) \;\leq\; \Gamma_L^\varepsilon(x, t; y, \tau) \;\leq\; C^+ \, \Gamma_K^{\lambda^+}(x, t; y, \tau).$$

We point out that, since the coefficients of $\mathscr{L}_\varepsilon$ are uniformly bounded by M, the coefficients of Theorem 5.1.2 do not depend on ε.

First of all, for every fixed $(\xi, \tau) \in S_T$ our aim is to show there exists a converging subsequence $(\Gamma_L^\varepsilon(\cdot, \cdot; \xi, \tau))_\varepsilon$, from now on simply $(\Gamma_L^\varepsilon)_\varepsilon$, in every compact subset of $\left(\mathbb{R}^N \setminus \{\xi\}\right) \times (\tau, T)$.

For this reason, we define a sequence of open subsets $(\Omega_p)_{p \in \mathbb{N}}$ of S_T

$$\Omega_p := \left\{ x \in \mathbb{R}^N \,:\, |x|^2 < p^2, |x - \xi|^2 > \frac{1}{2p} \right\} \times \Big(\tau + \frac{1}{p}, T - \frac{1}{p} \Big).$$

Note that $\Omega_p \subset\subset \Omega_{p+1}$ for every $p \in \mathbb{N}$. Moreover, $\cup_{p=1}^{+\infty} \Omega_p = \left(\mathbb{R}^N \setminus \{\xi\}\right) \times (\tau, T)$. Since $\Gamma_L^{\lambda^+}$ is a bounded function in Ω_p, we have that $(\Gamma_L^\varepsilon)_\varepsilon$ is an equibounded sequence in every Ω_p. Then, as the sequence $(\Gamma_L^\varepsilon)_\varepsilon$ is equibounded in Ω_2, it is equicontinuous in Ω_1 thanks to Theorem 2.1.5. Moreover, by Theorem 1.2.4 and Theorem 2.1.5, we also have that

$$\left(\tfrac{\partial \Gamma_L^\varepsilon}{\partial x} \right)_\varepsilon, \quad \left(\frac{\partial \Gamma_L^\varepsilon}{\partial \xi} \right)_\varepsilon, \quad \left(\frac{\partial^2 \Gamma_L^\varepsilon}{\partial x^2} \right)_\varepsilon, \quad \left(\frac{\partial^2 \Gamma_L^\varepsilon}{\partial \xi^2} \right)_\varepsilon, \quad (Y \Gamma_L^\varepsilon)_\varepsilon,$$

are bounded sequences in $C^0(\Omega_1)$, where Y is the Lie derivative defined in (1.1.5). Thus, there exists a subsequence $(\Gamma_L^{1,\varepsilon_1})_{\varepsilon_1}$ that converges uniformly to some function Γ_1 that satisfies (5.1.3) in Ω_1. Moreover, $\Gamma_1 \in C^2(\Omega_1)$ and the function $u(x, t) := \Gamma_1(x, t; \xi, \tau)$ is a.e. a classical solution to $\mathscr{L} u = 0$ in Ω_1, and hence a weak solution in the set Ω_1.

We next apply the same argument to the sequence $(\Gamma_L^{1,\varepsilon_1})_{\varepsilon_2}$ on the set Ω_2, and obtain a subsequence $(\Gamma_L^{2,\varepsilon_2})_{\varepsilon_2}$ that converges in $C^2(\Omega_2)$ to some function Γ_2, that belongs to $C^2(\Omega_2)$ and satisfies the bounds of Theorem 5.1.3 in Ω_2. Moreover, the function $u(x, t) := \Gamma_2(x, t; \xi, \tau)$ is a. e. a classical solution to $\mathscr{L} u = 0$ in the set Ω_2, and hence a weak solution, to $\mathscr{L} u = 0$ in the set Ω_2.

We next proceed by induction. Let us assume that the sequence $(\Gamma_L^{q-1,\varepsilon_{q-1}})_{\varepsilon_{q-1}}$ on the set Ω_q has been defined for some $q \in \mathbb{N}$. We extract from it a subsequence $(\Gamma_L^{q,\varepsilon_q})_{\varepsilon_q}$ converging in $C^2(\Omega_q)$ to some function Γ_q, satisfying Theorem 5.1.3 in Ω_q and it agrees with Γ_{q-1} on the set Ω_{q-1}.

Next, we define a function Γ_L in the following way: for every $\left(\mathbb{R}^N \setminus \{\xi\}\right) \times (\tau, T)$ we choose

$q \in \mathbb{N}$ such that $(x,t) \in \Omega_q$ and we set $\Gamma_L(x,t;\xi,\tau) := \Gamma_q(x,t;\xi,\tau)$.

This argument can be repeteadly applied to any choice of $(\xi,\tau) \in S_T$. Hence, it provides us with a non ambiguous definition of Γ_L. Indeed, for any given choice of $(\xi,\tau) \in S_T$, if $(x,t) \in \Omega_p$, then $\Gamma_p(x,t;\xi,\tau) = \Gamma_q(x,t;\xi,\tau)$ for every choice of $p,q \in \mathbb{N}$. In particular, we proved that Γ_L^ε converges compactly uniformly on S_T to a function Γ on a compactly generated space. Hence, $\Gamma(\cdot,\cdot;\xi,\tau)$ is continuous on $\mathbb{R}^N \times (\tau,T) \setminus \{(\xi,\tau)\}$ and a weak solution to $\mathscr{L}u = 0$ on $\mathbb{R}^N \times (\tau,T)$. Finally, Theorem 5.1.3 holds true for Γ_L because it is a weak solution to (4.1.1) in the sense of Definition 4.1.2.

Secondly, we verify that for any bounded function $\varphi \in C(\mathbb{R}^N)$ and any $x,\xi \in \mathbb{R}^N$ the function

$$u(x,t) = \int\limits_{\mathbb{R}^N} \Gamma_L(x,t;\xi,\tau)\,\varphi(\xi)\,d\xi \tag{5.3.1}$$

verifies the corresponding weak Cauchy problem in (5.1.1), hence it is a weak solution to (4.1.1) in $\mathbb{R}^N \times (\tau,T)$ and takes the initial datum when $t \to \tau$, with $t > \tau$. Note that u is well-defined given the Gaussian bounds of Theorem 5.1.3 and property 7. of Theorem 1.2.4. Then, considering that for every $\varepsilon \in (0,1]$

$$u_\varepsilon(x,t) := \int_{\mathbb{R}^N} \Gamma_L^\varepsilon(x,t;\xi,t_0)\,\varphi(\xi)\,d\xi. \tag{5.3.2}$$

satisfies $\mathscr{L}_\varepsilon u_\varepsilon = 0$ in the classical sense, see Theorem 1.2.4, thanks to the Dominated Lebesgue convergence theorem we get u defined in (5.3.1) is a weak solution to (4.1.1) in $\mathbb{R}^N \times (\tau,T)$.

Thus, we are left with the proof of the limiting property. By applying property 3. of Theorem 1.2.4 to the regularized operator $\mathscr{L}_\varepsilon$, we have that for every $\varepsilon \in (0,1]$ and for every $(\xi,\tau) \in \mathbb{R}^N \times (0,T)$ the following holds

$$\lim_{\substack{(x,t)\to(\xi,\tau)\\ t>\tau}} u_\varepsilon(x,t) = \varphi(\xi),$$

where u_ε is defined as above in (5.3.2). Now, thanks to Theorem 5.1.3 we are able to apply the Lebesgue Dominated Convergence Theorem, and thus for every $(\xi,\tau) \in \mathbb{R}^N \times (0,T)$ we have

$$\varphi(\xi) = \lim_{\varepsilon\to 0} \lim_{\substack{(x,t)\to(\xi,\tau)\\ t>\tau}} u_\varepsilon(x,t) = \lim_{\substack{(x,t)\to(\xi,\tau)\\ t>\tau}} u(x,t).$$

Finally, we are left with the proof the reproduction property listed in Theorem 5.1.2.

For every $\varepsilon > 0$, $x,\xi \in \mathbb{R}^N$ and $0 < \tau < s < t < T$ we get

$$\Gamma_L^\varepsilon(x,t;y,s)\Gamma_L^\varepsilon(y,s;\xi,\tau) \le C^+\,\Gamma_L^{\lambda^+}(x,t;y,s)C^+\,\Gamma_L^{\lambda^+}(y,s;\xi,\tau),$$

where the right-most inequality is obtained by applying the Gaussian upper bound in Theorem 5.1.3 to the fundamental solution Γ_ε. Hence, by applying the reproduction property of the fundamental solution $\Gamma_L^{\lambda^+}$ we get

$$\int_{\mathbb{R}^N} \Gamma_L^{\lambda^+}(x,t;y,s)\Gamma_L^{\lambda^+}(y,s;\xi,\tau)\,d\xi\,d\tau = \Gamma_L^{\lambda^+}(x,t;\xi,\tau),$$

which allows us to use the Lebesgue Dominated Convergence theorem. Thus, the property holds true.

We complete the proof by adapting these arguments to the adjoint operator $\mathscr{L}^*$ when considering the function v. $\square$

Part III

Relativistic generalization

In the last part of this thesis, we consider a class of second order degenerate kinetic operators $\mathscr{L}$ in the framework of special relativity. More precisely, we address the problem that particles are allowed to move faster than light in the model described by

$$\mathscr{K} u(p,y,t) = \frac{\partial^2 u}{\partial p^2}(p,y,t) - p\frac{\partial u}{\partial y}(p,y,t) - \frac{\partial u}{\partial t}(p,y,t) = 0 \qquad (p,y,t) \in \mathbb{R}^3, \qquad \text{(III.1)}$$

even though this is inconsistent with the theory of special relativity. The possible relativistic generalization of (III.1) that we study in this work is the following

$$\mathscr{L}u(p,y,t) = \sqrt{p^2+1}\,\tfrac{\partial}{\partial p}\left(\sqrt{p^2+1}\,\tfrac{\partial u}{\partial p}\right) - p\tfrac{\partial u}{\partial y} - \sqrt{p^2+1}\,\tfrac{\partial u}{\partial t} = 0, \qquad (p,y,t) \in \mathbb{R}^3. \quad \text{(III.2)}$$

In the next chapter, we explain why we believe that (III.2) is the suitable relativistic counterpart of (III.1) and we describe $\mathscr{L}$ as an Hörmander operator which is invariant with respect to Lorentz transformations. Then we prove a Lorentz-invariant Harnack type inequality, and we derive accurate asymptotic lower bounds for positive solutions to $\mathscr{L}u = 0$. As a consequence, we obtain a lower bound for the density of the relativistic stochastic process associated to $\mathscr{L}$. Throughout the part, we are only concerned with classical solutions to equation (III.2). These results are presented in Chapter 6 below and were obtained in collaboration with Anceschi and Polidoro in [10].

Chapter 6

Relativistic Fokker-Planck operator

6.1 Motivation

This chapter is devoted to the study of the following second order partial differential equation

$$\mathscr{L}u(p,y,t) = \sqrt{|p|^2+1}\,\mathrm{div}_p\left(\mathcal{D}\,D_p u\right) - \langle p, D_y u\rangle - \sqrt{|p|^2+1}\,\partial_t u = 0, \tag{6.1.1}$$

where $(p,y,t) \in \mathbb{R}^{2m+1}$ and $\mathcal{D}$ is the *relativistic diffusion matrix* given by

$$\mathcal{D} = \frac{1}{\sqrt{|p|^2+1}}\left(\mathbb{I}_m + p\otimes p\right).$$

Here and in the following, $\mathbb{I}_m$ denotes, as usual, the $m\times m$ identity matrix and $p\otimes p = (p_i p_j)_{i,j=1,\dots,m}$. In this context, a solution $u = u(p,y,t)$ to (6.1.1) denotes the density of particles in the phase space with momentum p and position y, at time t.

We observe that $\mathscr{L}$ is a strongly degenerate differential operator, since only second order derivatives with respect to the momentum variable $p \in \mathbb{R}^m$ appear. However, the first order part of $\mathscr{L}$ induces a strong regularizing property. More precisely, $\mathscr{L}$ is hypoelliptic in the sense of Definition 1.0.1, as we will prove in Appendix 6.A. As a consequence, we only need to consider classical solutions to $\mathscr{L}u = 0$. Indeed, as we will show in Appendix 6.A, we can write $\mathscr{L}$ in the form

$$\mathscr{L} := \sum_{j=1}^m X_j^2 + X_{m+1}, \tag{6.1.2}$$

with

$$X_j = \sum_{k=1}^m \left(\delta_{jk} + \tfrac{p_j p_k}{1+\sqrt{|p|^2+1}}\right)\tfrac{\partial}{\partial p_k}, \quad j=1,\dots,m, \quad \text{and} \quad X_{m+1} = \sum_{k=1}^m c_k(p) X_k - Y, \tag{6.1.3}$$

where $c_1, \dots c_m$ are smooth functions and

$$Y = \langle p, D_y u\rangle + \sqrt{|p|^2+1}\,\tfrac{\partial}{\partial t}. \tag{6.1.4}$$

Moreover, $\mathscr{L}$ does satisfy the Hörmander's rank condition (1.0.6), which in the present setting reads as follows

$$\operatorname{rank} \operatorname{Lie}\left\{X_1, \ldots, X_m, X_{m+1}\right\}(p, y, t) = 2m + 1, \qquad \forall (p, y, t) \in \mathbb{R}^{2m+1}, \tag{6.1.5}$$

and it is a well-known criterion for the hypoellipticity of an operator in the form (6.1.2) (see Chapter 1).

The aforementioned regularity property of operator $\mathscr{L}$ is related to a non-Euclidean structure on the space $\mathbb{R}^{2m+1}$ and its study needs to be addressed via an *ad hoc* approach. In particular, as we will see in the sequel, $\mathscr{L}$ is the relativistic version of a kinetic diffusion operator and it is invariant with respect to Lorentz transformations. Moreover, $\mathscr{L}$ is the Kolmogorov equation of a suitable relativistic stochastic process $(P_s, Y_s, T_s)_{s \geq 0}$, that is introduced in (6.1.12) below. Our main result is a lower bound for the density of the stochastic process $(P_s, Y_s, T_s)_{s \geq 0}$. This is the first step in developing a systematic study of $\mathscr{L}$ within the theory of PDEs. Indeed, our final aim is to extend the classical theory considered in [8] to the relativistic case. In particular, we plan to prove asymptotic results such as [12, 72, 73, 95] in this more general setting.

As we will see in Appendix 6.A, the treatment of operator $\mathscr{L}$ in dimension $m > 1$ involves cumbersome notation and computations. Thus, for the sake of simplicity, we restrict ourselves to the one-dimensional case. The corresponding generalization of our main result to higher dimension follows the same strategy but requires cumbersome computations and for this reason will be the content of a forthcoming work. In the one-dimensional case $\mathscr{L}$ writes in the following form

$$\mathscr{L} u(p, y, t) = \sqrt{p^2 + 1}\, \tfrac{\partial}{\partial p} \left(\sqrt{p^2 + 1}\, \tfrac{\partial u}{\partial p} \right) - p \tfrac{\partial u}{\partial y} - \sqrt{p^2 + 1}\, \tfrac{\partial u}{\partial t}, \qquad (p, y, t) \in \mathbb{R}^3, \tag{6.1.6}$$

and takes the Hörmander's form $\mathscr{L} = X^2 - Y$ if we set

$$X = \sqrt{p^2 + 1}\, \tfrac{\partial}{\partial p} \qquad \text{and} \qquad Y = p \tfrac{\partial f}{\partial y} + \sqrt{p^2 + 1}\, \tfrac{\partial f}{\partial t}. \tag{6.1.7}$$

6.1.1 Physical interpretation

Operator $\mathscr{L}$ is the relativistic version of the kinetic Fokker-Planck equation (1.0.1) with $\sigma = \sqrt{2}$, namely equation

$$\mathscr{K} u(p, y, t) = \frac{\partial^2 u}{\partial p^2}(p, y, t) - p \frac{\partial u}{\partial y}(p, y, t) - \frac{\partial u}{\partial t}(p, y, t) = 0 \qquad (p, y, t) \in \mathbb{R}^3. \tag{6.1.8}$$

In this chapter, we address a possible improvement of the model described in (6.1.8) which is in accordance with special relativity. Indeed, a questionable feature of (6.1.8) is that its diffusion term $\frac{\partial^2 u}{\partial p^2}(p, y, t)$ operates with infinite velocity, as in classical mechanics the velocity is proportional to the momentum. In particular, it is known that, if we consider a continuous, non-negative and compactly supported initial distribution $u(p, y, 0)$, then the unique non-negative solution $u(p, y, t)$ to the Cauchy problem relevant to (6.1.8) is strictly positive for every positive t (see, for instance, Theorem 1.2.4). In this scenario, there would be therefore a non-zero probability to find particles everywhere in space. This feature is clearly incompatible with the physical law that prevents particles from moving faster than light. To overcome this issue, we rely on the *relativistic velocity*

$$v = \frac{p}{\sqrt{p^2+1}}, \tag{6.1.9}$$

which clearly satisfies

$$\left| \frac{p}{\sqrt{p^2+1}} \right| < 1 \quad \text{for every } p \in \mathbb{R},$$

in accordance with the relativity principles[1]. We consider the *finite velocity Langevin process* analogous to (1.0.2)

$$\begin{cases} P_t = p_0 + \sqrt{2} \int\limits_0^t \sqrt{P_s^2+1}\, dW_s \\ Y_t = y_0 + \int\limits_0^t \frac{P_s}{\sqrt{P_s^2+1}} ds, \end{cases} \tag{6.1.10}$$

and we recall that, by applying the relativistic Itô calculus, Dunkel and Hänggi find in [43] the Kolmogorov equation

$$\widetilde{\mathscr{L}} u(p, y, t) = \frac{\partial}{\partial p} \left(\sqrt{p^2+1}\, \frac{\partial u}{\partial p}(p, y, t) \right) - \frac{p}{\sqrt{p^2+1}} \frac{\partial u}{\partial y}(p, y, t) - \frac{\partial u}{\partial t}(p, y, t) = 0 \tag{6.1.11}$$

satisfied by the density of the stochastic process $(P_t, Y_t)_{t \geq 0}$ in (6.1.10). We refer the reader to [35, 43] for an overview to the relativistic theory of Brownian motions and corresponding relativistic kinetic equations.

Alcàntara and Calogero find the same equation (6.1.11) in [4] following a different approach, i.e. by merely requiring that some relevant properties of the non-relativistic equation are preserved in the relativistic setting. More precisely, as the non-relativistic operator is known to be Galilean invariant, the first requirement is the invariance with respect to the equivalent relativistic transformations, namely the Lorentz transformations. In addition, the authors of [4] impose that the relativistic Maxwellian distribution (or Jüttner distribution) $e^{-\gamma\sqrt{p^2+1}}, \gamma > 0$, needs to be a stationary solution of equation (6.1.1) with friction, mirroring the fact that the Maxwellian distribution is a stationary solution of (6.1.8) with friction.

We emphasize that u is a solution to $\widetilde{\mathscr{L}} u = 0$ if, and only if, it is a solution to $\mathscr{L} u = 0$ with $\mathscr{L}$ defined in (6.1.6). We prefer to focus our attention on the differential operator $\mathscr{L}$ because it is invariant with respect to Lorentz transformations, as we will see in the following Subsection

[1]Here, we adopt a natural unit system with $c = 1$, where c is the speed of light.

6.1.2. We finally observe that (6.1.6) is the relativistic deterministic equation describing the density of the following stochastic process

$$\begin{cases} P_s = p_0 + \sqrt{2}\int\limits_0^s \sqrt{P_\tau^2+1}\, dW_\tau, \\ Y_s = y_0 + \int\limits_0^s P_\tau d\tau, \\ T_s = t_0 + \int\limits_0^s \sqrt{P_\tau^2+1}\, d\tau, \end{cases} \tag{6.1.12}$$

where the third component is the time, which is not an absolute quantity in the relativistic setting.

6.1.2 Invariance properties

As it will be widely used in the sequel, we now focus on the invariance properties of operators $\mathscr{L}$ and $\mathscr{K}$. As discussed in the Introduction of this work, it is well known that $\mathscr{K}$ is invariant with respect to the Galilean change of variables (5).

In a natural way, operator $\mathscr{L}$ satisfies the relativistic analogue of property (6), i.e. it is invariant under Lorentz transformations. To show that, let us first summarize basic definitions and a few properties of those transformations. We recall that the relativistic momentum $p(t)$ and energy $E(t)$ of a particle with position $y(t)$ and velocity $v(t) = dy(t)/dt$ are given by

$$p(t) = v\gamma(v(t)) = \frac{v(t)}{\sqrt{1-v(t)^2}}, \qquad E(t) = \sqrt{p(t)^2+1} = \frac{1}{\sqrt{1-v(t)^2}} = \gamma(v(t)),$$ [2]

respectively, with γ denoting the Lorentz factor

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}}.$$

We combine time with position, and energy with momentum, to obtain the contravariant four-vectors[3]

$$\begin{pmatrix} t \\ y \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} E \\ p \end{pmatrix}.$$

The above definitions refer to the intertial lab frame Σ, defined as the rest frame of the fluid. We now consider a second intertial frame $\widetilde{\Sigma}$, moving with constant velocity β with respect to Σ. According to Einstein's theory of special relativity, values of physical quantities in $\widetilde{\Sigma}$ can be related to those in Σ by means of the Lorentz transformations. In the one-dimensional

[2]We here assume that the rest mass of the test particle is one.

[3]We use the term "four-vector" independently of the actual number of spatial dimensions.

case, the Lorentz transformation matrix reads as follows

$$\Lambda(\beta) = \gamma(\beta) \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix},$$

and its inverse is $\Lambda(-\beta)$. The matrices are representations of the Lorentz group acting on the four-vectors. The transformation law of arbitrary four-vectors is computed as follows

$$\begin{aligned} \begin{pmatrix} \widetilde{t} \\ \widetilde{y} \end{pmatrix} &= \gamma(\beta) \begin{pmatrix} t - \beta y \\ y - \beta t \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{E} \\ \widetilde{p} \end{pmatrix} = \gamma(\beta) \begin{pmatrix} E - \beta p \\ p - \beta E \end{pmatrix} \\ \begin{pmatrix} t \\ y \end{pmatrix} &= \gamma(\beta) \begin{pmatrix} \widetilde{t} + \beta \widetilde{y} \\ \widetilde{y} + \beta \widetilde{t} \end{pmatrix}, \qquad \begin{pmatrix} E \\ p \end{pmatrix} = \gamma(\beta) \begin{pmatrix} \widetilde{E} + \beta \widetilde{p} \\ \widetilde{p} + \beta \widetilde{E} \end{pmatrix}. \end{aligned} \tag{6.1.13}$$

Let us consider the function, $v(\widetilde{p}, \widetilde{y}, \widetilde{t}) = u(p(\widetilde{p}), y(\widetilde{t}, \widetilde{y}), t(\widetilde{t}, \widetilde{y}))$ that represents the one-particle phase space probability density function measured in the moving frame $\widetilde{\Sigma}$, which, according to [113], transforms as a Lorentz scalar. If we set $w(p, y, t) := u(\widetilde{p}, \widetilde{y}, \widetilde{t})$ and $g(p, y, t) := f(\widetilde{p}, \widetilde{y}, \widetilde{t})$, then, applying the chain rule, we obtain

$$\begin{aligned} \frac{\partial w}{\partial p}(p, y, t) &= \gamma \left(1 - \frac{\beta p}{E}\right) \frac{\partial u}{\partial \widetilde{p}}(\widetilde{p}, \widetilde{y}, \widetilde{t}) \\ \frac{\partial w}{\partial y}(p, y, t) &= \gamma \left(\frac{\partial u}{\partial \widetilde{y}}(\widetilde{p}, \widetilde{y}, \widetilde{t}) - \beta \frac{\partial u}{\partial \widetilde{t}}(\widetilde{p}, \widetilde{y}, \widetilde{t})\right) \\ \frac{\partial w}{\partial t}(p, y, t) &= \gamma \left(-\beta \frac{\partial u}{\partial \widetilde{y}}(\widetilde{p}, \widetilde{y}, \widetilde{t}) + \frac{\partial u}{\partial \widetilde{t}}(\widetilde{p}, \widetilde{y}, \widetilde{t})\right). \end{aligned}$$

Then, the vector field X defined in (6.1.6) is invariant with respect to Lorentz transformations, since

$$\begin{aligned} X(u(\widetilde{p}, \widetilde{y}, \widetilde{t})) &= E \frac{\partial w}{\partial p}(p, y, t) = \gamma (E - \beta p) \frac{\partial u}{\partial p}(\widetilde{p}, \widetilde{y}, \widetilde{t}) \\ &= \widetilde{E} \frac{\partial u}{\partial \widetilde{p}}(\widetilde{p}, \widetilde{y}, \widetilde{t}) = (Xu)(\widetilde{p}, \widetilde{y}, \widetilde{t}). \end{aligned} \tag{6.1.14}$$

From (6.1.14), it immediately follows that the diffusion term $X^2 u$ in (6.1.6) is also invariant with respect to Lorentz transformations. As far as we are concerned with the drift term Y in (6.1.6), we obtain

$$\begin{aligned} Y(u(\widetilde{p}, \widetilde{y}, \widetilde{t})) &= p \frac{\partial w}{\partial y}(p, y, t) + E \frac{\partial w}{\partial t}(p, y, t) \\ &= \gamma (p - \beta E) \frac{\partial u}{\partial \widetilde{y}}(\widetilde{p}, \widetilde{y}, \widetilde{t}) + \gamma(-\beta p + E) \frac{\partial u}{\partial \widetilde{t}}(\widetilde{p}, \widetilde{y}, \widetilde{t}) \\ &= \widetilde{p} \frac{\partial u}{\partial \widetilde{y}}(\widetilde{p}, \widetilde{y}, \widetilde{t}) + \widetilde{E} \frac{\partial u}{\partial \widetilde{t}}(\widetilde{p}, \widetilde{y}, \widetilde{t}) = (Yu)(\widetilde{p}, \widetilde{y}, \widetilde{t}). \end{aligned} \tag{6.1.15}$$

In virtue of (6.1.14) and (6.1.15), operator $\mathscr{L}$ is invariant with respect to the Lorentz transformations (6.1.13), i.e.

$$\mathscr{L}u = f \quad \Longleftrightarrow \quad \mathscr{L}w = g, \quad \text{for every } (\widetilde{p}, \widetilde{y}, \widetilde{t}) \in \mathbb{R}^3. \tag{6.1.16}$$

Hence, owing to (6.1.16), operator $\mathscr{L}$ is invariant with respect to the following group operation on $\mathbb{R}^3$

$$(p_0, y_0, t_0) \circ_{\mathcal{L}} (p, y, t) = \left(p\sqrt{p_0^2+1} + p_0\sqrt{1+p^2}, y_0 + y\sqrt{p_0^2+1} + p_0 t, t_0 + t\sqrt{p_0^2+1} + p_0 y \right). \tag{6.1.17}$$

We remark that for small velocities $\sqrt{1+p_0^2} \approx 1$, and therefore (6.1.17) becomes precisely the non-relativistic composition law (5) for variables p and y.

Moreover, $\mathbb{G} := (\mathbb{R}^3, \circ_{\mathcal{L}})$ is a Lie group with identity e and inverse $(p, y, t)^{-1}$ defined as:

$$e = (0, 0, 0), \qquad (p, y, t)^{-1} = \left(-p, pt - \frac{y}{\sqrt{p^2+1}} - \frac{p^2 y}{\sqrt{p^2+1}}, -t\sqrt{p^2+1} + py \right).$$

Then, in particular, we have that

$$\begin{aligned}&(p_0, y_0, t_0)^{-1} \circ_{\mathcal{L}} (p, y, t) \\ &\quad = \left(p\sqrt{p_0^2+1} - p_0\sqrt{p^2+1}, \sqrt{p_0^2+1}(y-y_0) - p_0(t-t_0), \sqrt{p_0^2+1}(t-t_0) - p_0(y-y_0) \right),\end{aligned}$$

so that (6.1.16) is equivalent to

$$u(p, y, t) = w((p_0, y_0, t_0)^{-1} \circ_{\mathcal{L}} (p, y, t)). \tag{6.1.18}$$

To avoid confusion between the Galilean and the Lorentz change of variables, in this section we denote by $\circ_{\mathcal{G}}$ the composition law (5), i.e. we write

$$(p_0, y_0, t_0) \circ_{\mathcal{G}} (p, y, t) = (p_0 + p, y_0 + y + tp_0, t_0 + t) \qquad \text{for every } (p_0, y_0, t_0), (p, y, t) \in \mathbb{R}^3. \tag{6.1.19}$$

We conclude this section with a remark concerning operator $\widetilde{\mathscr{L}}$ in (6.1.11). As already noticed, $\widetilde{\mathscr{L}}u = 0$ if, and only if, $\mathscr{L}u = 0$. Moreover, $\widetilde{\mathscr{L}}$ looks simpler than $\mathscr{L}$, as the derivative with respect to the time variable $\frac{\partial u}{\partial t}$ appearing in $\mathscr{L}$ is multiplied by the coefficient $\sqrt{p^2+1}$, unlike $\widetilde{\mathscr{L}}$. However, operator $\widetilde{\mathscr{L}}$ is not invariant with respect to Lorentz transformations. To see this, we rewrite $\widetilde{\mathscr{L}}$ in the form $\widetilde{\mathscr{L}} = \widetilde{X}^2 - \widetilde{Y}$, with

$$\widetilde{X} = \sqrt[4]{p^2+1}\, \partial_p \quad \text{and} \quad \widetilde{Y} = \tfrac{p}{\sqrt{p^2+1}}\, \partial_p + \tfrac{p}{\sqrt{p^2+1}}\, \partial_y + \partial_t. \tag{6.1.20}$$

We now define $w(p, y, t) := u(\widetilde{p}, \widetilde{y}, \widetilde{t})$ as above and we check if the vector field $\widetilde{X}$ is invariant with respecto to Lorentz transformations. To this end, we perform the same computations as in (6.1.14) and we find

$$\begin{aligned}\widetilde{X}(u(\widetilde{p},\widetilde{y},\widetilde{t})) &= \sqrt{E}\frac{\partial w}{\partial p}(p,y,t) = \sqrt{E}\,\gamma\left(\frac{E-\beta p}{E}\right)\frac{\partial u}{\partial p}(\widetilde{p},\widetilde{y},\widetilde{t}) \\ &= \frac{\widetilde{E}}{\sqrt{E}}\frac{\partial u}{\partial \widetilde{p}}(\widetilde{p},\widetilde{y},\widetilde{t}) \neq (\widetilde{X}u)(\widetilde{p},\widetilde{y},\widetilde{t}).\end{aligned}$$

A similar computation shows that $\widetilde{Y}$ in (6.1.20) is also not invariant with respect to Lorentz transformations.

In a more formal way, we can prove that $\widetilde{\mathscr{L}}$ is not invariant with respect to (6.1.13) by taking advantage of a general result contained in [19]. Specifically, Bonfiglioli and Lanconelli prove in [19] a theorem for operators in the form (6.1.2), where $X_1,\dots,X_{m+1}$ are analytic Hörmander's vector fields, with the property that, for every $z\in\mathbb{R}^N$, the integral curves [4] $\exp(tX_1)z,\ \dots,\ \exp(tX_{m+1})z$ are defined for every $t\in\mathbb{R}$. They prove that operator $\mathscr{L}$ is invariant with respect to the left translation of some Lie group $\mathbb{G}=\left(\mathbb{R}^N,\circ\right)$ if, and only if, the Lie algebra generated by $X_1,\dots,X_{m+1}$, as a linear subspace of the smooth vector fields in $\mathbb{R}^N$, has dimension N. The same result was extended to C^∞ vector fields by Biagi and Bonfiglioli in [17].

If we apply this condition to operator $\mathscr{L}=X^2-Y$, with X and Y defined in (6.1.7), we find

$$[X,Y]=\sqrt{p^2+1}\,\partial_y+p\,\partial_t,\quad [X,[X,Y]]=Y\quad\text{and}\quad [Y,[X,Y]]=0,$$

so that the dimension of the Lie algebra generated by X and Y equals 3, which is the dimension of the space $\mathbb{R}^3$. On the other hand, if we write $\widetilde{\mathscr{L}}$ in the form $\widetilde{\mathscr{L}}=\widetilde{X}^2-\widetilde{Y}$, with $\widetilde{X}$ and $\widetilde{Y}$ as in (6.1.20), a direct computation shows that the dimension of the Lie algebra generated by $\widetilde{X}$ and $\widetilde{Y}$ is infinite. For this reason, we believe that $\mathscr{L}$ is the suitable relativistic generalization of (6.1.8).

6.1.3 Main results

Our main result is a lower bound for the density of the stochastic process (6.1.12). Since we prove it via an approach in the framework of PDE theory, it is natural to express it in terms of the *fundamental solution* Γ. We observe that a purely PDEs construction of a fundamental solution of $\mathscr{L}u=0$ is given in Theorem 1.3 of [19]. In order to expose the main result of this chapter, we therefore recall the definition of Γ below.

Definition 6.1.1. We say that a function $\Gamma:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ is a fundamental solution of operator $\mathscr{L}$ in (6.1.6) if it satisfies the following conditions:

1. for every $(p_0,y_0,t_0)\in\mathbb{R}^3$, the function $(p,y,t)\mapsto\Gamma(p,y,t;p_0,y_0,t_0)$ belongs to $L^1_{\text{loc}}(\mathbb{R})\cap C^\infty(\mathbb{R}^3\setminus\{(p_0,y_0,t_0)\})$ and it is a classical solution to $\mathscr{L}u=0$ in $\mathbb{R}^3\setminus\{(p_0,y_0,t_0)\}$;

[4] The integral curve $\gamma:I\to\mathbb{R}^N$ of a vector field X on $\mathbb{R}^N$ is defined by $\gamma'(s)=X(\gamma(s))$ for every $s\in I$.

2. for every $\varphi \in C_b(\mathbb{R}^2)$, the function

$$u(p,y,t) = \int_{\mathbb{R}^3} \Gamma(p,y,t;\xi,\eta,t_0)\varphi(\xi,\eta)d\xi\, d\eta$$

is a classical solution to the Cauchy problem

$$\begin{cases} \mathscr{L}u(p,y,t) = 0, & \text{in } \mathbb{R}^2 \times (t_0,+\infty) \\ f(p,y,t_0) = \varphi(p,y), & \text{in } \mathbb{R}^2; \end{cases}$$

In the statement of the following theorem, which is the main result of this chapter, the function Ψ is the *value function* of a suitable optimal control problem and is defined in equation (6.3.15) below.

Theorem 6.1.2. *Let Γ be the fundamental solution of $\mathscr{L}$ in* (6.1.6)*. Then for every $T > 0$ there exist three positive constants θ, c_T, C with $\theta < 1$, such that*

$$\Gamma(p_0,y_0,t_0;p_1,y_1,t_1) \geq \frac{c_T}{(t_0-t_1)^2} \exp\left\{-C\,\Psi\left(p_0,y_0,y_0;p_1,y_1,\theta^2 t_1 + (1-\theta^2)t_0\right)\right\}$$

for every $(p_0,y_0,t_0),(p_1,y_1,t_1) \in \mathbb{R}^3$ such that $0 < t_0 - t_1 < T$. The constants θ and C only depend on $\mathscr{L}$, while c_T also depends on T.

As far as the analogous upper bound is concerned, we believe it can be achieved by means of control theory in the same spirit of [31, 33]. As this problem needs to be studied in a different framework, this issue will be addressed in future research.

6.1.4 Outline of the chapter

This chapter is organized as follows. Section 6.2 is devoted to the proof of an invariant Harnack inequality for solutions to $\mathscr{L}u = 0$. Section 6.3 collects useful results on the optimal control problem associated to Ψ, while in Section 6.4 we give proof of our main result. Finally, in Appendix 6.A, we show how the higher dimensional operator (6.1.1) is related to Hörmander's theory.

6.2 Harnack inequality

This section is devoted to the proof of a scale-invariant Harnack inequality for solutions to (6.1.1), which is invariant with respect to Lorentz transformations. We introduce some notation necessary to state this result. For every positive r we introduce the cylinders

$$\begin{aligned} H_r(0) &:= \left\{(p,y,t) \in \mathbb{R}^3 \mid |p| < r, |y| < r^3, -r^2 < t < 0\right\}, \\ S_r(0) &:= \left\{(p,y,t) \in \mathbb{R}^3 \mid |p| < r, |y| < r^3, -r^2 \leq t \leq -r^2/2\right\}. \end{aligned} \tag{6.2.1}$$

Owing to (6.1.17), for every $z_0 = (p_0, y_0, t_0) \in \mathbb{R}^3$, we set

$$H_r^{\mathcal{L}}(z_0) := z_0 \circ_{\mathcal{L}} H_r(0), \qquad S_r^{\mathcal{L}}(z_0) := z_0 \circ_{\mathcal{L}} S_r(0). \tag{6.2.2}$$

We are now in a position to state the following result.

Theorem 6.2.1. *There exist two constants $C_H > 0$ and $\theta \in (0,1)$, only depending on $\mathscr{L}$, such that*

$$\sup_{S_{\theta r}^{\mathcal{L}}(z_0)} u \le C_H u(z_0),$$

for every $z_0 \in \mathbb{R}^3, r \in (0, 1/2]$, and for every non negative solution u to $\mathscr{L} u = 0$ in $H_r^{\mathcal{L}}(z_0)$.

The proof of Theorem 6.2.1 is obtained from the analogous Harnack inequality for the *non-relativistic* kinetic operator $\widetilde{\mathscr{K}}$ acting as

$$\widetilde{\mathscr{K}} u = a(p,y,t)\,\frac{\partial^2 u}{\partial p^2} + b(p,y,t)\frac{\partial u}{\partial p} - p\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t}. \tag{6.2.3}$$

In the following, $H_r^{\mathcal{G}}(z_0) = z_0 \circ_{\mathcal{G}} H_r(0) = \{z \in \mathbb{R}^3 : z = z_0 \circ_{\mathcal{G}} \zeta, \, \zeta \in H_r(0)\}$ and $S_r^{\mathcal{G}}(z_0) = z_0 \circ_{\mathcal{G}} S_r(0) = \{z \in \mathbb{R}^3 : z = z_0 \circ_{\mathcal{G}} \zeta, \, \zeta \in S_r(0)\}$, where $\circ_{\mathcal{G}}$ is the composition law defined in (6.1.19). We remark that, in contrast to the variable-coefficients operators studied in Chapters 4-5, operator $\widetilde{\mathscr{K}}$ in (6.2.3) is in trace form. For this reason, we recall the statement of the Harnack inequality for classical solutions to trace-form operators like (6.2.3) proved in [41].

Theorem 6.2.2. *Suppose that the coefficients a and b in* (6.2.3) *are Hölder continuous functions satisfying*

(H) *There exist two constants $\lambda^-, \lambda^+ > 0$ such that*

$$\lambda^- \le a(p,y,t) \le \lambda^+, \qquad |b(p,y,t)| \le \lambda^+ \qquad \text{for every } (p,y,t) \in \mathbb{R}^3.$$

Then there exist two constants $C_H > 0$ and $\theta \in (0,1)$, only depending on λ^- and λ^+ such that

$$\sup_{S_{\theta r}^{\mathcal{G}}(z_0)} u \le C_H u(z_0).$$

for every $z_0 \in \mathbb{R}^3, r \in \left(0, \frac{1}{2}\right)$, and for every non negative solution u to $\widetilde{\mathscr{K}} u = 0$ in $H_r^{\mathcal{G}}(z_0)$.

6.2.1 Change of variable

In order to prove Theorem 6.2.1, we perform an appropriate change of variable that allows us to write solutions to $\widetilde{\mathscr{L}} u = 0$ (and therefore to $\mathscr{L} u = 0$) in the form (6.2.3). To this end, we denote by φ the function defined as follows

$$\varphi(p) := \frac{p}{\sqrt{1+p^2}}, \qquad p \in \mathbb{R}, \tag{6.2.4}$$

where we remark that $\varphi(p)$ is actually the relativistic velocity defined in (6.1.9). By a direct computation, we easily obtain

$$\varphi'(p) := \frac{1}{(1+p^2)^{3/2}}, \qquad \varphi''(p) := -\frac{3p}{(1+p^2)^{5/2}}. \tag{6.2.5}$$

Moeover, it is easy to verify the function

$$\psi(x) := \frac{x}{\sqrt{1-x^2}} \tag{6.2.6}$$

is the inverse of φ, and the following identity holds

$$1-\varphi^2(p) = \frac{1}{1+p^2}, \qquad p \in \mathbb{R}. \tag{6.2.7}$$

We are now in a position to state and prove the following preliminary result.

Lemma 6.2.3. *Let u be a solution to $\mathscr{L}u = 0$. For every $(x,y,t) \in (-1,1) \times \mathbb{R}^2$, we define the function*

$$v(x,y,t) := u\left(\psi\left(x\right), y, t\right), \tag{6.2.8}$$

where ψ was defined in (6.2.6)*. Then v is a solution to the following equation*

$$\frac{\partial v}{\partial t}(x,y,t) + x\frac{\partial v}{\partial y}(x,y,t) = \left(1-x^2\right)^{5/2}\frac{\partial^2 v}{\partial x^2}(x,y,t) - 2x\left(1-x^2\right)^{3/2}\frac{\partial v}{\partial x}(x,y,t). \tag{6.2.9}$$

Proof. By inverting the change of variable in (6.2.8) we find that

$$x = \varphi(p), \tag{6.2.10}$$

and therefore

$$u(p,y,t) = v\left(\varphi(p), y, t\right).$$

Hence, from the chain rule it follows immediately

$$\frac{p}{\sqrt{1+p^2}}\frac{\partial u}{\partial y}(p,y,t) + \frac{\partial u}{\partial t}(p,y,t) = \varphi(p)\frac{\partial v}{\partial y}(x,y,t) + \frac{\partial v}{\partial t}(x,y,t). \tag{6.2.11}$$

Moreover, exploiting identities (6.2.5), (6.2.7) and (6.2.10), we obtain

$$\begin{aligned}
\frac{\partial u}{\partial p}(p,y,t) &= \varphi'(p)\frac{\partial v}{\partial x}(x,y,t),\\
\frac{\partial^2 u}{\partial p^2}(p,y,t) &= (\varphi'(p))^2\frac{\partial^2 v}{\partial x^2}(x,y,t) + \varphi''(p)\frac{\partial v}{\partial x}(x,y,t)\\
&= \frac{1}{(1+p^2)^3}\frac{\partial^2 v}{\partial x^2}(x,y,t) - \frac{3p}{(1+p^2)^{5/2}}\frac{\partial v}{\partial x}(x,y,t)
\end{aligned} \tag{6.2.12}$$

$$
\begin{aligned}
&= \left(1-\varphi^2(p)\right)^3 \frac{\partial^2 v}{\partial x^2}(x,y,t) - 3\varphi(p)\left(1-\varphi^2(p)\right)^2 \frac{\partial v}{\partial x}(x,y,t) \\
&= \left(1-x^2\right)^3 \frac{\partial^2 v}{\partial x^2}(x,y,t) - 3x\left(1-x^2\right)^2 \frac{\partial v}{\partial x}(x,y,t).
\end{aligned}
$$

As a consequence, the diffusion term in equation (6.1.11) becomes

$$
\begin{aligned}
&\sqrt{p^2+1}\frac{\partial^2 u}{\partial p^2}(p,y,t) + \frac{p}{\sqrt{p^2+1}}\frac{\partial u}{\partial p}(p,y,t) \\
&= \left(1-x^2\right)^{5/2} \frac{\partial^2 v}{\partial x^2}(x,y,t) - 3x\left(1-x^2\right)^{3/2} \frac{\partial v}{\partial x}(x,y,t) + x\left(1-x^2\right)^{3/2} \frac{\partial v}{\partial x}(x,y,t) \qquad (6.2.13) \\
&= \left(1-x^2\right)^{5/2} \frac{\partial^2 v}{\partial x^2}(x,y,t) - 2x\left(1-x^2\right)^{3/2} \frac{\partial v}{\partial x}(x,y,t).
\end{aligned}
$$

The claim then follows combining (6.2.11) and (6.2.13) and observing that $\mathscr{L}u = 0$ if and only if $\widetilde{\mathscr{L}}u = 0$. □

6.2.2 Proof of Theorem 6.2.1

We observe that the operator appearing in (6.2.9) writes in the form (6.2.3) if we choose

$$
a(x,y,t) = \left(1-x^2\right)^{5/2}, \qquad \text{and} \qquad b(x,y,t) = -2x\left(1-x^2\right)^{3/2}. \tag{6.2.14}
$$

Moreover, we remark that condition **(H)** is satisfied only on compact subsets of $(-1,1)\times\mathbb{R}^2$ as we have

$$
\inf_{-1<x<1} a(x,w,t) = 0.
$$

Proof of Theorem 6.2.1. Since $\mathscr{L}$ is invariant with respect to the Lorentz transformations (6.1.17), we first restrict ourselves to the case where $z_0 = (p_0, y_0, t_0) = (0,0,0)$. As a consequence, owing to (6.2.10), we have also $(x_0, y_0, t_0) = (0,0,0)$. We then observe that for every $p \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, there holds

$$
\left(\frac{1}{2}\right)^{5/2} \le \left(\frac{1}{1+p^2}\right)^{5/2} \le 1. \tag{6.2.15}
$$

We now apply the change of variable (6.2.8) and we observe that $a(0,0,0) = 1$, where $a(x,y,t)$ is the coefficient in (6.2.14). Keeping in mind that $x = \varphi(p)$, we find that for every point $(p,y,t) \in H_r^{\mathcal{L}}(0)$, the following inequality

$$
|x| = \left|\frac{p}{\sqrt{1+p^2}}\right| \le |p| \le r \tag{6.2.16}
$$

holds true. Thus, $(\varphi(p), y, t) \in H_r^{\mathcal{G}}(0)$ for every $(p,y,t) \in H_r^{\mathcal{L}}(0)$ and for every $r \in \left(0, \frac{1}{2}\right)$.

Furthermore, from definition (6.2.14) it follows that

$$a(\varphi(p), y, t) = \left(\frac{1}{1+p^2}\right)^{5/2},$$

and therefore

$$\left(\frac{1}{2}\right)^{5/2} \le a(x,y,t) \le 1, \qquad |b(\varphi(p), y, t)| = 2\frac{|p|}{\sqrt{1+p^2}}\frac{1}{(1+p^2)^{3/2}} \le 2,$$

for every $(x, y, t) = (\varphi(p), y, t)$ with $(p, y, t) \in H_r^{\mathcal{L}}(0)$. Thus, the coefficients appearing in (6.2.9) satisfy assumption **(H)** with $\lambda^- = \left(\frac{1}{2}\right)^{5/2}$ and $\lambda^+ = 2$. Since $(\varphi(p), y, t) \in H_r^{\mathcal{G}}(0)$ for every $(p, y, t) \in S_r^{\mathcal{L}}(0)$ and for every $r \in \left(0, \frac{1}{2}\right)$, our claim is proven for $z_0 = (0, 0, 0)$.

In order to prove our claim for an arbitrary point $z_0 \in \mathbb{R}^3$, we rely on the invariance of $\mathscr{L}$ with respect to (6.1.17). In particular, we apply the Lorentz change of variables (6.1.18) to a solution u to $\mathscr{L}u = 0$ in $H_r^{\mathcal{L}}(z_0)$ and we observe that the function w in (6.1.18) is a solution to $\mathscr{L}w = 0$ in $H_r^{\mathcal{L}}(0)$. Then the Harnack inequality holds for g and implies that

$$u(p, y, t) = w((p_0, y_0, t_0)^{-1} \circ_{\mathcal{L}} (p, y, t)) \le C_H w(0, 0, 0) = C_H u(p_0, y_0, t_0),$$

where C_H does not depend on z_0. This concludes the proof. □

Remark 6.2.4. We observe that the "cylinders" defined in (6.2.2) are the most natural geometric sets which can be defined starting from (6.2.1) and using the invariance group of $\mathscr{L}$. Finally, let us remark that, in virtue of (6.1.17), the sets (6.2.2) can be explicitely computed as follows

$$\begin{aligned} H_r^{\mathcal{L}}(z_0) := \Big\{ (p, y, t) \in \mathbb{R}^3 \mid & \left| p\sqrt{1+p_0^2} - p_0\sqrt{1+p^2} \right| < r, \\ & \left| \sqrt{1+p_0^2}(y - y_0) - p_0(t - t_0) \right| < r^3, \\ & -r^2 < \sqrt{1+p_0^2}(t - t_0) - p_0(y - y_0) < 0 \Big\}, \\ S_r^{\mathcal{L}}(z_0) := \Big\{ (p, y, t) \in \mathbb{R}^3 \mid & \left| p\sqrt{1+p_0^2} - p_0\sqrt{1+p^2} \right| < r, \\ & \left| \sqrt{1+p_0^2}(y - y_0) - p_0(t - t_0) \right| < r^3, \\ & -r^2 \le \sqrt{1+p_0^2}(t - t_0) - p_0(y - y_0) \le -r^2/2 \Big\}. \end{aligned} \tag{6.2.17}$$

However, we do not need the explicit expression (6.2.17) in our treatment, as it is sufficient to rely on definition (6.2.2) and on the invariance properties of $\mathscr{L}$.

6.3 Optimal control problem

This section is devoted to the proof of Theorem 6.1.2. In order to provide a clear treatment, we first recall some fundamental definitions from control theory and prove an equivalent statement of the Harnack inequality, more suitable to the construction of Harnack chains (see Proposition 6.3.6). We then prove an estimate for positive solutions to $\mathscr{L}u = 0$ (see Proposition 6.3.7) depending on the norm of the control. Finally, we conclude this section with a preliminary study of the optimal control problem associated to $\mathscr{L}$.

6.3.1 $\mathscr{L}$-admissible paths and Harnack chains

Along with the Harnack inequality Theorem 6.2.1, the main tool in the proof of our asymptotic estimates for the fundamental solution are *Harnack chains*, whose definition we recall below.

Definition 6.3.1 (Harnack chain)**.** Let Ω be an open subset of $\mathbb{R}^3$. We say that a finite set of points $\{z_0, z_1, \dots, z_k\} \in \Omega$ is a Harnack chain connecting z_0 to z_k if there exist positive constants $C_1, \dots, C_k$ such that

$$u(z_j) \le C_j u(z_{j-1}) \qquad j = 1, \dots, k,$$

for every positive solution u to $\mathscr{L}u = 0$.

In the present setting, we construct Harnack chains by connecting points belonging to appropriate trajectories, which naturally substitute segment lines in our non-Euclidean setting and are defined as follows.

Definition 6.3.2 ($\mathscr{L}$–admissible path)**.** A curve $\gamma(s) = (p(s), y(s), t(s)) : [0, T] \to \mathbb{R}^3$ is said to be a $\mathscr{L}$–admissible path if it is absolutely continuous and solves the following differential equation

$$\gamma'(s) = \omega(s) X(\gamma(s)) + Y(\gamma(s)), \tag{6.3.1}$$

for almost every $s \in [0, T]$, where X and Y are defined in (6.1.7). Moreover, we say that γ steers (p_0, y_0, t_0) to (p_1, y_1, t_1), with $t_0 > t_1$, if

$$\gamma(0) = (p_0, y_0, t_0), \qquad \gamma(T) = (p_1, y_1, t_1). \tag{6.3.2}$$

In the definition of $\mathscr{L}$–admissible path we assume $\omega \in L^2([0, T])$ and we refer to the function ω as the control of problem (6.3.1). Let us remark that, owing to (6.1.7), equation (6.3.1) can be explicitly written as follows

$$\begin{cases} p'(s) &= \omega(s)\sqrt{p^2(s)+1}, \\ y'(s) &= -p(s), \\ t'(s) &= -\sqrt{p^2(s)+1}, \end{cases} \tag{6.3.3}$$

for almost every $s \in [0, T]$.

Moreover, we observe that such optimal control problem is invariant with respect to the group operation (6.1.17). Indeed, let us consider a control $\omega(\cdot)$ steering (p_0, y_0, t_0) to (p_1, y_1, t_1) with trajectory $(p(s), y(s), t(s))$. Then, it is easy to prove the trajectory $(\widetilde{p}(s), \widetilde{y}(s), \widetilde{t}(s)) := (p_0, y_0, t_0)^{-1} \circ_{\mathcal{L}} (p(s), y(s), t(s))$ is a solution to (6.3.1)-(6.3.2) with the same control $\omega(\cdot)$. Additionally, the newly defined trajectory $(\widetilde{p}(s), \widetilde{y}(s), \widetilde{t}(s))$ satisfies the properties $(\widetilde{p}(0), \widetilde{y}(0), \widetilde{t}(0)) = (0, 0, 0)$.

Finally, we introduce the standard definition of attainable set from control theory.

Definition 6.3.3 (Attainable set)**.** For every $z_0 \in \Omega \subset \mathbb{R}^3$, the attainable set $\mathscr{A}_{z_0}$ of z_0 in Ω is defined as follows:

$$\mathscr{A}_{z_0} = \big\{ z \in \Omega \,:\, \textit{there exists } \bar{t} \in \mathbb{R}^+ \textit{ and a } \mathscr{L}\textit{-admissible path } \gamma : [0, \bar{t}] \to \Omega \\ \textit{such that } \gamma(0) = z_0,\ \gamma(\bar{t}) = z \big\}\,.$$

Now, our aim is to derive from Theorem 6.2.1 a statement of the Harnack inequality which is useful for the construction of Harnack chains. First of all, we define the positive cone

$$P_r(0) = \Big\{ (p, y, t) \in \mathbb{R}^3 : |p| < t^{\frac{1}{2}}, |y| < t^{\frac{3}{2}},\ -\theta^2 r^2 \le t < 0 \Big\}. \tag{6.3.4}$$

Moreover, in analogy with the definition of $H_r^{\mathcal{L}}(z_0)$ and $S_r^{\mathcal{L}}(z_0)$ in (6.2.2), we set $P_r^{\mathcal{L}}(z_0) := z_0 \circ_{\mathcal{L}} P_r(0)$. We are now in a position to state the following result, whose proof can be found in [100, Proposition 3.2].

Theorem 6.3.4. *Let Ω be an open set in $\mathbb{R}^3$ containing $H_r(z_0)$ for some $z_0 \in \mathbb{R}^3$ and $r \in \left(0, \frac{1}{2}\right)$. Then*

$$u(z_0 \circ_{\mathcal{L}} z) \le C_H u(z_0)$$

for every non negative solution u of $\mathscr{L}u = 0$ in Ω and for every $z \in P_r(0)$.

Next, we show that the trajectories defined in (6.3.1) belong to a certain positive cone provided a suitable choice of the parameter $s \in [0, T]$.

Lemma 6.3.5. *Let $s \in [0, T]$, $\omega \in L^2([0, T])$ be a control and let $\gamma(s) = (p(s), y(s), t(s))$ be an $\mathscr{L}$-admissible path starting from $z_0 = (p_0, y_0, t_0) \in \mathbb{R}^3$. Then for every $r \in \left(0, \frac{1}{2}\right)$ there exist two positive constants $k_0 := 2 \ln\left(\frac{3}{2}\right)$ and $\theta \in (0, 1)$, only depending on operator $\mathscr{L}$, such that*

$$\gamma(s) \in P_r^{\mathcal{L}}(z_0),$$

for every $s \in \left[0, \sqrt{\frac{2}{3}}\theta^2\, r^2\right]$ such that

$$\int_0^s |\omega(\tau)|^2 d\tau \le k_0^2.$$

Proof. Without loss of generality, we fix $z_0 = (0,0,0)$ and we give proof of this result for a given $\mathscr{L}$-admissible path starting from $(0,0,0)$. The general case directly follows from the translation invariance with respect to the group law (6.1.17).

Thus, we begin by considering the first component of $\gamma(s)$. In virtue of (6.3.3), for every $s > 0$, we have

$$\int_0^s p'(\tau)d\tau = \int_0^s \omega(\tau)\sqrt{p^2(\tau)+1}\,d\tau$$

and therefore

$$\int_0^s \omega(\tau)\,d\tau = \int_0^s \frac{p'(\tau)}{\sqrt{p^2(\tau)+1}}d\tau = \sinh^{-1}(p(s)) = \ln\left(p(s)+\sqrt{p^2(s)+1}\right). \tag{6.3.5}$$

Now, we apply Hölder's inequality and we estimate the L^2 norm of the control with k_0 to get

$$\left|\int_0^s \omega(\tau)\,d\tau\right| \le \left(\int_0^s |\omega(\tau)|^2\,d\tau\right)^{\frac{1}{2}}\sqrt{s} \le\ k_0\sqrt{s} \le \ln(1+\sqrt{s}), \qquad \forall s\in\left[0,\frac{1}{4}\right], \tag{6.3.6}$$

We observe that the last inequality follows from our choice of k_0 and the concavity of $\ln(1+x)$, which in particular implies that $\ln(1+x) \ge 2\ln(3/2)x$ for every $x\in[0,1/2]$. As a consequence,

$$\left|e^{\int_0^s \omega(\tau)\,d\tau}-1\right| \le e^{\left|\int_0^s \omega(\tau)\,d\tau\right|}-1\ \le \sqrt{s}, \qquad \forall s\in\left[0,\sqrt{\frac{2}{3}}\theta^2\,r^2\right]. \tag{6.3.7}$$

Then, combining (6.3.5), (6.3.6) and (6.3.7), we obtain

$$|p(s)| \le |p(s)+\sqrt{p^2(s)+1}-1| \le \sqrt{s}, \qquad \forall s\in\left[0,\sqrt{\frac{2}{3}}\theta^2\,r^2\right]. \tag{6.3.8}$$

Next, we consider the second component of $\gamma(s)$, that is

$$y(s) = -\int_0^s p(\tau)\,d\tau.$$

Owing to (6.3.8), we immediately get

$$|y(s)| \le \int_0^s \sqrt{\tau}\,d\tau = \frac{2}{3}s^{\frac{3}{2}} < s^{\frac{3}{2}}, \qquad \forall s\in\left[0,\sqrt{\frac{2}{3}}\theta^2\,r^2\right]. \tag{6.3.9}$$

By combining the above inequality, with (6.3.8), we obtain

$$0 \leq -t(s) = \int_0^s \sqrt{p^2(\tau)+1}\, d\tau \leq \int_0^s \sqrt{\tau+1} d\tau \leq \sqrt{\frac{3}{2}}\, s \leq \theta^2 r^2, \qquad \forall s \in \left[0, \sqrt{\frac{2}{3}}\theta^2\, r^2\right]. \tag{6.3.10}$$

Hence,

$$|p(s)| \leq s^{\frac{1}{2}}, \quad |y(s)| \leq s^{\frac{3}{2}}, \quad -\theta^2 r^2 \leq t(s) \leq 0, \qquad \forall s \in \left[0, \sqrt{\frac{2}{3}}\theta^2\, r^2\right].$$

This concludes the proof. □

Finally, we are in a position to prove a more suitable statement of the Harnack inequality for points of an admissible trajectory.

Proposition 6.3.6. *Let $T > 0$, $R > 0$ and $z_0 = (p_0, y_0, t_0) \in \mathbb{R}^3$. Let $s \in [0, T]$, $\omega \in L^2([0, T])$ be a control and let $\gamma(s) = (p(s), y(s), t(s))$ be an $\mathscr{L}$-admissible path starting from z_0. Then, for every non negative solution u to $\mathscr{L}u = 0$ in $H_R^{\mathscr{L}}(z_0)$, there exist three positive constants $k_0 := 2\ln\left(\frac{3}{2}\right)$, C_H and $\theta \in (0, 1)$, only depending on operator $\mathscr{L}$, such that*

$$u(\gamma(s)) \leq C_H u(z_0),$$

for every $s \in \left[0, \sqrt{\frac{2}{3}}\, \theta^2 r^2\right]$ such that

$$\int_0^s |\omega(\tau)|^2 d\tau \leq k_0^2.$$

Proof. The result directly follows by combining Theorem 6.2.1 with Proposition 6.3.5. □

6.3.2 Optimal control problem

We state and prove an useful intermediate result, which provides us with an estimate for any positive solution u to $\mathscr{L}u = 0$ at any point of a given $\mathscr{L}$-admissible path in terms of the L^2-norm of the control. Results of this kind are usually referred as *non local* Harnack inequalities. In particular, our result is an extension of [100, Theorem 1.1] and [22, Proposition 1.1]. For this reason, we hereby report only a sketch of the proof and for further details we refer the reader to [22, 100].

Proposition 6.3.7. *Let $z_0 = (p_0, y_0, t_0) \in \mathbb{R}^2 \times (T_0, T_1]$ and let $\omega \in L^2([0, T])$ be a control and $\gamma(s) = (p(s), y(s), t(s))$ be the corresponding $\mathscr{L}$-admissible path starting from $z_0 = (p_0, y_0, t_0) \in \mathbb{R}^3$. Moreover, let us fix $T_0 < t(s) < t_0 < T_1$, with $t_0 - t(s) \leq \theta^2(t_0 - T_0) \leq \frac{\theta^4}{4}$. Then, for every non negative solution u to $\mathscr{L}u = 0$ in $\mathbb{R}^2 \times (T_0, T_1]$, there exist three positive* *constants k_0, θ, C_H, only depending on operator $\mathscr{L}$, such that*

$$u(\gamma(s)) \le C_H^{\frac{\Phi(\omega)}{k_0^2}+1} u(z_0), \tag{6.3.11}$$

where

$$\Phi(\omega) = \int_0^s |\omega(\tau)|^2 d\tau. \tag{6.3.12}$$

Proof. Let k_0, C_H and θ be the constants of Proposition 6.3.6. We first observe that, if

$$\int_0^s |\omega(\tau)|^2 d\tau \le k_0^2,$$

then

$$\gamma(s) \in P_r^{\mathcal{L}}(z_0), \quad r := \sqrt{t_0 - T_0} \le \frac{1}{2},$$

in virtue of Proposition 6.3.5 and assumption $t_0 - t(s) \le \theta^2(t_0 - T_0)$. Since $H_r^{\mathcal{L}}(z_0) \subset \mathbb{R}^2 \times \times(T_0, T_1)$ thanks to our choice of r, Proposition 6.3.6 can be applied and there holds $u(p, y, t) \le C_H\, u(z_0)$, where C_H is the constant given by Theorem 6.2.1.

If the above inequality is not satisfied, we set

$$k = \max\left\{ j \in \mathbb{N} \,:\, \int_0^s |\omega(\tau)|^2 d\tau > j k_0^2 \right\} \tag{6.3.13}$$

and define recursively a sequence of times starting from $\sigma_0 \equiv 0$ as follows.

$$\sigma_j = \min\left\{ s,\, \inf\left\{ \sigma > 0 \,:\, \int_0^\sigma |\omega(\tau)|^2 d\tau > j k_0^2 \right\} \right\}, \tag{6.3.14}$$

for every $j = 1, \dots, k+1$. Thanks to (6.3.13), the sequence in (6.3.14) ends after a finite number of steps when the upper bound $\sigma_{k+1} \equiv s$ is reached. Moreover, for every $j = 0, \dots, k+1$, we define the sequence $t_j = t(\sigma_j)$, which satisfies $t(s) \equiv t_{k+1} < t_k < t_{k-1} < \dots t_1 < t_0$. We now observe that

$$H_{r_j}^{\mathcal{L}}(\gamma(\sigma_j)) \subset \mathbb{R}^2 \times (T_0, T_1), \quad \textit{for } r_j = \frac{\sqrt{t_j - t_{j+1}}}{\theta},\ j = 1, \dots, k.$$

In addition, we clearly have $t_j - t_{j+1} \le \theta^2 r_j^2$ and $r_j \le \frac{1}{2}$, since $\frac{t_j - t_{j+1}}{\theta^2} \le \frac{t_0 - T_0}{\theta^2} \le \frac{1}{4}$. Finally, as $\int_0^{\sigma_1} |\omega(\tau)|^2 d\tau \le k_0^2$, we can apply Proposition 6.3.6 and get $u(\gamma(\sigma_1)) \le C_H u(\gamma(0)) = C_H u(z_0)$. Similarly, owing to $\int_{\sigma_1}^{\sigma_2} |\omega(\tau)|^2 d\tau \le k_0^2$ and applying again Proposition 6.3.6 to the trajectory steering $(p_1, y_1, t_1) := \gamma(\sigma_1)$ to $(p_2, y_2, t_2) := \gamma(\sigma_2)$, we obtain $u(\gamma(\sigma_2)) \le C_H u(\gamma(\sigma_1)) \le C_H^2 u(z_0)$. We then iterate the above argument until at step $k+1$ and we obtain

$$u(\gamma(s)) \le C_H^{k+1} u(z_0).$$

We point out that the points $(\gamma(\sigma_j))_{j=1}^k$, chosen along the trajectory $\gamma(\cdot)$, define a Harnack chain. Finally, from (6.3.13), it follows that

$$k < \frac{\int_0^s |\omega(\tau)|^2 d\tau}{k_0^2},$$

and this concludes the proof of Proposition 6.3.7. □

Estimate (6.3.11) provides us with a bound dependent on the choice of the $\mathscr{L}$-admissible path steering z_0 to $\gamma(s)$. Hence, we introduce the *value function*

$$\Psi(p_0, y_0, t_0; p_1, y_1, t_1) := \inf_{\omega \in L^2([0,T])} \Phi(\omega), \tag{6.3.15}$$

where the infimum is taken over all the $\mathscr{L}$-admissible paths steering $z_0 := (p_0, y_0, t_0) \in \mathbb{R}^3$ to $z_1 := (p_1, y_1, t_1) \in \mathbb{R}^3$. Then, as a straightforward consequence of Proposition 6.3.7, we obtain

$$u(\gamma(s)) \leq M^{\frac{\Psi(p_0, y_0, t_0; p(s), y(s), t(s))}{k_0^2} + 1} u(z_0), \tag{6.3.16}$$

whenever u satisfies the assumptions of Proposition 6.3.7. As it will be clear in the following of this section, equation (6.3.16) is a key step in proving the lower bound for the fundamental solution of $\mathscr{L}$. Thus, in order to characterize the minimizing cost Ψ, and hence to obtain the best exponent in (6.3.11), we formulate the natural optimal control problem, i.e. we consider the function ω as the *control* of the path γ in (6.3.1) and we look for the one minimizing the *total cost* Φ defined in (6.3.12). As observed above, given a solution to (6.3.1)-(6.3.2), the same control steers $(0, 0, 0)$ to $(p_0, y_0, t_0)^{-1} \circ_{\mathcal{L}} (p_1, y_1, t_1)$. As the cost Φ depends on the control only, the two trajectories have the same cost. Hence,

$$\Psi(p_0, y_0, t_0; p_1, y_1, t_1) = \Psi\left(0, 0, 0; (p_0, y_0, t_0)^{-1} \circ_{\mathcal{L}} (p_1, y_1, t_1)\right).$$

As a consequence, we will fix the initial condition $(p_0, y_0, t_0) = (0, 0, 0)$ in (6.3.1)-(6.3.2) and then use the invariance property to solve it with a general initial condition. Thus, our aim is to study the optimal control problem

$$\inf_{\omega \in L^2([0,T])} \int_0^T \omega^2(\tau) d\tau \quad \textit{subject to the constraint}$$
$$\begin{cases} p'(s) &= \omega(s)\sqrt{p^2(s) + 1}, \\ y'(s) &= -p(s), \qquad 0 \leq s \leq T, \\ t'(s) &= -\sqrt{p^2(s) + 1}, \end{cases} \tag{6.3.17}$$
$$\textit{with } (p, y, t)(0) = (0, 0, 0), \quad (p, y, t)(T) = (p_1, y_1, t_1), \quad \textit{with } t_1 < 0.$$

To solve problem (6.3.17), one possible approach could be to apply the Pontryagin Maximum Principle (see [114, Chapter 6]) and to compute the Hamiltonian

$$H(p,y,t,\lambda_1,\lambda_2,\lambda_3,m_0,\omega) = \\ \lambda_1(s)\omega(s)\sqrt{p^2(s)+1} - \lambda_2(s)p(s) - \lambda_3(s)\sqrt{p^2(s)+1} + m_0\omega^2(s), \tag{6.3.18}$$

where λ_1, λ_2 and λ_3 are the coordinates of the covector λ.

We recall the first order optimality condition is ever considered to be sufficient, unless the *normality condition* holds, that is when the Lagrange multiplier m_0 is not vanishing, see [2]. Hence, we first show the normality condition holds true in the case of our interest.

Proposition 6.3.8. *Problem* (6.3.17) *admits no abnormal extremals.*

Proof. We argue by contradiction by assuming $m_0 = 0$ in (6.3.18). Given this choice of m_0, (6.3.18) now reads as follows

$$H(p,y,t,\lambda_1,\lambda_2,\lambda_3,0,\omega) = \lambda_1(s)\omega(s)\sqrt{p^2(s)+1} - \lambda_2(s)p(s) - \lambda_3(s)\sqrt{p^2(s)+1}.$$

In this case, the maximization of the Hamiltonian reads as follows

$$\frac{\partial H}{\partial \omega}(p,y,t,\lambda_1,\lambda_2,\lambda_3,0,\omega) = \lambda_1(s)\sqrt{p^2(s)+1} = 0 \quad \Rightarrow \quad \lambda_1(s) = 0, \quad \forall s \in [0,T].$$

Moreover, owing to $\lambda_1(s) = 0$, for every $s \in [0,T]$ there holds

$$\lambda_1'(s) = -\frac{\partial H}{\partial p}(p,y,\lambda_1,\lambda_2,\lambda_3,0,\omega) = \lambda_2(s) + \frac{\lambda_3(s)p(s)}{\sqrt{p^2(s)+1}} = 0 \quad \Rightarrow \quad \lambda_2(s) = -\frac{\lambda_3(s)}{\sqrt{p^2(s)+1}}.$$

Additionally, as $\lambda_2'(s) = -\frac{\partial H}{\partial y} = 0$ and $\lambda_3'(s) = -\frac{\partial H}{\partial t} = 0$, we directly compute $\lambda_2'(s)$ and we obtain

$$\lambda_2'(s) = -\frac{\lambda_3(s)}{(p^2(s)+1)^{3/2}} = 0 \quad \Rightarrow \quad \lambda_3(s) = 0 \quad \Rightarrow \quad \lambda_2(s) = 0, \quad \forall s \in [0,T].$$

Thus, we conclude that

$$(\lambda_1(s), \lambda_2(s), \lambda_3(s), m_0) = (0,0,0,0), \quad \forall s \in [0,T],$$

which contradicts the fact that $(\lambda_1(s), \lambda_2(s), \lambda_3(s), m_0)$ is never vanishing. □

Since no abnormal extramals occur, we choose $m_0 = -\frac{1}{2}$ and we compute the optimal control ω^* as the unique minimizer of $H(p,y,t,\lambda_1,\lambda_2,\lambda_3,-\frac{1}{2},\omega)$, i.e.

$$\omega^*(s) = \lambda_1(s)\sqrt{p^2(s)+1}. \tag{6.3.19}$$

As a consequence, the maximized Hamiltonian H^* is

$$H^*(p, y, t, \lambda_1, \lambda_2, \lambda_3, -\frac{1}{2}, \omega^*) = \frac{1}{2}\lambda_1^2(s)(p^2(s)+1) - \lambda_2(s)p(s) - \lambda_3(s)\sqrt{p^2(s)+1}, \quad (6.3.20)$$

and the corresponding Hamiltonian system reads as follows

$$\begin{cases} p'(s) &= \lambda_1(s)\left(p^2(s)+1\right), \\ y'(s) &= -p(s), \\ t'(s) &= -\sqrt{p^2(s)+1} \\ \lambda_1'(s) &= -p(s)\lambda_1^2(s) + \lambda_2(s) + \frac{\lambda_3(s)}{\sqrt{p^2(s)+1}}, \\ \lambda_2'(s) &= 0 \\ \lambda_3'(s) &= 0. \end{cases} \quad (6.3.21)$$

We observe that, from the last equation in (6.3.21), it follows

$$\lambda_2(s) = c_2, \;\; \lambda_3(s) = c_3, \qquad \forall s \in [0, T].$$

Moreover, we choose the parameter $k := \lambda_1(0)$ as the initial condition for the first extremal, which is the unique solution to (6.3.21), with initial condition

$$(p, y, t, \lambda_1, \lambda_2, \lambda_3)(0) = (0, 0, 0, k, c_2, c_3).$$

Furthermore, as the Hamiltonian is a constant of motion, we set

$$E := \lambda_1^2(s)\left(p^2(s)+1\right) - 2\lambda_2(s)p(s) - 2\lambda_3(s)\sqrt{p^2(s)+1} = k^2 - 2c_3. \quad (6.3.22)$$

Moreover, in virtue of (6.3.19) and equations $y'(s) = -p(s)$, $t'(s) = -\sqrt{p^2(s)+1}$, we can compute the cost for extremals as follows

$$\begin{aligned} C(\omega(\cdot)) &= \int_0^T \omega^2(\tau)d\tau = \int_0^T \lambda_1^2(\tau)\left(p^2(\tau)+1\right)d\tau \\ &= \int_0^T \left(E - 2c_2y'(\tau) - 2c_3t'(\tau)\right)d\tau = ET - 2c_2y_1 - 2c_3t_1. \end{aligned} \quad (6.3.23)$$

Remark 6.3.9. Since solving analytically (6.3.21) is a real challenge, we are not able to further proceed in our characterization of the optimal control. For this reason, the statement of Theorem 6.1.2 explicitly reports the value function Ψ. In the future, it will be interesting to study this problem from a numerical perspective, as already proposed in [95] for the pricing problem for Asian Options.

6.4 Proof of Theorem 6.1.2

In this section, we prove the main result of this chapter, i.e. a lower bound for the fundamental solution Γ of $\mathscr{L}$. We follow the approach proposed in [31], where an analogous result is proved about an operator arising in Finance. A key tool in this argument is a lower bound for a Green function G for operator $\widetilde{\mathscr{K}}$ introduced in (6.2.3).

First of all, we consider the functions a and b in (6.2.14) and we modify them for $|x| > \frac{1}{2}$ in order to have continuous coefficients satisfying assumption **(H)**. It is sufficient to set

$$\begin{aligned} &a(x,y,t) = \left(1-x^2\right)^{5/2} \quad \text{and} \quad b(x,y,t) = -2x\left(1-x^2\right)^{3/2}, \quad \text{for} \quad -\tfrac{1}{2} \le x \le \tfrac{1}{2}, \\ &a(x,y,t) = \left(\tfrac{3}{4}\right)^{5/2} \quad \text{and} \quad b(x,y,t) = -\text{sign}(x)\left(\tfrac{3}{4}\right)^{3/2}, \quad \text{for} \quad |x| \ge \tfrac{1}{2}. \end{aligned} \tag{6.4.1}$$

Then, [102, Theorem 1.1] provides us with a fundamental solution $\Gamma_{\widetilde{\mathscr{K}}}$ of the Kolmogorov operator $\widetilde{\mathscr{K}}$ introduced in (6.2.3). We are now in a position to define a Green function for operator $\widetilde{\mathscr{K}}$ in a suitable cylinder $\mathcal{H}$ defined as follows.

$$\mathcal{H} = \mathcal{S} \times (0,T), \qquad \text{with} \quad \mathcal{S} = B((1,0),3/2) \cap B((-1,0),3/2),$$

where $B((x_0,w_0),r)$ denotes the the Euclidean ball of $\mathbb{R}^2$ centered at (x_0,w_0) and of radius r, and T is a positive constant. In [41, Section 4] it is proved that the Dirichlet problem for $\widetilde{\mathscr{K}}$ is well-posed on $\mathcal{H}$, i.e. for every bounded continuous function f defined on $\mathcal{H}$ and for every bounded continuous function φ defined on $\partial\mathcal{H}$, there exists a unique classical solution u to equation $\widetilde{\mathscr{K}}u = f$ in $\mathcal{H}$. Moreover, f attains continuously the boundary condition at every point of the parabolic boundary $\partial_P\mathcal{H}$ of $\mathcal{H}$, that is

$$\partial_P\mathcal{H} = (\mathcal{S} \times \{0\}) \cup (\partial\mathcal{S} \times [0,T]).$$

The Green function for $\widetilde{\mathscr{K}}$ on $\mathcal{H}$ is defined as the function $G : \overline{\mathcal{H}} \times \mathcal{H} \to [0,+\infty)$ such that

$$G(x,y,t;\xi,\eta,\tau) := \Gamma_{\widetilde{\mathscr{K}}}(x,y,t;\xi,\eta,\tau) - h(x,y,t;\xi,\eta,\tau),$$

where $h(x,y,t;\xi,\eta,\tau)$ is the solution to the Dirichlet problem:

$$\begin{cases} \widetilde{\mathscr{K}}u = 0 & \text{in } \mathcal{H}, \\ u = \Gamma_{\widetilde{\mathscr{K}}}(x,y,t;\xi,\eta,\tau) & \text{in } \partial_P\mathcal{H}. \end{cases} \tag{6.4.2}$$

We now recall the most important property of function G. For every $g \in C_0^\infty(\mathcal{H})$ and $\varphi \in C_0^\infty(\mathcal{S})$, the function

$$v(x,y,t) := \int_{\mathcal{H}} G(x,y,t;\xi,\eta,\tau) f(\xi,\eta,\tau)\, d\xi\, d\eta\, d\tau + \int_{\mathcal{S}} G(x,y,t;\xi,\eta,0)\varphi(\xi,\eta)\, d\xi\, d\eta$$

is a classical solution to the Dirichlet problem

$$\begin{cases} \widetilde{\mathscr{K}} u = -f & \text{in } \mathcal{H}, \\ u = \varphi & \text{in } \mathcal{S} \times \{0\}, \\ u = 0 & \text{in } \partial\mathcal{S} \times [0,T]. \end{cases} \tag{6.4.3}$$

We point out that the above property is stated in [41, Section 4] only for $\varphi = 0$. The validity of (6.4.3) follows from well-known properties of the fundamental solution $\Gamma_{\widetilde{\mathscr{K}}}$. We finally recall the statement of a local lower bound for the Green function given in [41, Theorem 4.3].

Lemma 6.4.1. *There exists two positive constants $c_G > 0$ and $\delta_G \in (0,1]$, only depending on the constants appearing in assumption **(H)**, such that*

$$G(0,0,t;0,0,0) \geq \frac{c_G}{t^2}, \quad \forall t \in (0, \delta_G)\,.$$

We are now in a position to prove a local lower bound for the fundamental solution Γ of the relativistic operator $\mathscr{L}$. The proof of this result is an adaptation of [31, Lemma 4.3] to the case of our interest.

Lemma 6.4.2. *For every positive constant T, there exists a psitive constant κ_T, only depending on the constants appearing in assumption **(H)**, such that*

$$\Gamma(0,0,t;0,0,0) \geq \frac{\kappa_T}{t^2}, \qquad \forall t \in (0,T)\,.$$

Proof. In order to prove our claim, we just need to show that there holds

$$\Gamma(p,y,t;\xi,\eta,\tau) \geq G(p,y,t;\xi,\eta,\tau) \qquad \forall (p,y,t;\xi,\eta,\tau) \in \overline{\mathcal{H}} \times \mathcal{H}. \tag{6.4.4}$$

Indeed, if (6.4.4) holds true, then the result for $0 < t < \delta_G$ is a straightforward consequence of Lemma 6.4.1. The result for any $T > \delta_G$ follows from the fact that Γ is a continuous strictly positive function.

Thus, it is only left to prove inequality (6.4.4). To this end, for every non-negative $\varphi \in C_0^\infty(\mathcal{S})$ and for every $(p,y,t) \in \overline{\mathcal{H}}$, we set

$$\begin{aligned} v(p,y,t) &:= \int_{\mathcal{S}} G(p,y,t;\xi,\eta,0)\varphi(\xi,\eta)\, d\xi\, d\eta, \\ w(p,y,t) &:= \int_{\mathcal{S}} \Gamma(p,y,t;\xi,\eta,0)\varphi(\xi,\eta)\, d\xi\, d\eta, \end{aligned}$$

where Γ is the fundamental solution of $\mathscr{L}$ and G is the Green function of $\widetilde{\mathscr{K}}$ in $\mathcal{H}$. By Definition 6.1.1 and (6.4.3), both v and w are solution to $\mathscr{L} u = 0$ in $\mathcal{H}$, or equivalently to

$\widetilde{\mathscr{K}} u = 0$. Then by (6.4.3) and comparison principle we find $w \geq v$ in $\mathcal{H}$. Hence, this implies

$$\int\limits_{\mathcal{S}} \left(\Gamma(p, y, t; \xi, \eta, 0) - G(p, y, t; \xi, \eta, 0)\right) \varphi(\xi, \eta)\, d\xi\, d\eta \geq 0$$

for every non-negative $\varphi \in C_0^\infty(\mathcal{S})$ and for every $(p, y, t) \in \overline{\mathcal{H}}$. This concludes the proof. $\square$

Proof of Theorem 6.1.2. By choosing $T_0 = 0$ and $T = T_1 = t_0$, we apply Proposition 6.3.7 and Lemma 6.4.2 and obtain

$$\begin{aligned} \Gamma(p_0, y_0, t_0; 0, 0, 0) &\geq C_H^{-\frac{\Psi(p_0, y_0, t_0; 0, 0, (1-\theta^2)t_0)}{k_0^2} - 1} \Gamma\left(0, 0, (1-\theta^2)t_0; 0, 0, 0\right) \\ &\geq C_H^{-\frac{\Psi(p_0, y_0, t_0; 0, 0, (1-\theta^2)t_0)}{k_0^2} - 1} \frac{\kappa_T}{(1-\theta^2)^2 t_0^2} \end{aligned}$$

for every $(p_0, y_0, t_0) \in \mathbb{R}^3$ such that $t_0 \leq \frac{\theta^2}{2}$. This proves Theorem 6.1.2 for $(p_1, y_1, t_1) = (0, 0, 0)$, where

$$c_T = C_H^{-1} \frac{\kappa_T}{(1-\theta^2)^2}.$$

The statement for a general point $(p_1, y_1, t_1) \in \mathbb{R}^3$ follows from the traslation invariance of $\mathscr{L}$ with respect to (6.1.17).

Appendix

6.A Higher dimensional case

6.A.1 Hörmander's operators

In this section, we check that the m-dimensional operator $\mathscr{L}$ in (6.1.1) can be written] in the form (6.1.2) and satisfies the Hörmander's condition (6.1.5).

We first explain how to choose the vector fields $X_1, \dots, X_m$ in (6.1.3). As a first step, we observe that equation (6.1.1) can be written in its non-divergence form

$$\mathscr{L}u(p,y,t) = \mathrm{Tr}\left((\mathbb{I}_m + p\otimes p)\, D_p^2 u\right) + \langle m\,p, D_p u\rangle - Yu = 0. \tag{6.A.1}$$

We consider the $m \times m$ symmetric matrix $\mathbf{X}$

$$\mathbf{X}(p) = (X_1(p), \dots, X_m(p)),$$

whose columns are the coefficients of the vector fields $X_1, \dots, X_m$. We have

$$\sum_{j=1}^{m} X_j u = \mathbf{X}\, D_p u, \qquad \text{and} \qquad \sum_{j=1}^{m} X_j^2 u = \mathbf{X}^2\, D_p^2 u + \langle \widetilde{c}, D_p \rangle u,$$

for some vector $\widetilde{c} = \widetilde{c}(p)$. We then determine $\mathbf{X}$ such that $\mathbf{X}^2 = \mathbb{I}_d + p \otimes p$. To do this, we recall that, for any given $q \in \mathbb{R}^m$, we have

$$(\mathbb{I}_m + q\otimes q)^2 = \mathbb{I}_m + (2+|q|^2)\, q \otimes q.$$

Then,

$$(\mathbb{I}_m + q\otimes q)^2 = \mathbb{I}_m + p\otimes p \tag{6.A.2}$$

if we choose

$$q = \alpha\, p \qquad \text{for some } \alpha \text{ such that} \qquad (2+|q|^2)|q|^2 = |p|^2. \tag{6.A.3}$$

Direct computations show that the second equality in (6.A.3) implies that

$$1 + |q|^2 = \sqrt{|p|^2+1} \tag{6.A.4}$$

and therefore

$$\alpha = \frac{1}{\sqrt{1+\sqrt{|p|^2+1}}}. \tag{6.A.5}$$

Hence, by choosing

$$\mathbf{X} = \mathbb{I}_m + q \otimes q, \qquad q = \frac{1}{\sqrt{1+\sqrt{|p|^2+1}}}\, p$$

we find the vector fields $X_1, \dots, X_m$ introduced in (6.1.3). Moreover, the components of $\widetilde{c}$ are

$$\widetilde{c}_j(p) = \sum_{i,k=1}^{m} (\delta_{ik} + q_i q_k)\, \frac{\partial(q_j q_k)}{\partial p_i}, \quad j = 1, \dots, m. \tag{6.A.6}$$

Thus, from (6.A.1) and (6.A.2) we obtain the following identity

$$\mathscr{L} = \sum_{j=1}^{m} X_j^2 + \langle (m\, p - \widetilde{c}(p)), D_p \rangle - Y.$$

In order to conclude that $\mathscr{L}$ writes in the form (6.1.2), we observe that the matrix $\mathbb{I}_m - \frac{1}{1+|q|^2}\, q \otimes q$ is the inverse of $\mathbb{I}_m + q \otimes q$. As a consequnce, we have

$$\left(\mathbb{I}_m - \tfrac{1}{1+|q|^2}\, q \otimes q\right)(X_1(p), \dots, X_m(p)) = D_p.$$

This concludes the proof of (6.1.2), where the vector $c(p) = (c_1(p), \dots, c_m(p))$ is defined as

$$c(p) = (m\, p - \widetilde{c}(p))^T \left(\mathbb{I}_m - \tfrac{1}{1+|q|^2}\, q \otimes q\right)$$

and has smooth coefficients, in virtue of (6.A.3) and (6.A.5).

We next prove that $\mathscr{L}$ does satisfy the Hörmander's condition (6.1.5). We first note that the Lie algebra generated by $X_1, \dots, X_m, X_{m+1}$ agrees with the Lie algebra generated by $X_1, \dots, X_m, Y$, and then we claim that

$$\operatorname{rank} \operatorname{Lie}\{X_1, \dots, X_m, Y\}(p, y, t) = 2m + 1, \qquad \forall (p, y, t) \in \mathbb{R}^{2m+1}. \tag{6.A.7}$$

We compute the commutator $[X_j, Y]$ for $j = 1, \dots, m$. We find that

$$[X_j, Y]u := X_j Y u - Y X_j u = \sum_{k=1}^{m} (\delta_{jk} + q_j q_k)\, \frac{\partial u}{\partial y_k} + p_j \frac{\partial u}{\partial t}.$$

We now consider the $(2m + 1) \times (2m + 1)$ matrix $\mathbf{M}$ whose columns are the coefficients of $X_1, \dots, X_m$, $[X_1, Y], \dots, [X_m, Y], Y$ and we prove that

$$\det \mathbf{M} = \sqrt{|p|^2 + 1}. \tag{6.A.8}$$

We have

$$\mathbf{M} = \Big(X_1, \ldots, X_m, [X_1, Y], \ldots, [X_m, Y], Y\Big) = \begin{pmatrix} \mathbb{I}_m + q \oplus q & \mathbb{O}_m & 0_m \\ \mathbb{O}_m & \mathbb{I}_m + q \oplus q & p \\ 0_m^T & p^T & \sqrt{|p|^2 + 1} \end{pmatrix},$$

where $\mathbb{O}_m$ is the $m \times m$ matrix whose entries are zeros, and 0_m is the zero column vector of $\mathbb{R}^m$. Up to a change of basis in $\mathbb{R}^m$, it is not restrictive to assume that $q = |q|\,\mathbf{e}_m$, being $\mathbf{e}_m$ the m-th vector of the canonical basis of $\mathbb{R}^m$. Then the matrix $\mathbf{M}$ takes the simpler form

$$\mathbf{M} = \begin{pmatrix} \mathbf{D} & 0_{2m-1} & 0_{2m-1} \\ 0_{2m-1}^T & 1 + |q|^2 & |p| \\ 0_{2m-1}^T & |p| & \sqrt{|p|^2 + 1} \end{pmatrix},$$

where $\mathbf{D} = \mathbb{I}_{2m-1} + (1 + |q|^2)\,\mathbf{e}_m \otimes \mathbf{e}_m$. Thus, (6.A.8) follows from the first equality in (6.A.5).

6.A.2 Lorentz invariance

The invariance with respect to Lorentz transformations is also preserved in the higher dimensional case. Indeed, it is sufficient to observe that the diffusion operator in (6.1.1) is the Laplace-Beltrami operator over the Riemannian manifold $(\mathbb{R}^m, g)$, where g is the metric induced by the Minkonwski metric over the hyperboloid $\mathfrak{g} = \{(E,p) : E = \sqrt{|p|^2 + 1}\}$. We recall that the Laplace-Beltrami operator is invariant with respect to isometries. Then, the invariance with of $\mathscr{L}$ follows from the fact that the Lorentz transformation in the momentum component corresponds to a translation over $\mathfrak{g}$. Moreover, the invariance of the drift term Y in (6.1.1) follows immediately from (6.1.15), which clearly still holds true in the higher dimensional case.

We conclude this appendix by remarking that, as already mentioned in Section 6.1, we expect that Theorem 6.1.2 holds true also in higher dimension. However, the proof of this result in this more general setting would require some cumbersome calculations and for this reason it will be the content of a forthcoming paper.

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