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DOTTORATO DI RICERCA IN MATEMATICA

CICLO XXXV

Nonlinear problems with biological and physical insights

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Abstract

Nonlinear models are a fundamental tool related to mathematical analysis. This is due to the wideness of problems which might be formulated according to them and the corresponding questions which arise concerning e.g. existence, uniqueness and regularity of solutions. In fact, they are often tied hand in glove to nonlinear PDEs. Although they have been studied for quite a long time, an infinite number of different assumptions can unlock different problems, which require unknown strategies and are fascinating for theoretical purposes, but not only. Indeed, nonlinear models have largely been used as a tool to describe and validate multiple phenomena and theories about what surrounds us, for example physics, biology and medicine.

The aim of this thesis is to present three different problems in this framework, whose study involves different techniques and approaches. This work is therefore divided into three Parts.

In **Part I**, we present two non-isothermal Cahn-Hilliard models. The first one is a two-dimensional PDE system describing the phase separation behaviour of a two-component fluid in a bounded domain. In particular, we are interested in studying the existence, uniqueness and regularity of solutions. This is a starting point to introduce a second model, namely a three-dimensional non-isothermal Cahn-Hilliard system describing tumor growth.

Part II is devoted to study the mechanism of breathing. In particular, we consider the lungs as a viscoelastic deformable porous medium and breathing as an isothermal periodic process, which also takes into account the phenomenon of hysteresis.

Eventually, in **Part III** we move to the field of Calculus of Variations and in particular to the study of obstacle problems. Namely, we focus on higher differentiability properties of solutions to obstacle problems with nonstandard growth conditions. Our analysis takes into account a particular class of double phase functionals. These are a useful tool to study the behaviour of strongly anisotropic materials whose hardening properties are strongly dependent on the point and connected to the exponent ruling the growth of the gradient variable.

The main results contained in this thesis have been obtained during my Ph.D. studies. In particular, **Part I** includes the content of the two papers

▶ E. Ipocoana and A. Zafferi, *Further regularity and uniqueness results for a non-isothermal Cahn-Hilliard equation*, Comm. Pure Appl. Anal., **20**(2), 763–782 (2021)

▶ E. Ipocoana, *On a non-isothermal Cahn-Hilliard model for tumor growth*, J. Math. Anal. Appl **506**(2) 125665 (2022)

Part II is based on the submitted paper

▶ E. Ipocoana and P. Krejčí, *A model for assisted periodic breathing with degenerate permeability*, submitted (2022)

which is a further development of my Master's thesis results, published in

▶ M. Eleuteri, E. Ipocoana, J. Kopfová, P. Krejčí, *Periodic solutions of a hysteresis model for breathing*, ESAIM: Mathematical Modelling and Numerical Analysis **54** 255–271, (2020)

Eventually, **Part III** gathers the main results obtained in

▶ A.G. Grimaldi and E. Ipocoana, *Higher differentiability results in the scale of Besov spaces to a class of double-phase obstacle problems*, ESAIM: COCV **28** 51, (2022)

▶ A.G. Grimaldi and E. Ipocoana, *Higher fractional differentiability for solutions to a class of obstacle problems with non-standard growth conditions*, Adv. Calc. Var. (2022)

In fact, these are not the only products of my research. Indeed, during my Ph.D. I also worked on regularity results for the Kolmogorov equation, namely

▶ E. Ipocoana and A. Rebucci, *Pointwise estimates for degenerate Kolmogorov equations with L^p-source term*, J. Evol. Equ. **22**(1), 2 (2022)

and other currently ongoing projects whose content is not part of this thesis.

Contents

Part I

Non-isothermal Cahn-Hilliard models

The first Part of this work is devoted to present models in the Cahn-Hilliard framework. The Cahn-Hilliard equation was first formulated by J.W Cahn and J.Hilliard in [18, 19] and it describes phase separation of multi-component mixtures, that is the segregation of the system into spatial domains.
The Cahn-Hilliard equation reads

$$\varphi_t = \text{div} \left[M(\varphi)\nabla \left[\frac{1}{\varepsilon} F'(\varphi) - \varepsilon\Delta\varphi \right] \right], \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{I.1}$$

complemented with appropriate initial and boundary conditions.
The process of phase separation is described by the order parameter $\varphi = \varphi(x,t)$, which represents the concentration of one of the two components in a binary system occupying a volume Ω. $M = M(\varphi)$ is a *mobility* coefficient and $F(\varphi)$ is the *free homogeneous energy function*. We remark that we here stand in the case of the so-called *diffuse interface models*. Namely, we allow a partial mixing of components in a narrow interfacial region, whose thickness is represented by a (small) parameter ε.
In the main part of our discussion, we assume the domain Ω to be bounded and with a boundary $\partial\Omega$ smooth enough. We will explain in details the needed regularity according to the different cases we consider in Chapters 1 and 2 respectively. Besides, in our analysis we suppose that the mobility coefficient M is constant. In this particular case, the Cahn-Hilliard equation (I.1) can be formulated as the system

$$\varphi_t = \Delta\mu \tag{I.2}$$

$$\mu = M\left(-\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) \right), \tag{I.3}$$

where we introduced the auxiliary variable μ, known in the literature as *chemical potential*.
According to both the formulations (I.1) and system (I.2)-(I.3), we notice that the temperature does not appear in the Cahn-Hilliard equation. This is because in general this phase separation model is isothermal, so it does not consider the possible variation of temperature during the process.
However, following the spirit of [46, 47, 39] we here also take into account the effects of the (absolute) temperature θ. Indeed, the analysis of non-isothermal models has been used to describe the evolution of several types of substances, such as, e.g., plastic materials, shape memory alloys and liquid cristals [41, 42, 50].

In particular, in Chapter 1 we model the phase separation occurring in a two-component fluid occupying a bounded spatial domain. The first issue we address is the derivation of the model. We present two different approaches. The first one follows the lead of a more general case, presented in [46], based on the strategy proposed in [55]. In the second one, we adapt the method based on the balance of microforces in [69] to the non-isothermal case. The existence and uniqueness of the solution to a more general system were established respectively in [47] and [39]. However, we here focus on proving further regularity results for the solution, which allows us to give a simpler proof of the uniqueness. The well-posedness results presented in Chapter 1 are published in [77].

In Chapter 2 we use Cahn-Hilliard equations to describe tumor growth. In the context of diffuse interface models, the tumor is seen as a mass of cells surrounded by healthy tissue, with a thin layer separating the tumoral and healthy regions. The main novelty of this model is that we here consider the effects of temperature variations on the tumor. After presenting a thermodynamically consistent derivation of our system, we prove the existence of a weak (entropy) solution. The main results of Chapter 2 are published in [75].

Chapter 1

Two-dimensional non-isothermal Cahn-Hilliard model

We aim to establish new regularity properties for a non-isothermal Cahn-Hilliard model describing the phase separation of a two-component fluid occupying a bounded domain $\Omega \subseteq \mathbb{R}^2$. The model we consider consists of a PDE system describing the evolution of the unknown variables, namely the order parameter φ, chemical potential μ and absolute temperature θ. That is

$$\varphi_t = \Delta\mu, \tag{1.1}$$

$$\mu = -\Delta\varphi + F'(\varphi) - \theta, \tag{1.2}$$

$$\theta_t + \theta\varphi_t - \text{div}\,(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2, \tag{1.3}$$

and it corresponds to the Cahn-Hilliard system for phase separation (cnfr. system (I.2)-(I.3)) coupled with the internal energy equation describing the evolution of temperature. This is a nonlinear system whose main source of difficulty is directly related to the termodynamic consistency of the model. Namely, it is represented by the quadratic term in the right-hand side of (1.3). The analysis is carried out in the 2-dimensional torus $\Omega = [0,1] \times [0,1]$, therefore we choose periodic boundary conditions for all the unknowns. The function F, whose derivative appears in (1.2), is a possibly non-convex potential whose minima represent the least energy configuration of the phase variable. Here, we assume that F is smooth and with power-like growth at ∞. Moreover the function $\kappa(\theta)$ in (1.3) denotes the heat conductivity coefficient, assumed to grow at ∞ as a power of θ, as it has been recently considered in several contributions, for instance [47].

Our system is part of the more general model

$$\mathrm{div}\mathbf{u} = 0, \tag{1.4}$$

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \Delta\mathbf{u} - \mathrm{div}(\nabla\varphi\otimes\nabla\varphi), \tag{1.5}$$

$$\varphi_t + \mathbf{u}\cdot\nabla\varphi = \Delta\mu, \tag{1.6}$$

$$\mu = -\Delta\varphi + F'(\varphi) - \theta, \tag{1.7}$$

$$\theta_t + \mathbf{u}\cdot\nabla\theta + \theta\big(\varphi_t + \mathbf{u}\cdot\nabla\varphi\big) - \mathrm{div}(\kappa(\theta)\nabla\theta) = |\nabla\mathbf{u}|^2 + |\nabla\mu|^2 \tag{1.8}$$

where the Cahn-Hilliard equation and the internal energy equation are coupled with a Navier-Stokes equation. This model has been derived and studied in [46] where the existence of solutions was shown in the 3D case (under some slightly different assumptions on the coefficients) in a very general and weak formulation. Then in [47] also the 2D case was analyzed, obtaining the existence of strong solutions. Eventually in [39] the authors were able to improve the previous results by defining a class of slightly smoother solutions and by proving that uniqueness holds in that class (and therefore well-posedness results have been proved). The key point on which these well-posedness results are proved is the following. The right hand side of (1.8) lies exactly in $L^2(0,T;L^2(\Omega))$ and this information apparently does not seem to be sufficient to get additional regularity for θ, which is essential in order to be able, for instance, to test the equation for the temperature by θ_t. In particular, a L^∞-bound is lacking because Moser iterations do not work for L^2 on the right hand side and this would be crucial in order to manage some coefficients growing like powers of θ. Therefore much efforts have been adopted in the already mentioned papers [47, 39] to overcome this difficulty and be able anyway to get a control of the gradient of θ in $L^2(\Omega)$, uniformly in time.
Motivated by these works, we aim to show that, assuming a null velocity vector field, the Moser iteration scheme works, so that the crucial L^∞-estimate for θ is now available. As a consequence, we are thus able to present a simplified proof of uniqueness for the solution to our non-isothermal Cahn-Hilliard model (1.1)-(1.3).

The main results concerning further regularity and uniqueness of solutions presented in this Chapter are contained in the published paper [77].
Chapter 1 is then structured as follows. After introducing the suitable notation in Section 1.1, we proceed showing a thermodynamically consistent derivation of the model. In particular, in Section 1.2 we show two different approaches on how to derive our system. Namely, on one hand in Section 1.2.1 we present a first strategy based on the fact that the spatial mean of the order parameter φ is conserved in time. On the other hand, in Section 1.2.2 we follow Gurtin's approach based on microforces, which does not need the

condition on mass conservation. Then we focus on proving the well-posedness results. Thus, Section 1.3 provides the assumptions for our mathematical problem. The core of this Chapter is presented in Section 1.4, where the main result, Theorem 1.4.1, namely existence and uniqueness of a solution for our problem (1.1)-(1.3), together with the additional regularity for θ, is proved.

1.1 Setting

Let us introduce some notation which will be useful in Section 1.2 and then adapted and extended in Section 1.4 to study the well-posedness results.
We suppose that a two-component fluid occupies a bounded spatial domain $\Omega \subset \mathbb{R}^2$, with a sufficiently regular boundary $\partial\Omega$. We let $\mathbf{n}$ denote the outer normal unit vector to $\partial\Omega$. Moreover, $\varphi(x,t)$ is the *order parameter*, representing the concentration difference of the fluid, or the concentration of one component, and $\theta(x,t)$ is the *absolute temperature.*

The symbol $\|\cdot\|_X$ will denote the norm in a generic Banach space. We set $H := L^2(\Omega)$ and $V := H^1(\Omega)$, $(\cdot,\cdot)$ stands for the usual standard product in H. For any function $v \in H$, we set

$$v_\Omega = \frac{1}{|\Omega|}\int_\Omega v \mathrm{d}x = \int_\Omega v \mathrm{d}x,$$

to indicate the spatial mean of v, being $|\Omega| = 1$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V' and V and $|\Omega|$ stands for the Lebesgue measure of Ω. We note as H_0, V_0 and V_0' the closed subspaces of functions (or functionals) having zero mean value in H, V, and, respectively, in V'. If the integral is replaced with the duality, the above can be extended to $v \in V'$. We denote as H_0, V_0 and V_0' the closed subspaces of functions (or functionals) having zero mean value in H, V, and, respectively, in V'. Then

$$\|v\|_{V_0} := \left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)^{1/2}$$

represents a norm on V_0, which is equivalent to the norm inherited from V by the subsequent Poincaré-Wirtinger inequality (A.1f). In particular $\|\cdot\|_{V_0}$ is a Hilbert norm and we can introduce the associated Riesz isomorphism mapping $J : V_0 \to V_0'$ by setting, for $u, v \in V_0$,

$$\langle Ju, v\rangle := ((u,v))_{V_0} := \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x. \tag{1.9}$$

For $f \in H_0$ it is easy to check that $u = J^{-1}f \in H^2(\Omega)$. Actually, u is the (unique) solution to the elliptic problem

$$u \in H_0, \quad -\Delta u = f, \quad \nabla u \cdot \mathbf{n}_{|\partial\Omega} = 0.$$

Moreover, if u is as above, then

$$\langle J(u - u_\Omega), v \rangle = -\int_\Omega v \Delta u \mathrm{d}x$$

for all $v \in V_0$. Finally, we can identify H_0 with H_0' by means of the scalar product on H obtaining the Hilbert triplet $V_0 \subset H_0 \subset V_0'$, where inclusions are continuous and dense. In particular, if $z \in V$ and $v \in V_0$, it holds that

$$\int_\Omega \nabla z \cdot \nabla (J^{-1} v) \mathrm{d}x = \int_\Omega (z - z_\Omega) v \mathrm{d}x = \int_\Omega z v \mathrm{d}x. \tag{1.10}$$

1.2 Derivation of the model

We suppose that a two-component fluid occupies an open spatial domain $\Omega \subset \mathbb{R}^2$. We denote by $\varphi(x,t)$ the concentration of one of the components of the fluid and $\theta(x,t)$ is the absolute temperature.

We present here two different thermodynamically consistent approaches to derive our model. The first one relies on the fact that the scalar function $\varphi = \varphi(x,t)$ satisfies the mass conservation constraint. The second strategy is on the other hand based on a microforces balance and does not require the condition on the conservation of mass.

We recall and collect here some general considerations, which will be exploited in both approaches.

According to the Ginzburg-Landau theory for phase transitions, we postulate the free energy density ψ and the energy functional Ψ respectevely in the form

$$\psi = \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}F(\varphi) + f(\theta) - \theta\varphi \tag{1.11}$$

$$\Psi = \int_\Omega \psi \mathrm{d}x. \tag{1.12}$$

Here, ε is a positive constant depending on the interface thickness. The function F in (1.11) penalizes the deviation of the length $|\varphi|$ from its natural value 1. We refer for instance to the double-well potential and the logarithmic potential. The term f in (1.11) describes the part of free energy which is purely caloric and is related to the specific heat

$c_V(\theta) = Q'(\theta)$ through relation

$$Q(\theta) = f(\theta) - \theta f'(\theta),$$

where Q is the entropy flux. In this Chapter (and in Chapter 2 as well) we assume that the specific heat is such that $c_V \equiv 1$. Assuming it constant is reasonable since in many materials the specific heat has small fluctuations around a single value. Moreover, we recall that it holds

$$q = Q\theta, \tag{1.13}$$

where q denotes the heat flux. The internal energy density of the system is given by Gibbs's relation

$$e = \psi + \theta s. \tag{1.14}$$

Here, s denotes the entropy of the system, which has the following expression, according to (1.11)

$$s = -\frac{\partial \psi}{\partial \theta} = -f'(\theta) + \varphi. \tag{1.15}$$

Combining the expression of internal energy density (1.14) and entropy formula (1.15), we infer

$$\frac{\partial e}{\partial t} = \frac{\partial \psi}{\partial t} + \theta \frac{\partial s}{\partial t} + s \frac{\partial \theta}{\partial t} = \frac{\partial \psi}{\partial \varphi} \frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial \nabla \varphi} \frac{\partial \nabla \varphi}{\partial t} + \theta \frac{\partial s}{\partial t} \tag{1.16}$$

and consequently

$$\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} = \frac{\partial \psi}{\partial \varphi} \frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial \nabla \varphi} \frac{\partial \nabla \varphi}{\partial t}. \tag{1.17}$$

Both of the approaches rely on the use of the thermodynamic principles. In particular, we exploit the second law of thermodynamics in the form of the Clausius-Duhem inequality

$$\theta \left(\frac{\partial s}{\partial t} + \operatorname{div} Q \right) \geq 0. \tag{1.18}$$

1.2.1 Conserved order parameter approach

We here follow the general approach presented in [55]. We note that our case can be interpreted as a particular one of [46]. However, for sake of completeness, we report here

all the details.
We denote by

$$E = (\varphi, \nabla\varphi, \theta)$$

the set of the state variables, which characterize the configuration of the material. On the other hand, the path along which the system tends to dissipate energy is described by the set of dissipative variables

$$\delta E = (\varphi_t, \nabla\theta).$$

The free energy density $\psi(E)$ and the energy functional $\Psi(E)$ are given respectevely in the form (1.11) and (1.12).
As already mentioned, the approach we present here is based on the mass conservation of the order parameter. In order to impose this constraint, we write

$$\varphi = \varphi^0 + m_0, \tag{1.19}$$

where m_0 denotes the mean value of the initial datum φ_0. Then, the conservation of mass corresponds to prescribe φ^0 to take its values in H_0 during the whole evolution of the system. The conservation of mass $\varphi(t,\cdot) = \varphi(0,\cdot)$ a.e. for $t \in (0,T)$ implies

$$\frac{D}{Dt}\int_\Omega \varphi \mathrm{d}x = \int_\Omega \varphi_t \mathrm{d}x = 0, \tag{1.20}$$

where we remark that in our case the material derivative and time derivative coincide because we assume the fluid to have zero velocity.
We also notice that condition (1.20) translates into φ_t having zero spatial mean.
Supposing that φ has a suitable regularity such that $\varphi_t \in V_0'$ a.e. in time, we define

$$\mu^0 := -J^{-1}\varphi_t, \quad \text{so that } \varphi_t = -J\mu^0 = \Delta\mu^0 \quad \text{in } V_0', \tag{1.21}$$

which entails $\mu^0 \in V_0$.
Moreover, if $\varphi_t \in H_0$, then, by elliptic regularity, we have

$$\mu^0 \in V_0 \cap H^2(\Omega) \text{ and } \nabla\mu^0 \cdot \mathbf{n}_{|\partial\Omega} = 0.$$

This setting allows us to introduce the pseudo-potential of dissipation Φ. This functional characterizes the evolution of the system and it is supposed to be non-negative and convex with respect to the dissipative variables. Its expression is given by

$$\Phi(\delta E, E) = \int_\Omega \phi(\delta E, E)\mathrm{d}x + \langle \varphi_t, J^{-1}\varphi_t \rangle, \tag{1.22}$$

where the "local component" ϕ of this *dissipation density* is given by

$$\phi(\delta E, E) = \frac{\kappa(\theta)}{2\theta}|\nabla\theta|^2, \tag{1.23}$$

where $\kappa(\theta) > 0$ is the heat conductivity.
The last term in (1.22), which is linked to the mass conservation, is nonstandard. In fact, it corresponds to a squared $V_0'-$ norm of the partial derivative of φ and thus depends in a nonlocal way on the dissipative variable.
We notice that we can equivalently rewrite the pseudo-potential Φ as follows

$$\Phi(\delta E, E) = \int_\Omega \tilde{\phi}(\delta E, E)\mathrm{d}x, \quad \text{where } \tilde{\phi}(\delta E, E) = \phi(\delta E, E) + \frac{1}{2}|\nabla\mu^0|^2. \tag{1.24}$$

Indeed, integrating by parts in space and using the definition of J, it turns out that

$$\int_\Omega |\nabla\mu^0|^2\mathrm{d}x = -\int_\Omega \Delta\mu^0\mu^0\mathrm{d}x = \int_\Omega J(\mu^0)\mu^0\mathrm{d}x = \langle \varphi_t, J^{-1}(\varphi_t)\rangle.$$

We also remark that we define the functionals Φ and Ψ for all sets of variables E and δE for which they make sense. Namely, the class of admissible state variables is given by the condition of finiteness of Ψ and Φ.

Balance equations and constitutive relations.
According to [55], by the principle of virtual power it follows that

$$\mathrm{div}\mathbf{H} - B = 0, \tag{1.25}$$

where the energy density B and the energy flux $\mathbf{H}$ are assumed to decompose as their non-dissipative and dissipative components. That is

$$B = B^{nd} + B^d,$$
$$\mathbf{H} = \mathbf{H}^{nd} + \mathbf{H}^d$$

where, taking to account (1.11), we have

$$B^{nd} = \frac{\partial\psi}{\partial\varphi} = \frac{1}{\varepsilon}F'(\varphi) - \theta, \tag{1.26}$$

$$B^d = \delta_{H_0,\varphi_t}\Phi = J^{-1}(\varphi_t), \tag{1.27}$$

According to (1.27), B^d is defined as the sub-differential of Φ with respect to φ_t in the space H_0. This coincides in particular with $J^{-1}(\varphi_t)$. Indeed, for any $v \in H_0$, we have

$$\begin{aligned}\big(J^{-1}(\varphi_t), v - \varphi_t\big) &= \langle v - \varphi_t, J^{-1}(\varphi_t)\rangle = ((v - \varphi_t, \varphi_t))_{V_0'} \\ &\leq \frac{1}{2}\,\|v\|^2_{V_0'} - \frac{1}{2}\,\|\varphi_t\|^2_{V_0'} = \frac{1}{2}\langle v, J^{-1}v\rangle - \frac{1}{2}\langle \varphi_t, J^{-1}\varphi_t\rangle.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbf{H}^{nd} &= \frac{\partial\psi}{\partial\nabla\varphi} = \varepsilon\nabla\varphi \\ \mathbf{H}^d &\equiv 0.\end{aligned}$$

Analogously, we decompose the heat flux q and the entropy flux Q as

$$\begin{aligned}q &= q^d + q^{nd}, \\ Q &= Q^d + Q^{nd},\end{aligned}$$

with $q^d = \theta Q^d$ and $q^{nd} = \theta Q^{nd}$, where, in view of (1.23),

$$q^d = \theta Q^d = \theta\frac{\partial\phi}{\partial\nabla\theta} = \kappa(\theta)\nabla\theta, \tag{1.28}$$

while the non-dissipative component is determined a posteriori in order to comply with the second law of themodynamics.
In the sequel we also ask the heat conductivity to grow as a power of the absolute temperature. This choice is mainly motivated by mathematical reasons, however it is also coherent with physical interpretations (see [108]).

The order parameter equation (1.1) is derived according to the virtual power principle (1.25). In fact, we need to include the boundary conditions and the conservation mass constraint. Therefore we first rewrite Ψ, according to (1.12) , as

$$\Psi(E) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla\varphi^0|^2 + \frac{1}{\varepsilon}F(\varphi^0 + m_0) + f(\theta) - \theta(\varphi^0 + m_0)\right)\mathrm{d}x,$$

where we used the decomposition $\varphi = \varphi^0 + m_0$ introduced in (1.19). Thus we express (1.25) as a generalized gradient flow problem in H_0, namely

$$B^d + \delta_{H_0,\varphi^0}\Psi = \delta_{H_0,\varphi_t}\Phi + \delta_{H_0,\varphi^0}\Psi = 0. \tag{1.29}$$

We remark that requiring φ^0 to lie in the domain of the differential $\delta_{H_0,\varphi^0}\Psi$ means that there exists a (unique) function $z \in H_0$ such that $\delta_{H_0,\varphi^0}\Psi(\varphi^0)$ can be represented by z in the scalar product of H_0 (and so of H). This leads to the fact that (1.29) contains both the homogeneous Neumann boundary conditions for φ and the mass conservation constraint.
Additionally, we observe that such a function z must be of the form

$$z = -\varepsilon\Delta\varphi^0 + \frac{1}{\varepsilon}\left(F'(\varphi^0 + m_0) - F'(\varphi^0 + m_0)_\Omega\right) - \theta + \theta_\Omega. \tag{1.30}$$

Now, putting together (1.29) and (1.30) with (1.27), we then obtain

$$J^{-1}(\varphi_t) = \varepsilon\Delta\varphi^0 - \frac{1}{\varepsilon}\left(F'(\varphi^0 + m_0) - F'(\varphi^0 + m_0)_\Omega\right) + \theta - \theta_\Omega.$$

Eventually, we apply the distributional Laplace operator to both hand sides, recalling that $-\Delta J^{-1}v = v$ for any $v \in H_0$, as stated in Section 1.1. Then we introduce the auxiliary variable μ such that $\mu^0 = \mu - (\mu)_\Omega$, in accordance with (1.21). This entails

$$\varphi_t = \Delta\mu, \tag{1.31}$$

$$\mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta. \tag{1.32}$$

Therefore, choosing $\varepsilon = 1$, we then recover system (1.1)-(1.2).
By the first law of thermodynamics, it follows that

$$\frac{\partial e}{\partial t} + \mathrm{div} q = B\varphi_t + \mathbf{H}\cdot\nabla\varphi_t + \Pi. \tag{1.33}$$

The last term in (1.33) does not appear in the standard theory proposed in [55]. Indeed, Π is needed to balance the nonlocal dependence with respect to the dissipative variable φ_t of the last term appearing in the expression (1.22) of the psudopotential of dissipation Φ.
We now aim to determine the expressions of Π and the nondissipative component of the heat flux q^{nd}. This is done exploiting the second law of thermodynamics in the form of Clausius-Duhem inequality (1.18).
Before doing so, we need to gain some useful relations. We first notice that, combining (1.11), (1.14) and (1.15) we infer

$$e = \psi + \theta s = \frac{1}{\varepsilon}F(\varphi) + \frac{\varepsilon}{2}|\nabla\varphi|^2 + Q(\theta). \tag{1.34}$$

We also remark that the latter term in (1.17) is given by

$$\frac{\partial \psi}{\partial \nabla \varphi}(\nabla \varphi)_t = \mathbf{H}^{nd} \cdot (\nabla \varphi_t)\,. \tag{1.35}$$

Besides, by (1.30)-(1.32), we get that

$$\varphi_t J^{-1}(\varphi_t) = -\Delta\mu(\mu - \mu_\Omega) = -\frac{1}{2}\Delta(\mu - \mu_\Omega)^2 + |\nabla\mu|^2. \tag{1.36}$$

We are now ready to focus on the second law of thermodynamics. Developing the left hand side of (1.18), we get

$$\begin{aligned}
\theta\,(s_t + \mathrm{div} Q) &\overset{(1.34),(1.13)}{=} e_t + \mathrm{div} q - \frac{\partial \psi}{\partial \varphi} - \theta_t s - Q \cdot \nabla\theta \\
&\overset{(1.15)}{=} e_t + \mathrm{div} q - \frac{\partial \psi}{\partial \varphi}\varphi_t - \frac{\partial \psi}{\partial \nabla\varphi} \cdot (\nabla\varphi)_t - Q \cdot \nabla\theta \\
&\overset{(1.33),(1.26),(1.28)}{=} B\varphi_t + \mathbf{H} \cdot (\nabla\varphi)_t + \Pi - \frac{\partial \psi}{\partial \nabla\varphi} \cdot (\nabla\varphi)_t \\
&\quad - B^{nd}\varphi_t + \frac{\kappa(\theta)}{\theta}|\nabla\theta|^2 - Q^{nd} \cdot \nabla\theta \\
&\overset{(1.27),(1.35),(1.36)}{=} |\nabla\mu|^2 - \frac{1}{2}\Delta(\mu - \mu_\Omega)^2 + \Pi + \frac{\kappa(\theta)}{\theta}|\nabla\theta|^2 - Q^{nd} \cdot \nabla\theta.
\end{aligned}$$

Then, in order to obtain the non-negativity of the right hand side of (1.18), we can assume, e.g., the following constitutive relations

$$q^{nd} = 0, \quad \Pi = \frac{1}{2}\Delta(\mu - \mu_\Omega)^2. \tag{1.37}$$

With these choices, we get $\int_\Omega \Pi(x)\mathrm{d}x = 0$, as expected. Moreover, rewriting the internal energy balance (1.33) as

$$(Q(\theta))_t + \theta\varphi_t - \mathrm{div}(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2, \tag{1.38}$$

and choosing the specific heat equal to one, we recover the temperature equation (1.3). Notice that the dissipation terms on the right hand side are perfectly in agreement with the expression (1.24) of the pseudo-potential of dissipation Ψ. Indeed, as already mentioned, one has $\mu^0 = \mu - \mu_\Omega$ due to (1.31)-(1.32).

1.2.2 Gurtin's microforces approach

In this Section, we follow Gurtin's approach proposed in [69] in order to derive our system (1.1)-(1.3). Namely, we treat separately the balance laws and the constitutive relations, moreover we consider the following balance law for internal microforces

$$\operatorname{div}\zeta + \pi = 0, \tag{1.39}$$

where ζ is a vector representing the microstress and π is a scalar corresponding to the internal microforces. Microforces describe the forces associated with microscopic configurations of atoms, differently from standard forces, which are associated with macroscopic length scales. These different length scales are the reason why a separate balance law for microforces is needed. Besides, since in Cahn-Hilliard models the kinematics is associated with the order parameter, it is natural to infer that the working of microforces has effects on φ. Such interatomic forces may be mirrored on the macroscopic level by fields which perform work when the order parameter undergoes changes. Therefore this working can be described in terms of φ_t, which explains why the microforces are scalar rather than vector quantities.

We complement the balance law for microforces with the fundamental balance laws. Since we do not consider external mass sources, the mass balance law reads

$$\varphi_t = -\operatorname{div} h, \tag{1.40}$$

where h is the mass flux.

The derivation of our system is based on the first and second fundamental laws of thermodynamics. We proceed showing first the derivation of the Cahn-Hilliard system (1.1)-(1.2) and then the one of the temperature equation (1.3).

Cahn-Hilliard system.
According to [69], we write the first law in the form

$$\frac{d}{dt}\int_R e\mathrm{d}x = -\int_{\partial R} q\cdot\nu\mathrm{d}\eta + \mathcal{W}(R) + \mathcal{M}(R), \tag{1.41}$$

where R is the control volume, ν is the outward unit normal to ∂R and

$$\mathcal{W}(R) = \int_{\partial R}(\zeta\cdot\nu)\frac{\partial\varphi}{\partial t}\mathrm{d}\eta, \tag{1.42}$$

$$\mathcal{M}(R) = -\int_{\partial R}\mu h\cdot\nu\mathrm{d}\eta \tag{1.43}$$

are the rate of working and the rate at which free energy is added to R (assuming no heat supply) respectively. Using Green's formula, we can rewrite (1.41) as

$$\frac{\partial e}{\partial t} = -\operatorname{div} q + \frac{\partial \varphi}{\partial t} \operatorname{div} \zeta + \zeta \cdot \nabla \frac{\partial \varphi}{\partial t} - h \cdot \nabla \mu - \mu \operatorname{div} h, \tag{1.44}$$

where we introduced the heat flux q in (1.13). Since the control volume R is arbitary, exploiting the mass balance (1.40) and the microforce balance (1.39), we infer

$$\frac{\partial e}{\partial t} = -\operatorname{div} q + (\mu - \pi) \frac{\partial \varphi}{\partial t} + \zeta \cdot \nabla \frac{\partial \varphi}{\partial t} - h \nabla \mu. \tag{1.45}$$

We now impose the validity of the second law of thermodynamics in the form of the Clausius-Duhem inequality (1.18). We develop the left hand side of (1.18) as follows

$$\begin{aligned}
\theta \left(\frac{\partial s}{\partial t} + \operatorname{div} Q \right) &\overset{(1.14)}{=} \frac{\partial e}{\partial t} - \frac{\partial \psi}{\partial t} - s \frac{\partial \theta}{\partial t} + \theta \operatorname{div} Q \\
&\overset{(1.13)}{=} \frac{\partial e}{\partial t} - \frac{\partial \psi}{\partial t} - s \frac{\partial \theta}{\partial t} + \operatorname{div} q - Q \cdot \nabla \theta \\
&\overset{(1.45)}{=} (\mu - \pi) \frac{\partial \varphi}{\partial t} + \zeta \cdot \nabla \frac{\partial \varphi}{\partial t} - h \nabla \mu - \frac{\partial \psi}{\partial t} - s \frac{\partial \theta}{\partial t} - Q \cdot \nabla \theta \\
&\overset{(1.17)}{=} \left(\mu - \pi - \frac{\partial \psi}{\partial \varphi} \right) \frac{\partial \varphi}{\partial t} + \left(\zeta - \frac{\partial \psi}{\partial \nabla \varphi} \right) \frac{\partial \nabla \varphi}{\partial t} - h \nabla \mu - Q \cdot \nabla \theta.
\end{aligned}$$

In order to satisfy relation (1.18), we impose

$$\mu - \pi - \frac{\partial \psi}{\partial \varphi} = 0, \tag{1.46}$$

$$\zeta = \frac{\partial \psi}{\partial \nabla \varphi}, \tag{1.47}$$

$$h \nabla \mu + Q \cdot \nabla \theta \leq 0, \tag{1.48}$$

where in particular in order for (1.48) to hold, we exploited Fourier's law

$$q = -\kappa(\theta) \nabla \theta, \tag{1.49}$$

with $\kappa = \kappa(\theta) > 0$ heat conductivity.

The combination of (1.11) and (1.47) straightly gives

$$\zeta = \varepsilon \nabla \varphi, \tag{1.50}$$

which leads to, according to (1.11), (1.39) and (1.46),

$$\mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta. \tag{1.51}$$

Eventually, inequality (1.48) can be satisfied choosing $h = -\nabla\mu$, which is a suitable assumption according to [69]. Therefore equation (1.40) reads

$$\varphi_t = \Delta\mu. \tag{1.52}$$

Temperature equation.
We start from the internal energy equation (1.45), taking advantage of (1.47) and of the expression for the chemical potential (1.46), therefore

$$\frac{\partial e}{\partial t} = -\operatorname{div} q + \frac{\partial\psi}{\partial\varphi}\frac{\partial\varphi}{\partial t} + \frac{\partial\psi}{\partial\nabla\varphi}\frac{\partial\nabla\varphi}{\partial t} - h\nabla\mu.$$

Now, exploiting the assumption $h = -\nabla\mu$ and Fourier's law (1.49), we infer

$$\frac{\partial e}{\partial t} = \operatorname{div}(\kappa(\theta)\nabla\theta) + \frac{\partial\psi}{\partial\varphi}\frac{\partial\varphi}{\partial t} + \frac{\partial\psi}{\partial\nabla\varphi}\frac{\partial\nabla\varphi}{\partial t} + |\nabla\mu|^2$$

and by identity (1.16),

$$\theta\frac{\partial s}{\partial t} - \operatorname{div}(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2. \tag{1.53}$$

From (1.15), we might write

$$\theta\frac{\partial s}{\partial t} = \theta(-f'(\theta))_t + \theta\varphi_t.$$

On the other hand, according to the definition of Q, it holds $(Q(\theta))_t = (-f'(\theta))_t$, with in particular $(Q(\theta))_t = Q'(\theta)\theta_t$. Since we supposed that we are considering the case in which the specific heat $c_V = 1$, it follows that $Q'(\theta) = 1$. This implies that

$$\theta s_t = \theta_t + \theta\varphi_t.$$

Thus, equation (1.53) reads

$$\theta_t + \theta\varphi_t - \operatorname{div}(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2. \tag{1.54}$$

1.3 Assumptions

In this Section we expand and adapt the notation presented in Section 1.1, in order to study the well-posedness of our system (1.1)-(1.3).

We consider our PDE system taking place in the two-dimensional flat torus with periodic boundary conditions, namely $\Omega = [0,1] \times [0,1]$ and $\varphi|_{x_i=0} = \varphi|_{x_i=1}$ $i = 1,2 \quad \forall t \in (0,T)$.
We denote as $H := L^2_{\text{per}}(\Omega)$ the space of functions in $L^2(\mathbb{R}^2)$ which are Ω-periodic (i.e., 1-periodic both in x_1 and x_2). Analogously, we set $V := H^1_{\text{per}}(\Omega)$. The spaces H and V are endowed with the norms of $L^2(\Omega)$ and $H^1(\Omega)$, respectively. For brevity, the norm of H will be simply indicated by $\|\cdot\|$. Still for brevity, we omit the variables of integration. We will specify them when there could be a misinterpretation. The symbol $\langle \cdot, \cdot \rangle$ will indicate the duality between V' and V and $(\cdot,\cdot)$ will stand for the standard product of H. We also write $L^p(\Omega)$ instead of $L^p_{\text{per}}(\Omega)$, and the same for other spaces; indeed, no confusion should arise since periodic boundary conditions are assumed to hold for all unknowns. We denote $H^m_{\text{per}}(\Omega)$ the space of functions which are $H^m_{\text{loc}}(\Omega)$ and Ω-periodic, for $m \in \mathbb{R}, m \geq 0$. In particular, for $m = 0$ we have $H^0_{\text{per}}(\Omega) = L^2_{\text{per}}(\Omega)$.

We now focus on the mathematical hypotheses needed on the nonlinear terms.
We ask the configuration potential F to satisfy:

$$F \in C^3(\mathbb{R};\mathbb{R}), \quad \liminf_{|r|\to\infty} \frac{F(r)}{|r|} > 0, \tag{1.55}$$

$$F''(r) \geq -\lambda \text{ for some } \lambda \geq 0, \text{ and all } r \in \mathbb{R}, \tag{1.56}$$

$$|F'''(r)| \leq \tilde{c}_F(1 + |r|^{p_F - 1}) \text{ for some } \tilde{c_F} \geq 0, p_F \geq 1, \text{ and all } r \in \mathbb{R}. \tag{1.57}$$

We remark that (1.57) implies

$$|F''(r)| \leq c_F(1 + |r|^{p_F}) \text{ for some } c_F \geq 0, p_F \geq 0, \text{ and all } r \in \mathbb{R}.$$

Assumption (1.55) postulates regularity and coercivity of F, (1.56) is λ-convexity and (1.57) prescribes a polynomial growth at infinity. Note that (1.55) implies that

$$F(s) \geq -c_0 \quad \forall s \in \mathbb{R}$$

for some constant $c_0 > 0$. We assume moreover the heat conductivity to be given by

$$\kappa(r) = 1 + r^q, \quad q \in [2,\infty), \quad r \geq 0. \tag{1.58}$$

Correspondingly, we define

$$K(r) := \int_0^r \kappa(s)ds = r + \frac{1}{1+q} r^{1+q}, \quad r \geq 0. \tag{1.59}$$

We aim to estimate the norm $\|K(\theta)\|_V^2$, since it will be needed in the sequel. We observe that, for some $k_q > 0$,

$$\int_\Omega \kappa(\theta)^2 |\nabla\theta|^2 \,\mathrm{dx} = \|\nabla K(\theta)\|^2 \geq \|\nabla\theta\|^2 + k_q \left\|\nabla\theta^{q+1}\right\|^2. \tag{1.60}$$

Then, by (A.2) with $p = 2$, we infer

$$\|K(\theta)\|_V^2 \leq c_q \left(\int_\Omega (\theta + \theta^{q+1})\, dx \right)^2 + c_q \left(\int_\Omega \left| \nabla\theta + \nabla \left(\frac{\theta^{q+1}}{q+1} \right) \right|^2 dx \right) =: I + II \tag{1.61}$$

for some $c_q > 0$. From (A.2), choosing $p = 2(q+1)$, it holds

$$\begin{aligned} I = c_q \bigg(\int_\Omega \theta + \int_\Omega \theta^{q+1} \bigg)^2 &\leq c_q \|\theta\|_{L^1(\Omega)}^2 + c_q \left(\|\theta\|_{L^1(\Omega)}^{2(q+1)} + \|\nabla\theta^{q+1}\|^2 \right) \\ &\leq c_q \left(1 + \|\theta\|_{L^1(\Omega)}^{2(q+1)} + \|\nabla\theta^{q+1}\|^2 \right). \end{aligned} \tag{1.62}$$

We then estimate II according to (1.60). Therefore we get

$$\|K(\theta)\|_V^2 \leq C \left(1 + \|\theta\|_{L^1(\Omega)}^{2(q+1)} + \int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \right), \tag{1.63}$$

for some C depending on q.

Complying with the boundary conditions and the lack of external forces, we have the conservation of mass and of the total energy $\mathcal{E}$ that will be rigourously defined in (1.69), namely

$$\varphi(t)_\Omega = \varphi(0)_\Omega \qquad \mathcal{E}(\varphi(t), \theta(t)) = \mathcal{E}(\varphi(0), \theta(0)).$$

It comes natural to define the "energy-entropy space" of data as:

$$\mathcal{H} = \left\{ z = (\varphi, \theta) \in V \times L^1(\Omega) : \theta > 0 \text{ a.e. in } \Omega, \log\theta \in L^1(\Omega) \right\}.$$

In our space we omitted the chemical potential μ, in view of the fact that μ can be regarded as an auxiliary variable, and sometimes, depending on the situation, it will be more convenient to "exclude" μ. This can be easily achieved rewriting the system (1.1)-(1.2) as a single equation where μ no longer appears.

Now, in agreement with [47] and [39], we define the set

$$\mathcal{V} := \left\{ z = (\varphi, \theta) \in \mathcal{H} \cap \left(H^3(\Omega) \times V \right) : K(\theta) \in V, 1/\theta \in L^1(\Omega), \theta > 0 \text{ a.e.} \right\},$$

where the requirement $K(\theta) \in V$ yields $\theta \in V$.
Eventually, we recall here a result which will be useful in order to reach regularity in Section 1.4.2. The proof of this Lemma can be found in [47].

Lemma 1.3.1. Let $\mathcal{O}$ a smooth bounded domain in $\mathbb{R}^2$. Then, there exists $c > 0$ depending only on $\mathcal{O}$ such that

$$\|\xi\|_{H^1(\mathcal{O})'} \leq c\Big(1 + \|\xi\|_{L^1(\mathcal{O})} \log^{1/2}\left(e + \|\xi\|_{L^2(\mathcal{O})}\right)\Big) \tag{1.64}$$

for any $\xi \in L^2(\mathcal{O})$.

1.4 Well posedness results

1.4.1 Main result

We are now ready to present the main result of this Chapter, namely

Theorem 1.4.1. *Let us assume* (1.55)-(1.57) *and* (1.58). *Let also* $T > 0$. *Then given* $z_0 \in \mathcal{V}$ *there exists a unique solution to our problem, namely a triple* (φ, μ, θ) *with the regularity*

$$\varphi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^\infty(0,T;H^3(\Omega)), \tag{1.65}$$

$$\mu \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)), \tag{1.66}$$

$$\theta \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;V), \tag{1.67}$$

$$\theta > 0 \text{ a.e. in } (0,T) \times \Omega,$$

$$K(\theta) \in L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;V), \tag{1.68}$$

satisfying equations (1.1)-(1.3) *a.e. in* $(0,T) \times \Omega$ *and complying with the initial conditions*

$$\varphi|_{t=0} = \varphi_0, \qquad \theta|_{t=0} = \theta_0$$

almost everywhere in Ω.

An existence result for our model can be proved by means of the solution of an approximating problem and then on the application of Schauder's fixed point theorem. The complete proof can be found in [47].

Here, we focus o regularity and uniqueness results. Concerning the regularity, in Section 1.4.2, we recover the basic regularity already shown in [47] for the general model, while in Section 1.4.3 we obtain further regularity with respect to [39]. We then exploit the regularity obtained to prove the uniqueness of the solution in Section 1.4.4.

1.4.2 Initial regularity

In this section we recover the basic regularity already obtained for the general model in [47] and we sketch the main points. We observe that, considering here null velocity, as one may expect, also this part of the proof turns to be simplified. In particular this is evident in two points: in the complementary estimates (1.82) and (1.84), that can be easily derived one from the other (due to the absence of convective terms) and in the key estimate of the term (θ_t, φ_t). This last achievement is obtained by means of the control of two terms: the estimate of $\int \kappa^2(\theta)|\nabla\theta|^2$ follows exactly as in [47] while the estimate of the term $\int |\nabla\mu|^2\varphi_t$ can be heavily simplified, even if the idea of relying on conjugate functions still is necessary, for the presence of the quadratic term in the right hand side of (1.3).

Energy and entropy estimates

The energy estimate is obtained by testing (1.1) by μ, (1.2) by $-\varphi_t$, (1.3) by 1 and then integrating over Ω. We then sum up all the obtained relations. Therefore we infer

$$\frac{d}{dt}\mathcal{E}(\varphi,\theta) = 0, \qquad \text{where } \mathcal{E}(\varphi,\theta) := \int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 + F(\varphi) + \theta\right) \tag{1.69}$$

which is the *total energy* of the system, given by the sum of the *interfacial*, *configuration*, and *thermal* energies (the three terms in $\mathcal{E}$). From relation (1.69) we infer the following a priori estimates

$$\|\varphi\|_{L^\infty(0,T;V)} \le c, \tag{1.70}$$

$$\|\theta\|_{L^\infty(0,T;L^1(\Omega))} \le c, \tag{1.71}$$

where we exploited (1.55) in order to obtain (1.70) and we used the nonnegativity of θ to get (1.71) from (1.69). Moreover, from (1.70) and Sobolev's embeddings, we also have

$$\|\varphi\|_{L^\infty(0,T;L^p(\Omega))} \le c_p \quad \text{for all } p \in [1,\infty). \tag{1.72}$$

On the other hand, integrating (1.1) over Ω, and using the periodic boundary conditions, we observe

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi = 0 \quad \text{a.e. in } (0,T). \tag{1.73}$$

The entropy estimate corresponds to the entropy production principle. In order to obtain it, we test (1.3) by $-\theta^{-1}$ and integrate over Ω, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(-\log\theta-\varphi)+\int_\Omega\frac{1}{\theta}|\nabla\mu|^2+\int_\Omega\frac{\operatorname{div}(\kappa(\theta)\nabla\theta)}{\theta}=0.$$

The last term on the left hand side can be written as

$$\begin{aligned}\int_\Omega\frac{\operatorname{div}(\kappa(\theta)\nabla\theta)}{\theta}&=\int_\Omega\frac{\operatorname{div}(\nabla\theta)}{\theta}+\int_\Omega\frac{\operatorname{div}(\theta^q\nabla\theta)}{\theta}\\&=\int_\Omega\frac{\Delta\theta}{\theta}+\int_\Omega\frac{q\theta^{q-1}|\nabla\theta|^2+\theta^q\Delta\theta}{\theta}\\&=\int_\Omega|\nabla\log\theta|^2+k_q|\nabla\theta^{q/2}|^2.\end{aligned}$$

Therefore we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(-\log\theta-\varphi)+\int_\Omega\frac{1}{\theta}|\nabla\mu|^2+\int_\Omega\left(|\nabla\log\theta|^2+k_q|\nabla\theta^{q/2}|^2\right)=0, \tag{1.74}$$

with $k_q>0$ only depending on the exponent q introduced in (1.58). We now integrate in time. Recalling that $|\log r|\le r-\log r \quad \forall r>0$ and owing to (1.70) and (1.71), we get the a priori bounds

$$\|\log\theta\|_{L^\infty(0,T;L^1(\Omega))}+\|\log\theta\|_{L^2(0,T;V)}\le c, \tag{1.75}$$

$$\|\nabla\theta^{q/2}\|_{L^2(0,T;H)}\le c. \tag{1.76}$$

In particular, from (1.75) we see that the strict positivity of θ is preserved a.e. in $(0,T)\times\Omega$ also in the limit. Moreover, the combination of inequality (A.2) with estimates (1.71) and (1.76) gives

$$\|\theta^{q/2}\|_{L^2(0,T;V)}\le c,$$

which implies in particular

$$\|\theta\|_{L^2(0,T;H)}\le c$$

and, being $q\ge 2$, according to (1.58),

$$\|\theta\|_{L^2(0,T;V)}\le c. \tag{1.77}$$

First estimates for μ, φ and φ_t

From equation (1.3) and periodic boundary conditions, we get

$$\int_\Omega |\nabla\mu|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta + \int_\Omega \theta\varphi_t. \tag{1.78}$$

Our aim is to control the terms on the right hand side.
In order to do so, we first integrate in time, and then we estimate the first one thanks to (1.71). On the other hand, by using (1.1) and Hölder's and Young's inequalities, the second integral can be controlled as follows

$$\int_\Omega \theta\varphi_t = \int_\Omega \theta\Delta\mu = -\int_\Omega \nabla\theta\cdot\nabla\mu \le \frac{1}{2}\big(\|\nabla\mu\|^2 + \|\nabla\theta\|^2\big). \tag{1.79}$$

The first term on the right hand side is absorbed by the corresponding one on the left hand side of (1.78), while we use (1.77) to estimate the latter. Hence, we obtain

$$\|\nabla\mu\|_{L^2(0,T;H)} \le c. \tag{1.80}$$

We now integrate (1.2) in space, combine (1.57), (1.71) and (1.72) and then take the (essential) supremum with respect to time; we infer

$$\|\mu_\Omega\|_{L^\infty(0,T)} \le c. \tag{1.81}$$

This estimate, combined with (1.80), gives

$$\|\mu\|_{L^2(0,T;V)} \le c. \tag{1.82}$$

Now, testing (1.1) by nonzero $v \in V$, we can notice that

$$\langle\varphi_t, v\rangle = -\int_\Omega \nabla\mu\cdot\nabla v \le \|\nabla\mu\|\|\nabla v\| \le c\|\nabla\mu\|\|v\|_V. \tag{1.83}$$

Hence, dividing by $\|v\|_V$, passing to the supremum with respect to $v \in V\setminus\{0\}$, squaring, integrating in time, and using (1.82), we infer

$$\|\varphi_t\|_{L^2(0,T;V')} \le c. \tag{1.84}$$

On the other hand, we test equation (1.1) by $-\mu$ and note that $\langle\varphi_t, \mu_\Omega\rangle = 0$, since φ_t has zero (generalized) mean. Therefore the use of the Poincaré-Wirtinger inequality yields

$$\begin{aligned}\|\nabla\mu\|^2 &= -\int_\Omega \varphi_t\mu = -\int_\Omega \varphi_t(\mu-\mu_\Omega) \le \|\mu-\mu_\Omega\|_V\|\varphi_t\|_{V'}\\ &\le \frac{1}{2}\|\nabla\mu\|^2 + c\|\varphi_t\|^2_{V'},\end{aligned}$$

which allows us to get

$$\|\nabla\mu\|^2 \le c\|\varphi_t\|^2_{V'}. \tag{1.85}$$

Finally, if we test (1.2) by $\Delta^2\varphi$ and integrate over Ω, by recalling (1.57), we get

$$\|\nabla\Delta\varphi\|^2 \le c\big(1+\|\nabla\theta\|^2+\|\nabla\mu\|^2\big)$$

Integrating this inequality in time and using (1.77) and (1.80), we then obtain

$$\|\varphi\|_{L^2(0,T;H^3(\Omega))} \le c. \tag{1.86}$$

Key estimate: control of the term (θ_t,φ_t)

First of all we take (1.1), differentiate it with respect to time, and test the result by $J\varphi_t$, where J was first introduced in (1.9). Correspondingly, we differentiate (1.2) in time and test the result by $-\varphi_t$. Therefore we have

$$\begin{aligned}&\big\langle\varphi_{tt}, J\varphi_t\big\rangle = \langle\Delta\mu_t, J\varphi_t\rangle\\ &\big\langle\mu_t,-\varphi_t\big\rangle = \langle-\Delta\varphi_t,-\varphi_t\rangle + \big\langle F''(\varphi)\varphi_t,-\varphi_t\big\rangle + (\theta_t,\varphi_t)\end{aligned}$$

Summing the obtained relations, noting that a couple of terms cancel in view of

$$\langle\Delta\mu_t, J\varphi_t\rangle = -\big\langle(-\Delta)(\mu_t-(\mu_t)_\Omega),(-\Delta)^{-1}\varphi_t\big\rangle = -\big\langle\mu_t-(\mu_t)_\Omega,\varphi_t\big\rangle = -\langle\mu_t,\varphi_t\rangle, \tag{1.87}$$

by (A.3) and (1.73), we then get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|^2_{V'} + \|\nabla\varphi_t\|^2 + \int_\Omega (F''(\varphi)+\lambda)|\varphi_t|^2\\ =&\lambda\|\varphi_t\|^2 + (\theta_t,\varphi_t) \le \frac{1}{8}\|\nabla\varphi_t\|^2 + c\|\varphi_t\|^2_{V'} + (\theta_t,\varphi_t).\end{aligned} \tag{1.88}$$

Reabsorbing, this is equivalent to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|^2_{V'} + \frac{7}{8}\|\nabla\varphi_t\|^2 + \int_\Omega (F''(\varphi)+\lambda)|\varphi_t|^2 \le c\|\varphi_t\|^2_{V'} + (\theta_t,\varphi_t). \tag{1.89}$$

On the other hand, testing (1.3) by φ_t yields

$$
\begin{aligned}
(\theta_t, \varphi_t) + \int_\Omega \theta \varphi_t^2 =& - \int_\Omega \kappa(\theta) \nabla\theta \cdot \nabla\varphi_t + \int_\Omega |\nabla\mu|^2 \varphi_t \\
\leq& \frac{1}{16}\|\nabla\varphi_t\|^2 + 4\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 + \int_\Omega |\nabla\mu|^2\varphi_t.
\end{aligned}
\tag{1.90}
$$

To treat the term in $\nabla\theta$, we test (1.3) by $6K(\theta)$ introduced in (1.59) (the coefficient 6 is suitable for reabsorbing the second term in the right hand side of the previous inequality by the left hand side) and working exactly as in [47] we deduce the following estimate

$$
\begin{aligned}
&6\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\theta) + 5\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \\
\leq& c(1 + \|\varphi_t\|_{V'}^4) + \frac{1}{8}\|\nabla\varphi_t\|^2 + 6\int_\Omega K(\theta)|\nabla\mu|^2.
\end{aligned}
\tag{1.91}
$$

where we set

$$
\mathcal{J}(r) := \int_0^r K(s)\,\mathrm{d}s = \frac{r^2}{2} + \frac{1}{(q+1)(q+2)} r^{q+2}, \quad r \geq 0. \tag{1.92}
$$

Summing (1.90) and (1.91) we get

$$
\begin{aligned}
&(\theta_t, \varphi_t) + \int_\Omega \theta\varphi_t^2 + 6\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\theta) + \int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \\
\leq& c\big(1 + \|\varphi_t\|_{V'}^4\big) + \frac{3}{16}\|\nabla\varphi_t\|^2 + \int_\Omega (6K(\theta) + \varphi_t)|\nabla\mu|^2.
\end{aligned}
\tag{1.93}
$$

Then, adding together (1.89) and (1.93) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|_{V'}^2 + 6\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\theta) + \frac{11}{16}\|\nabla\varphi_t\|^2 + \int_\Omega (F''(\varphi) + \lambda)|\varphi_t|^2 \\
&+ \int_\Omega \theta\varphi_t^2 + \int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \\
\leq& c\|\varphi_t\|_{V'}^2 + c\big(1 + \|\varphi_t\|_{V'}^4\big) + \int_\Omega (6K(\theta) + \varphi_t)|\nabla\mu|^2.
\end{aligned}
\tag{1.94}
$$

Neglecting some positive terms in the left hand side and rearranging, we then arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|_{V'}^2 + 6\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{J}(\theta) + \frac{11}{16}\|\nabla\varphi_t\|^2 + \int_\Omega \kappa^2(\theta)|\nabla\theta|^2 \\
\leq& c(1 + \|\varphi_t\|_{V'}^2)^2 + \int_\Omega (6K(\theta) + \varphi_t)|\nabla\mu|^2.
\end{aligned}
\tag{1.95}
$$

We now focus on controlling of the last term in the right hand side of (1.95), which represents the most difficult part of our argument. In order to do so, we use the embedding inequality (1.64) setting $\xi = |\nabla\mu|^2$. Then, exploiting (1.63), (1.73) and the Poincaré-Wirtinger inequality, we infer

$$\begin{aligned}\int_\Omega (6K(\theta)+\varphi_t)|\nabla\mu|^2 \leq & c\big(\|K(\theta)\|_V + \|\nabla\varphi_t\|\big)\big\||\nabla\mu|^2\|_{V'} \\ \leq & c + \frac{1}{2}\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 + \frac{1}{8}\|\nabla\varphi_t\|^2 + c\big\||\nabla\mu|^2\|^2_{V'} \\ \leq & c + \frac{1}{2}\int_\Omega \kappa^2(\theta)|\nabla\theta|^2 + \frac{1}{8}\|\nabla\varphi_t\|^2 \\ & + c\big\||\nabla\mu|^2\big\|^2_{L^1(\Omega)}\log\big(e+\big\||\nabla\mu|^2\big\|_{L^2(\Omega)}\big).\end{aligned} \tag{1.96}$$

Next, we consider the functions $\psi(r) = e^r$, $r \in \mathbb{R}$ and $\psi^*(s) = s(\log s - 1)$, $s > 0$ (extended by continuity to $s = 0$ by setting $\psi^*(0) = 0$), which are *convex conjugate*. This means that $\forall r \in \mathbb{R}$, $s \geq 0$, it holds $rs \leq \psi(r) + \psi^*(s)$, as we can see for example in [10, Sec. 1.4]. If we now set $r = \log\big(e + \||\nabla\mu|^2\|_{L^2(\Omega)}\big)$ and $s = c\big\||\nabla\mu|^2\big\|^2_{L^1(\Omega)}$, we can estimate the last term in (1.96) as follows

$$\begin{aligned} & c\big\||\nabla\mu|^2\big\|^2_{L^1(\Omega)}\log\big(e+\big\||\nabla\mu|^2\big\|_{L^2(\Omega)}\big) \\ \leq & c\big\||\nabla\mu|^2\big\|^2_{L^1(\Omega)}\Big(\log\big(c\big\||\nabla\mu|^2\big\|^2_{L^1(\Omega)}\big) - 1\Big) + e + \big\||\nabla\mu|^2\big\|_{L^2(\Omega)} \\ \leq & c + c\|\nabla\mu\|^4\log\big(e+\|\nabla\mu\|^2\big) + \|\nabla\mu\|^2_{L^4(\Omega)},\end{aligned} \tag{1.97}$$

where we used the fact that $\big\||\nabla\mu|^2\big\|^2_{L^1(\Omega)} = \|\nabla\mu\|^4$ and elementary inequalities concerning logarithms.

The first non-constant term on the right hand side of (1.97) can be estimated by using (1.85) as follows

$$\begin{aligned} c\|\nabla\mu\|^4\log\big(e+\|\nabla\mu\|^2\big) \leq & c\big(1+\|\varphi_t\|^4_{V'}\big)\log\Big(e + c\big(1+\|\varphi_t\|^2_{V'}\big)\Big) \\ \leq & c(1+\|\varphi_t\|^2_{V'})^2\log\big(e+\|\varphi_t\|^2_{V'}\big)\end{aligned} \tag{1.98}$$

while the second one can be controlled by using equation (1.1) and inequalities (A.1a) and (A.3) as

$$\begin{aligned}\|\nabla\mu\|^2_{L^4(\Omega)} \leq & c\|\nabla\mu\|\|\mu\|_{H^2(\Omega)} \leq c\|\nabla\mu\|\big(\|\mu\|_V + \|\Delta\mu\|\big) \\ \leq & c\|\mu\|^2_V + c\|\varphi_t\|^2 \ \leq c\|\mu\|^2_V + \frac{1}{8}\|\nabla\varphi_t\|^2 + c\|\varphi_t\|^2_{V'}.\end{aligned} \tag{1.99}$$

Then, setting $M(t) := c\|\mu\|^2_V$, plugging (1.97)-(1.99) in (1.96) and in turn the result into (1.95) we finally deduce

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|_{V'}^2+6\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\mathcal{J}(\theta)+\frac{7}{16}\|\nabla\varphi_t\|^2+\frac{1}{2}\int_\Omega\kappa^2(\theta)|\nabla\theta|^2\\ \leq& c(e+\|\varphi_t\|_{V'}^2)^2[1+\log\left(e+\|\varphi_t\|_{V'}^2\right)]+M(t) \\ \leq& c(e+\|\varphi_t\|_{V'}^2)^2\log\left(e+\|\varphi_t\|_{V'}^2\right)+M(t).\end{aligned}\tag{1.100}$$

Let us now set

$$\Phi(t):=e+\frac{1}{2}\|\varphi_t\|_{V'}^2,\qquad \Theta(t):=6\int_\Omega\mathcal{J}(\theta(t)).\tag{1.101}$$

Hence, (1.100) reads

$$\Phi'(t)+\Theta'(t)\leq\ c\,[\Phi(t)]^2\log(\Phi(t))+M(t).\tag{1.102}$$

We define $Z(t):=e+\Phi(t)+\Theta(t)$, then we divide both hand sides of (1.102) by $Z\log Z$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\log\log Z(t)=\frac{Z'(t)}{Z(t)\log Z(t)}\\ &\leq M(t)\frac{\Phi(t)}{(e+\Phi(t)+\Theta(t))\log(e+\Phi(t)+\Theta(t))}+\frac{M(t)}{Z(t)\log Z(t)}\\ &+c\Phi\frac{\Phi\log(e+\Phi(t))}{(e+\Phi(t)+\Theta(t))\log(e+\Phi(t)+\Theta(t))}\\ &\leq M(t)+\frac{M(t)}{Z(t)\log Z(t)}+c\Phi(t).\end{aligned}$$

Thus we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\log Z(t)=\frac{Z'(t)}{Z(t)\log Z(t)}\leq\Phi(t)+\frac{M(t)}{Z(t)\log Z(t)},\tag{1.103}$$

where we recall that $\|\Phi\|_{L^1(0,T)}+\|M(t)\|_{L^1(0,T)}\leq c$ in view of the a-priori estimates (1.82) and (1.84). Moreover, working in a similar way as in [47] in order to estimate the initial condition, we have $Z(0)<\infty$, hence we can integrate (1.103) over $(0,T)$ to obtain

$$\|Z\|_{L^\infty(0,T)}\leq c.\tag{1.104}$$

Consequences

From (1.92), (1.104) reads

$$\|\varphi_t\|_{L^\infty(0,T;V')}\leq c,\tag{1.105}$$

$$\|\theta\|_{L^\infty(0,T;L^{q+2}(\Omega))}\leq c.\tag{1.106}$$

Combining (1.1) with (1.81), we get

$$\|\mu\|_{L^\infty(0,T;V)} \leq c. \tag{1.107}$$

According to the above relations and using (1.59), (1.73), after integrating (1.100) over $(0,T)$, we infer

$$\|\varphi_t\|_{L^2(0,T;V)} \leq c, \tag{1.108}$$
$$\|K(\theta)\|_{L^2(0,T;V)} \leq c. \tag{1.109}$$

Now we read (1.1) as a time-dependent family of elliptic problems. Combining standard regularity results with (1.108), we have

$$\|\mu\|_{L^2(0,T;H^3(\Omega))} \leq c. \tag{1.110}$$

We conclude by providing some estimates for the terms μ_t and φ_t. We (formally) differentiate (1.2) with respect to time and use (1.3), therefore we infer

$$\mu_t = -\Delta\varphi_t + F''(\varphi)\varphi_t + \theta\varphi_t - \Delta K(\theta) - |\nabla\mu|^2. \tag{1.111}$$

We now test the above relation by nonzero $v \in V$. Recalling the boundary conditions, we obtain

$$\begin{aligned}\langle \mu_t, v\rangle &= \int_\Omega \nabla(\varphi_t + K(\theta)) \cdot \nabla v + \left\langle F''(\varphi)\varphi_t + \theta\varphi_t - |\nabla\mu|^2, v\right\rangle \\ &\leq \|v\|_V \Big(\big\|\nabla(\varphi_t + K(\theta))\big\| + \big\|F''(\varphi)\varphi_t + \theta\varphi_t - |\nabla\mu|^2\big\|_{L^{3/2}(\Omega)}\Big).\end{aligned} \tag{1.112}$$

Then, dividing by $\|v\|_V$, passing to the supremum with respect to $v \in V \setminus \{0\}$, squaring, and integrating in time, we get

$$\|\mu\|_{H^1(0,T;V')} \leq c. \tag{1.113}$$

Indeed, according to (1.108)-(1.109), it holds

$$\big\|\nabla(\varphi_t + K(\theta))\big\|_{L^2(0,T;H)} \leq c. \tag{1.114}$$

and moreover it holds

$$\big\|F''(\varphi)\varphi_t + \theta\varphi_t - |\nabla\mu|^2\big\|_{L^2(0,T;L^{3/2}(\Omega))} \leq c \tag{1.115}$$

where the exponent $3/2$ is chosen just for simplicity (any number strictly greater than 1 would be allowed, indeed).

This last inequality can be proved considering each term in the norm separately. We know that $F''(\varphi)$ grows as a power of φ, whose regularity is given by (1.70) and (1.86). Hence, integrating in time and exploiting (1.70) and (1.108) we infer

$$\|F''(\varphi)\varphi_t\|_{L^2(0,T;L^{3/2}(\Omega))} \leq c.$$

On the other hand, integrating in time, from (1.106) and (1.108), we obtain

$$\|\theta\varphi_t\|_{L^2(0,T;L^{3/2}(\Omega))} \leq c.$$

For the last term, integrating once more in time and taking advantage of (1.110) we get

$$\left\| |\nabla\mu|^2 \right\|_{L^2(0,T;L^{3/2}(\Omega))} \leq c.$$

Combining the previous bounds we eventually gain (1.115).
Eventually, testing (1.3) by a nonzero $v \in V$ and proceeding similarly as above, it follows

$$\|\theta\|_{H^1(0,T;V')} \leq c.$$

1.4.3 Further regularity

Thanks to the estimates obtained in the previous section, we are now able to prove the regularity presented in Theorem 1.4.1.
First of all we focus our attention on the estimate

$$\theta \in L^\infty(0,T;L^\infty(\Omega)) \tag{1.116}$$

obtained by a Moser's iteration technique, as in [87].

We start multiplying (1.3) by θ^p, where p is a positive exponent which will be specified later on, and then we integrate over Ω. Therefore, we have, in view of (1.58),

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega \theta^{p+1} + \frac{4p}{(p+1)^2}\int_\Omega \left|\nabla\theta^{\frac{p+1}{2}}\right|^2 + \frac{4p}{(p+q+1)^2}\int_\Omega \left|\nabla\theta^{\frac{p+q+1}{2}}\right|^2 \\ \leq \int_\Omega |\Delta\mu|\theta^{p+1} + \int_\Omega |\nabla\mu|^2\theta^p.\end{aligned} \tag{1.117}$$

This entails, using (A.2),

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \theta^{p+1} + \frac{4p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|_V^2 \\ &\leq \frac{4p}{(p+1)}\|\theta\|_{L^1(\Omega)}^{p+1} + (p+1)\int_\Omega |\Delta\mu|\theta^{p+1} + (p+1)\int_\Omega |\nabla\mu|^2\theta^p\end{aligned}$$

$$\leq (p+1)\int_\Omega (|\Delta\mu|+1)\theta^{p+1} + (p+1)\int_\Omega |\nabla\mu|^2\theta^p =: I_1 + I_2, \tag{1.118}$$

where we observed that $\frac{4p}{p+1} \leq p+1$ and where c_p denotes the Poincaré constant in (A.2). Now,

$$\begin{aligned}
I_1 :=& (p+1)\int_\Omega (|\Delta\mu|+1)\theta^{p+1} = (p+1)\int_\Omega \theta^{\frac{(p+1)}{2}}(|\Delta\mu|+1)\theta^{\frac{(p+1)}{2}}\\
\leq& (p+1)\left\|\theta^{\frac{p+1}{2}}\right\|_V \left\|\theta^{\frac{(p+1)}{2}}(|\Delta\mu|+1)\right\|_{V'}\\
\leq& \frac{2p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|_V^2 + C(p+1)^2\left\|\theta^{\frac{p+1}{2}}(|\Delta\mu|+1)\right\|_{L^{6/5}(\Omega)}^2
\end{aligned}$$

where the positive constant C is allowed to vary from line to line.

At this point we use Hölder's inequality with exponents 5 and 5/4, therefore we get

$$\begin{aligned}
\left\|\theta^{\frac{p+1}{2}}(|\Delta\mu|+1)\right\|_{L^{6/5}(\Omega)}^2 =& \left(\int_\Omega \theta^{\frac{3}{5}(p+1)}(|\Delta\mu|+1)^{\frac{6}{5}}\right)^{\frac{5}{3}}\\
\leq& \||\Delta\mu|+1\|_{L^6(\Omega)}^2 \left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}\\
\leq& c(1+\|\mu\|_{H^3(\Omega)}^2)\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}.
\end{aligned}$$

Eventually we deduce

$$I_1 \leq \frac{2p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|_V^2 + C(p+1)^2(1+\|\mu\|_{H^3(\Omega)}^2)\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}, \tag{1.119}$$

On the other hand, observing that $\theta^p \leq \theta^{p+1}+1$ and recalling Sobolev's embedding theorem, thanks to (1.107) we are led to

$$\begin{aligned}
I_2 :=& (p+1)\int_\Omega |\nabla\mu|^2\theta^p \leq (p+1)\int_\Omega |\nabla\mu|^2\theta^{p+1} + (p+1)\|\nabla\mu\|_{L^2(\Omega)}^2\\
\leq& c(p+1)\left\|\theta^{\frac{p+1}{2}}\right\|_V \left\||\nabla\mu|^2\theta^{\frac{p+1}{2}}\right\|_{V'} + C(p+1)\\
\leq& \frac{2p}{c_p(p+1)}\left\|\theta^{\frac{p+1}{2}}\right\|_V^2 + C(p+1)^2\left\||\nabla\mu|^2\theta^{\frac{p+1}{2}}\right\|_{L^{5/4}(\Omega)}^2 + C(p+1).
\end{aligned}$$

Now, applying Hölder's inequality with exponents 6/5, 6 and interpolation inequality (A.1e) we obtain

$$\left\||\nabla\mu|^2\theta^{\frac{p+1}{2}}\right\|_{L^{5/4}(\Omega)}^2 = \left(\int_\Omega \theta^{\frac{5}{8}(p+1)}|\nabla\mu|^{\frac{5}{2}}\right)^{\frac{8}{5}} \leq \|\nabla\mu\|_{L^{15}(\Omega)}^4\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}\right)^{\frac{4}{3}}$$

$$\leq C \|\mu\|^2_{H^3(\Omega)} \left(\int_\Omega \theta^{\frac{3}{4}(p+1)} \right)^{\frac{4}{3}}.$$

Therefore, we have the following inequality for I_2:

$$I_2 \leq \frac{2p}{c_p(p+1)} \left\| \theta^{\frac{p+1}{2}} \right\|^2_V + C(p+1)^2 \|\mu\|^2_{H^3(\Omega)} \left(\int_\Omega \theta^{\frac{3}{4}(p+1)} \right)^{\frac{4}{3}} + C(p+1). \tag{1.120}$$

Using (1.118), (1.119) and (1.120) yields

$$\frac{d}{dt} \int_\Omega \theta^{p+1} \leq C(p+1)^2 \left(1 + \|\mu\|^2_{H^3(\Omega)} \right) \left(\int_\Omega \theta^{\frac{3}{4}(p+1)} \right)^{\frac{4}{3}} + C(p+1),$$

then, by a further integration over $(0,t), t \in (0,T]$,

$$\int_\Omega \theta^{p+1}(t) \leq C\,(p+1)^2 \left(\sup_{[0,T]} \left(\int_\Omega \theta^{\frac{3}{4}(p+1)} \right)^{\frac{4}{3}} + K_{p,T} \right), \tag{1.121}$$

where we took advantage of (1.110) and where with $K_{p,T}$ we denote a term cointaining the information on the initial datum $\theta^{p+1}(0)$ which possibly depends on T. This term depends exponentially on p, but this difficulty is later overcome taking the $(1/p)$-power. In order to apply Moser's iteration, we consider the sequence $(p_k)_k$ of real numbers defined by

$$p_0 = 3, \quad p_{k+1} = \frac{4}{3} p_k, \quad k \in \mathbb{N}.$$

Let us take $p = p_{k+1} - 1$ in (1.121). We then have

$$\int_\Omega \theta^{p_{k+1}}(t) \leq C\, p_{k+1}^2 \left(\sup_{[0,T]} \left(\int_\Omega \theta^{p_k} \right)^{\frac{4}{3}} + K_{p,T} \right),$$

hence

$$\sup_{[0,T]} \int_\Omega \theta^{p_{k+1}} \leq C\, p_{k+1}^2 \max \left\{ \sup_{[0,T]} \left(\int_\Omega \theta^{p_k} \right)^{\frac{4}{3}}, K_{p,T} \right\}.$$

Thanks to (1.106), we already have $\theta \in L^\infty(0,T;L^{q+2}(\Omega))$, where q was introduced in (1.58). Therefore, we can apply the Moser lemma and get

$$\forall k \in \mathbb{N}, \qquad \sup_{[0,T]} \|\theta\|_{L^{p_k}(\Omega)} \leq C.$$

Taking the limit as k goes to infinity leads to (1.116). This also immediately entails that $K(\theta) \in L^\infty(0,T;L^\infty(\Omega))$.

We are now able to prove that $\theta \in L^\infty(0,T;V)$. In order to do so, we formally multiply

(1.3) by $\partial_t K(\theta) = \kappa(\theta)\theta_t$. We obtain

$$\begin{aligned}&\int_\Omega \left|\sqrt{\kappa(\theta)}\theta_t\right|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla K(\theta)|^2 \\ =& -\int_\Omega \theta\Delta\mu\kappa(\theta)\theta_t + \int_\Omega |\nabla\mu|^2\kappa(\theta)\theta_t = \int_\Omega \kappa(\theta)\theta_t[|\nabla\mu|^2 - \theta\,\Delta\mu]\,\mathrm{dx}\end{aligned}$$

Owing to (1.116), (1.107) and (A.1a), the right hand side can be controlled as

$$\begin{aligned}&\int_\Omega \kappa(\theta)\theta_t[|\nabla\mu|^2 - \theta\,\Delta\mu]\,\mathrm{dx} \\ =&\int_\Omega \sqrt{\kappa(\theta)}\sqrt{\kappa(\theta)}\theta_t\left(\theta\Delta\mu + |\nabla\mu|^2\right) \\ \le&\|\sqrt{\kappa(\theta)}\|_{L^\infty(\Omega)}\|\sqrt{\kappa(\theta)}\theta_t\|\left(\|\theta\|_{L^\infty(\Omega)}\|\Delta\mu\| + \|\nabla\mu\|^2_{L^4(\Omega)}\right) \\ \le& c\|\sqrt{\kappa(\theta)}\theta_t\|\|\mu\|_{H^2(\Omega)} \le \frac{1}{2}\|\sqrt{\kappa(\theta)}\theta_t\|^2 + c\|\mu\|^2_{H^2(\Omega)},\end{aligned}$$

where in the last row we controlled $\|\nabla\mu\|^2_{L^4(\Omega)}$ by (A.1a) and (1.107). Then, on account of (1.110), we get

$$\theta \in H^1(0,T;L^2(\Omega)) \qquad K(\theta) \in L^\infty(0,T;V)$$

and this last estimate entails the desired result $\theta \in L^\infty(0,T;V)$.
Finally, by reading (1.2) as

$$-\Delta\varphi + F'(\varphi) = \theta + \mu \in L^\infty(0,T;V),$$

we deduce the thesis using (1.65), that is $\varphi \in L^\infty(0,T;H^3(\Omega))$.

1.4.4 Uniqueness

We now address the uniqueness of solution in Theorem 1.4.1. Let $z_0 \in \mathcal{V}$ and let $(\varphi_i, \mu_i, \theta_i), i = 1,2$, be a couple of (stable) solutions both emanating from z_0 over the interval $(0,T)$. Taking $(\varphi, \mu, \theta) := (\varphi_1 - \varphi_2, \mu_1 - \mu_2, \theta_1 - \theta_2)$, we can readily obtain

$$\varphi_t = \Delta\mu, \tag{1.122}$$
$$\mu = -\Delta\varphi + F'(\varphi_1) - F'(\varphi_2) - \theta, \tag{1.123}$$
$$\theta_t + \theta_1\Delta\mu + \theta\Delta\mu_2 - \Delta[K(\theta_1) - K(\theta_2)] = (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu \tag{1.124}$$

coupled with null initial data. This guarantees e.g. $\varphi_\Omega(t) = 0 \quad \forall t \ge 0$. By the regularity (1.65)-(1.67), we observe

$$\|\varphi_i(t)\|_{H^3(\Omega)} + \|\mu_i(t)\|_V + \|\theta_i(t)\|_V \le c, \quad t \in (0,T)$$

$$\|\mu_i\|_{L^2(0,T;H^3(\Omega))} + \|\theta_i\|_{L^\infty(0,T;L^\infty(\Omega))} \le c \tag{1.125}$$

for some positive constant c depending on T and on the initial data. These properties will be frequently used in the following.

The proof we show is based on the application of Gronwall's Lemma to the functional

$$\mathcal{Z}(t) := \varepsilon \|\nabla\varphi(t)\|^2 - 2\varepsilon \langle \theta(t) - \theta_\Omega(t), \varphi(t) \rangle + \|\theta(t) - \theta_\Omega(t)\|_{V'}^2 + \theta_\Omega(t)^2, \tag{1.126}$$

which is zero for $t = 0$. We notice that for $\varepsilon > 0$ small enough,

$$\mathcal{Z}(t) \ge c_\varepsilon(\|\nabla\varphi(t)\|^2 + \|\theta(t) - \theta_\Omega(t)\|_{V'}^2 + \theta_\Omega(t)^2). \tag{1.127}$$

Therefore we need to estimate the terms $\frac{d}{dt}\|\nabla\varphi(t)\|^2$, $\frac{d}{dt}\|\theta(t) - \theta_\Omega(t)\|_{V'}^2$ and $\frac{d}{dt}\theta_\Omega(t)^2$. The first one will be addressed in Section 1.4.4, the second in Section 1.4.4 and the third in Section 1.4.4.

Preliminary estimates

First of all we control $\Delta\varphi$ by testing (1.123) by $-\Delta\varphi$:

$$\begin{aligned} \|\Delta\varphi\|^2 =& - \langle \mu - \mu_\Omega, \Delta\varphi \rangle - \langle \theta - \theta_\Omega, \Delta\varphi \rangle + \langle F'(\varphi_1) - F'(\varphi_2), \Delta\varphi \rangle \\ \le& c \|\Delta\varphi\| (\|\nabla\mu\| + \|\theta - \theta_\Omega\| + \|F'(\varphi_1) - F'(\varphi_2)\|) \end{aligned}$$

Exploiting Hölder's inequality with exponents 3 and 3/2 and recalling (1.57), (1.125), we obtain

$$\begin{aligned} \|F'(\varphi_1) - F'(\varphi_2)\|^2 \le& c \left\langle (1 + |\varphi_1|^{2p_F} + |\varphi_2|^{2p_F}), \varphi^2 \right\rangle \\ \le& c \|\varphi\|_{L^3(\Omega)}^2 \le c \|\varphi\|_V^2 . \end{aligned}$$

Eventually, according to (1.85), we conclude that

$$\|\Delta\varphi\|^2 \le c(\|\nabla\mu\|^2 + \|\theta - \theta_\Omega\|^2 + \|\varphi\|_V^2) \le c(\|\varphi_t\|_{V'}^2 + \|\theta - \theta_\Omega\|^2 + \|\varphi\|_V^2). \tag{1.128}$$

Difference in order parameters for Gronwall's argument (1.126)

Testing (1.122) by $J^{-1}\varphi_t$, we get

$$\|\varphi_t\|_{V'}^2 + \langle \mu, \varphi_t \rangle = 0,$$

where, multiplying (1.123) by φ_t, we can write the second term as

$$\langle \mu, \varphi_t \rangle = \frac{1}{2}\frac{d}{dt}(\|\nabla\varphi\|^2 - 2\langle \theta - \theta_\Omega, \varphi\rangle) + \langle F'(\varphi_1) - F'(\varphi_2), \varphi_t \rangle + \langle \theta_t, \varphi \rangle .$$

Combining the previous relations we infer

$$\begin{aligned}&\|\varphi_t\|_{V'}^2 + \frac{1}{2}\frac{d}{dt}(\|\nabla\varphi\|^2 - 2\langle \theta - \theta_\Omega, \varphi\rangle)\\ &= -\langle F'(\varphi_1) - F'(\varphi_2), \varphi_t \rangle - \langle \theta_t, \varphi \rangle .\end{aligned} \tag{1.129}$$

We take care of second term on the right hand side by multiplying (1.124) by φ:

$$\begin{aligned}\langle \theta_t, \varphi\rangle =& \langle K(\theta_1) - K(\theta_2), \Delta\varphi\rangle + \langle \nabla\mu, \nabla\theta_1\varphi\rangle + \langle \nabla\mu, \theta_1\nabla\varphi\rangle\\ &- \langle \theta\Delta\mu_2, \varphi\rangle + \langle (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \varphi\rangle .\end{aligned}$$

We notice that a direct estimate of the first term in the right hand side could have been provided due to (1.128) and the regularity achieved on $K(\theta)$. However in this way, it turns to be difficult to reabsorb the term $C\|\varphi_t\|_{V'}$ in the left hand side. Therefore the gain of regularity in θ apparently does not simplify this part of the proof and we need to proceed as in [39].

We notice that, exploiting (1.123) two times, the first term on the right hand side reads

$$\begin{aligned}&\langle K(\theta_1) - K(\theta_2), \Delta\varphi\rangle\\ &= \langle K(\theta_1) - K(\theta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - \theta + \theta_\Omega\rangle\\ &\quad - (\mu_\Omega + \theta_\Omega)\int_\Omega [K(\theta_1) - K(\theta_2)]\\ &= \langle K(\theta_1) - K(\theta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega\rangle\\ &\quad - \langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle .\end{aligned}$$

Therefore we have,

$$\begin{aligned}&\langle \theta_t, \varphi\rangle\\ &= \langle K(\theta_1) - K(\theta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega\rangle\\ &\quad - \langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle + \langle \nabla\mu, \nabla\theta_1\varphi\rangle + \langle \nabla\mu, \theta_1\nabla\varphi\rangle - \langle \theta\Delta\mu_2, \varphi\rangle\\ &\quad + \langle (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \varphi\rangle .\end{aligned}$$

Owing (1.129) and combining the above relations, we obtain

$$\|\varphi_t\|_{V'}^2 + \frac{1}{2}\frac{d}{dt}\Big(\|\nabla\varphi\|^2 - 2\langle \theta - \theta_\Omega, \varphi\rangle \Big) - \langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle$$

$$
\begin{aligned}
=& -\langle F'(\varphi_1) - F'(\varphi_2), \varphi_t\rangle \\
&- \langle K(\theta_1) - K(\theta_2), -\mu + \mu_\Omega + F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega\rangle \\
&- \langle \nabla\mu, \nabla\theta_1\varphi + \theta_1\nabla\varphi\rangle + \langle \theta\Delta\mu_2, \varphi\rangle \\
&- \langle (\nabla\mu_1 + \nabla\mu_2)\cdot\nabla\mu, \varphi\rangle =: I_3 + I_4 + I_5 + I_6 + I_7.
\end{aligned} \tag{1.130}
$$

First of all using (1.57), we have

$$
\begin{aligned}
&\|\nabla(F'(\varphi_1) - F'(\varphi_2))\| \\
&\qquad \le c \left\| |\nabla\varphi|(1 + |\varphi_1^{p_F}|) + |\varphi|(1 + |\varphi_1|^{p_F-1} + |\varphi_2|^{p_F-1})|\nabla\varphi_2| \right\| \le c\,\|\varphi\|_V .
\end{aligned} \tag{1.131}
$$

Therefore, as $\langle \varphi_t, 1\rangle = 0$, we deduce

$$
\begin{aligned}
I_3 :=& -\langle F'(\varphi_1) - F'(\varphi_2), \varphi_t\rangle \\
\le& C\,\|\nabla(F'(\varphi_1) - F'(\varphi_2))\|\,\|\varphi_t\|_{V'} \le c\,\|\varphi\|_V\,\|\varphi_t\|_{V'} .
\end{aligned}
$$

On the other hand, by (1.116),

$$
\|K(\theta_1) - K(\theta_2)\|_{3/2} \le c(\|\theta - \theta_\Omega\| + |\theta_\Omega|),
$$

and, according to (1.131), it follows

$$
\begin{aligned}
I_4 :=& -\langle K(\theta_1) - K(\theta_2), \mu - \mu_\Omega - F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega\rangle \\
\le& \|K(\theta_1) - K(\theta_2)\|_{L^{3/2}(\Omega)} (\|\mu - \mu_\Omega\|_{L^3(\Omega)} \\
&+ \|F'(\varphi_1) - F'(\varphi_2) - (F'(\varphi_1) - F'(\varphi_2))_\Omega\|_{L^3(\Omega)}) \\
\le& c(\|\theta - \theta_\Omega\| + |\theta_\Omega|)(\|\nabla\mu\| + \|\varphi\|_V).
\end{aligned}
$$

Owing to (A.1a), (A.1b) and (A.1f)

$$
\begin{aligned}
I_5 :=& -\langle \nabla\mu, \nabla\theta_1\varphi + \theta_1\nabla\varphi\rangle \\
\le& c\,\|\nabla\mu\|\,(\|\nabla\theta_1\|\,\|\varphi\|_{L^\infty(\Omega)} + \|\nabla\varphi\|_{L^4(\Omega)}\,\|\theta_1\|_{L^4(\Omega)}) \\
\le& c\,\|\nabla\mu\|\left(\|\varphi\|^{1/2}\,\|\varphi\|^{1/2}_{H^2(\Omega)} + \|\nabla\varphi\|^{1/2}\,\|\varphi\|^{1/2}_{H^2(\Omega)}\right) \\
\le& c\,\|\nabla\mu\|\,\|\nabla\varphi\|^{1/2}\,\|\varphi\|^{1/2}_{H^2(\Omega)} .
\end{aligned}
$$

Now, using (A.1f) and the injection $V \subset L^p(\Omega)$, for $p \ge 1$,

$$
\begin{aligned}
I_6 :=& \langle \theta\Delta\mu_2, \varphi\rangle = \theta_\Omega\,\langle \Delta\mu_2, \varphi\rangle + \langle (\theta - \theta_\Omega)\Delta\mu_2, \varphi\rangle \\
\le& c|\theta_\Omega|\,\|\nabla\mu_2\|\,\|\nabla\varphi\| + c\,\|\theta - \theta_\Omega\|\,\|\Delta\mu_2\|_{L^4(\Omega)}\,\|\varphi\|_{L^4(\Omega)}
\end{aligned}
$$

$$\leq c|\theta_\Omega| \, \|\nabla\varphi\| + c\,\|\theta-\theta_\Omega\| \, \|\mu_2\|_{H^3(\Omega)} \|\nabla\varphi\| \, .$$

Finally combining the previous strategy with (A.1a), we get

$$\begin{aligned} I_7 :=& -\langle(\nabla\mu_1+\nabla\mu_2)\cdot\nabla\mu,\varphi\rangle \leq \|\nabla\mu_1+\nabla\mu_2\|_{L^4(\Omega)} \|\nabla\mu\| \, \|\varphi\|_{L^4(\Omega)} \\ \leq& \big(\|\mu_1\|^{1/2}_{H^2(\Omega)} + \|\mu_2\|^{1/2}_{H^2(\Omega)} \big) \|\nabla\mu\| \, \|\nabla\varphi\| \, . \end{aligned}$$

Eventually, the above computations and Young's inequality yield

$$\begin{aligned} & I_3+I_4+I_5+I_6+I_7 \\ \leq& c\,\|\varphi_t\|_{V'} \|\varphi\|_V + c(\|\theta-\theta_\Omega\|+|\theta_\Omega|)(1+\|\mu_2\|_{H^3(\Omega)})\,\|\varphi\|_V \\ &+ c\,\|\nabla\mu\|\,\|\nabla\varphi\|^{1/2}\,\|\varphi\|^{1/2}_{H^2(\Omega)} + c(\|\theta-\theta_\Omega\|+|\theta_\Omega|)\,\|\nabla\mu\| \\ &+ c\left(1+\|\mu_1\|^{1/2}_{H^2(\Omega)}+\|\mu_2\|^{1/2}_{H^2(\Omega)}\right)\|\nabla\mu\|\,\|\nabla\varphi\| \\ \leq& \frac{1}{2}\|\varphi_t\|^2_{V'} + \frac{\alpha}{2}\|\nabla\mu\|^2 + \frac{c}{\alpha}\|\theta-\theta_\Omega\|^2 + \frac{\alpha}{2}\|\Delta\varphi\|^2 + g(t)(\|\varphi\|^2_V+\theta^2_\Omega) \\ \leq& \left(\frac{1}{2}+c\alpha\right)\|\varphi_t\|^2_{V'} + c\left(\frac{1}{\alpha}+\alpha\right)\|\theta-\theta_\Omega\|^2 + g(t)(\|\varphi\|^2_V+\theta^2_\Omega). \end{aligned}$$

Where in the last passage we took advantage of (1.128) to estimate the term depending on $\Delta\varphi$. Moreover, we have defined

$$g(t) := c[1+\|\mu_1\|^2_{H^3(\Omega)}+\|\mu_2\|^2_{H^3(\Omega)}], \tag{1.132}$$

with (large) constant $c>0$ also depending on the choice of the small constant $\alpha>0$ Combining the previous estimates with (1.130), we finally get

$$\begin{aligned} &\|\varphi_t\|^2_{V'} + \frac{d}{dt}(\|\nabla\varphi\|^2 - 2\langle\theta-\theta_\Omega,\varphi\rangle) - 2\langle K(\theta_1)-K(\theta_2),\theta-\theta_\Omega\rangle \\ &\leq \left(\frac{1}{2}+c\alpha\right)\|\varphi_t\|_{V'} + c\left(\frac{1}{\alpha}+\alpha\right)\|\theta-\theta_\Omega\|^2 + g(t)(\|\varphi\|^2_V+\theta^2_\Omega). \end{aligned} \tag{1.133}$$

As mentioned before, the aim of these calculations is to apply Gronwall's Lemma to a specific functional already introduced. In order to do that, we are trying to obtain the derivative of such functional on the left-hand side and the functional itself on the right. Thus all terms arising from that must be either integrable over $(0,T)$ (which is the $g(t)$) or they must be balanced with some term on the left side as $\|\varphi_t\|^2_{V'}$ or $\|\theta-\theta^2_\Omega\|$ (which will arise from $\langle K(\theta_1)-K(\theta_2),\theta-\theta_\Omega\rangle$).

Difference of temperatures for Gronwall's argument (1.126)**.**

We test (1.124) by $J^{-1}(\theta-\theta_\Omega)$ and integrate by parts, therefore we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta-\theta_\Omega\|_{V'}^2+\langle K(\theta_1)-K(\theta_2),\theta-\theta_\Omega\rangle\\
=&\big\langle\nabla\theta_1\cdot\nabla\mu, J^{-1}(\theta-\theta_\Omega)\big\rangle+\big\langle\theta_1\nabla\mu,\nabla J^{-1}(\theta-\theta_\Omega)\big\rangle\\
&-\big\langle(\theta-\theta_\Omega)\Delta\mu_2, J^{-1}(\theta-\theta_\Omega)\big\rangle+\theta_\Omega\langle\mu_2-(\mu_2)_\Omega,\theta-\theta_\Omega\rangle\\
&+\big\langle(\nabla\mu_1+\nabla\mu_2)\cdot\nabla\mu, J^{-1}(\theta-\theta_\Omega)\big\rangle=:I_8+I_9+I_{10}.
\end{aligned}$$

First of all we have

$$\begin{aligned}
I_8:=&\big\langle\theta_1\nabla\mu,\nabla J^{-1}(\theta-\theta_\Omega)\big\rangle\\
\le& c\|\nabla\mu\|\big\|\nabla J^{-1}(\theta-\theta_\Omega)\big\|\le c\|\nabla\mu\|\|\theta-\theta_\Omega\|_{V'}.
\end{aligned}$$

From (A.1b),

$$\big\|J^{-1}(\theta-\theta_\Omega)\big\|_{L^\infty(\Omega)}\le c\|\theta-\theta_\Omega\|_{V'}^{1/2}\|\theta-\theta_\Omega\|^{1/2}.$$

Thus, we infer

$$\begin{aligned}
I_9:=&-\big\langle\nabla\theta_1\cdot\nabla\mu-(\theta-\theta_\Omega)\Delta\mu_2, J^{-1}(\theta-\theta_\Omega)\big\rangle\\
&+\big\langle(\nabla\mu_1+\nabla\mu_2)\cdot\nabla\mu, J^{-1}(\theta-\theta_\Omega)\big\rangle\\
\le&\big\|J^{-1}(\theta-\theta_\Omega)\big\|_{L^\infty}[\|\theta_1\|_V\|\nabla\mu\|\\
&+\|\Delta\mu_2\|\|\theta-\theta_\Omega\|+(\|\nabla\mu_1\|+\|\nabla\mu_2\|)\|\nabla\mu\|]\\
\le& c\|\theta-\theta_\Omega\|_{V'}^{1/2}\|\theta-\theta_\Omega\|^{1/2}\|\nabla\mu\|+c\|\mu_2\|_{H^2(\Omega)}\|\theta-\theta_\Omega\|_{V'}^{1/2}\|\theta-\theta_\Omega\|^{3/2}.
\end{aligned}$$

Eventually,

$$\begin{aligned}
I_{10}:=&\theta_\Omega\langle\mu_2-(\mu_2)_\Omega,\theta-\theta_\Omega\rangle\\
\le& c|\theta_\Omega|\|\mu_2-(\mu_2)_\Omega\|_V\|\theta-\theta_\Omega\|_{V'}\le c|\theta_\Omega|\|\theta-\theta_\Omega\|_{V'}.
\end{aligned}$$

Combining the estimates of I_8, I_9, I_{10} and exploiting Young's inequality, we finally have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta-\theta_\Omega\|_{V'}^2+\langle K(\theta_1)-K(\theta_2),\theta-\theta_\Omega\rangle\le\delta\varepsilon\|\theta-\theta_\Omega\|^2\\
&+\alpha\varepsilon\|\nabla\mu\|^2+C_*(1+\|\mu_2\|_{H^2(\Omega)}^2)\|\theta-\theta_\Omega\|_{V'}^2+c|\theta_\Omega|^2,
\end{aligned}\tag{1.134}$$

with the (large) constant C_* depending on the small constants $\alpha,\delta,\varepsilon$ which will be specified at the end.

Difference of temperatures' means for Gronwall's argument (1.126)

Integrating (1.124) over Ω we obtain

$$|\Omega|(\theta_\Omega)_t = \langle \nabla\theta_1, \nabla\mu\rangle - \langle \theta - \theta_\Omega, \Delta\mu_2\rangle + \langle (\nabla\mu_1 + \nabla\mu_2), \nabla\mu\rangle\,, \tag{1.135}$$

and by (1.125) it yields

$$|(\theta_\Omega)_t| \leq c(\|\nabla\mu\| + \|\mu_2\|_{H^3(\Omega)}\|\theta - \theta_\Omega\|_{V'}).$$

Moreover, multiplying (1.135) by θ_Ω we have, for (small) $\alpha > 0$ we will choose later and corresponding (large) $c > 0$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\theta_\Omega^2 \leq & c|\theta_\Omega|(\|\nabla\mu\| + \|\mu_2\|_{H^3(\Omega)}\|\theta - \theta_\Omega\|_{V'}^2) & (1.136)\\ \leq & \alpha\varepsilon\|\nabla\mu\|^2 + c(\theta_\Omega^2 + \|\mu_2\|_{H^3(\Omega)}^2\|\theta - \theta_\Omega\|_{V'}^2). & (1.137)\end{aligned}$$

Conclusion

We recall the definition of the functional we want to use

$$\mathcal{Z}(t) := \varepsilon\|\nabla\varphi\|^2 - 2\varepsilon\langle\theta - \theta_\Omega, \varphi\rangle + \|\theta - \theta_\Omega\|_{V'}^2 + \theta_\Omega^2,$$

Summing (1.134), (1.136) and $\frac{\varepsilon}{2}$ times (1.133), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\mathcal{Z} + (1-\varepsilon)\langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle + \frac{\varepsilon}{2}\|\varphi_t\|_{V'}^2\\ \leq & \varepsilon(c + c\alpha^2 + \delta)\|\theta - \theta_\Omega\|^2 + 2\alpha\varepsilon\|\nabla\mu\|^2 + g(t)\mathcal{Z},\end{aligned}$$

where g was introduced in (1.132).

Next, we take care of the second term in the left hand side. From (1.59), we obtain

$$\langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle = \|\theta - \theta_\Omega\|^2 + \frac{1}{q+1}\langle l(\theta_1) - l(\theta_2), \theta - \theta_\Omega\rangle$$

where $l(\theta_i) = \theta_i^{q+1}, i = 1, 2$. Now

$$\begin{aligned} l(\theta_1) - l(\theta_2) &= \int_0^1 \frac{d}{ds} l(s\theta_1 + (1-s)\theta_2)ds\\ &= \int_0^1 l'(s\theta_1 + (1-s)\theta_2)(\theta_1 - \theta_2)ds = \omega(\theta_1, \theta_2)\theta,\end{aligned}$$

where we set

$$\omega(\theta_1,\theta_2) := \int_0^1 l'(s\theta_1 + (1-s)\theta_2)ds.$$

We observe that it holds $\omega(\theta_1,\theta_2) \geq 0$ almost everywhere. Moreover, we notice that (1.116) implies,

$$|\omega(\theta_1,\theta_2)| \leq c \tag{1.138}$$

Therefore we infer

$$\begin{aligned}\langle K(\theta_1) - K(\theta_2), \theta - \theta_\Omega\rangle &= \|\theta-\theta_\Omega\|^2 + \frac{1}{1+q}\langle l(\theta_1) - l(\theta_2), \theta-\theta_\Omega\rangle \\ &= \|\theta-\theta_\Omega\|^2 + \frac{1}{q+1}\int_\Omega \omega(\theta_1,\theta_2)\theta(\theta-\theta_\Omega) \\ &= \|\theta-\theta_\Omega\|^2 + \frac{1}{q+1}\int_\Omega \omega(\theta_1,\theta_2)|\theta-\theta_\Omega|^2 \\ &\quad + \frac{1}{q+1}\int_\Omega \omega(\theta_1,\theta_2)\theta_\Omega(\theta-\theta_\Omega) \\ &\geq \|\theta-\theta_\Omega\|^2 + \frac{1}{q+1}\int_\Omega \omega(\theta_1,\theta_2)\theta_\Omega(\theta-\theta_\Omega).\end{aligned}$$

Hence, exploiting (1.138), we get

$$\left|\frac{1}{q+1}\int_\Omega \omega(\theta_1,\theta_2)\theta_\Omega(\theta-\theta_\Omega)\right| \leq c|\theta_\Omega|\,\|\theta-\theta_\Omega\| \leq \frac{1}{2}\|\theta-\theta_\Omega\|^2 + c\theta_\Omega^2,$$

and moreover

$$\langle K(\theta_1) - K(\theta_2), \theta-\theta_\Omega\rangle \geq \frac{1}{2}\|\theta-\theta_\Omega\|^2 - c\theta_\Omega^2.$$

Putting everything together we finally have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\mathcal{Z} + \frac{1-\varepsilon}{2}\|\theta-\theta_\Omega\|^2 + \frac{\varepsilon}{2}\|\varphi_t\|_{V'}^2 \\ \leq &\varepsilon(c + c\alpha^2 + \delta)\|\theta-\theta_\Omega\|^2 + 3\alpha\varepsilon\|\nabla\mu\|^2 + g(t)\mathcal{Z} \\ &\overset{(1.85)}{\leq} \varepsilon(c + c\alpha^2 + \delta)\|\theta-\theta_\Omega\|^2 + 3\alpha\varepsilon c\|\varphi_t\|_{V'}^2 + g(t)\mathcal{Z}.\end{aligned} \tag{1.139}$$

The constant c on the right hand side of (1.139) only depends on the regularity properties of the solutions collected in (1.125). In particular, it is *independent* of the parameters $\alpha, \delta, \varepsilon$. Therefore, chosing $\alpha > 0$ small enough, the second term on the right hand side of (1.139) can be absorbed the corresponding quantities in the left hand side. Moreover, taking ε sufficiently small (it might depend on other parameters), the first term on the right hand side can be absorbed, too. As a matter of fact, we are able to semplify (1.139) as follows

$$\frac{d}{dt}\mathcal{Z} + \kappa_0(\|\theta - \theta_\Omega\|^2 + \|\varphi_t\|_{V'}^2) \leq g(t)\mathcal{Z},$$

where $\kappa_0 > 0$ and g was defined in (1.132), hence, exploiting (1.125), it is summable over the interval $(0, T)$. Since $\mathcal{Z}(0) = 0$, then by Gronwall's Lemma and (1.127) we eventually see that $\mathcal{Z}$ is identically 0 over $(0, T)$, which gives us the assert.

Chapter 2

Non-isothermal Cahn-Hilliard model for tumor growth

The study of tumor growth processes has become of great interest also for mathematicians in recent years [5, 9, 21, 28, 92, 104, 118]. Indeed, mathematical models might be able to give further insights in tumor growth behaviour. In particular, the framework of diffuse interface modeling with Cahn-Hilliard equations [19] has received increasing attention. In this context, the tumor is seen as an expanding mass surrounded by healthy tissues. Its evolution is assumed to be governed by mechanisms such as proliferation of cells via nutrient consumption, apoptosis [56, 78, 102] and, in more complex models like [58, 59, 60, 72], also chemotaxis and active transport of specific chemical species effects. Moreover it is possible to include the effects of fluid flow into the evolution of the tumor, which brings to the so-called Cahn-Hilliard-Darcy models (see [60, 78]). However, up to our knowledge it seems that even if the effects of variations of temperature have been studied for Cahn-Hilliard equations [39, 46, 77], they have been neglected so far in the analysis of tumor growth. From the medical point of view, the effects of temperature on tumor growth have not been completely understood yet, although they have been investigated since the very beginning of the 20th century [110]. The general tendency of the scientific community seems to support the thesis that hyperthermia can lead to partial or complete destruction of tumor cells [15, 98, 109, 115]. In fact, it has also been observed that low ambient temperature influences the production of particular nutrients for the tumor [88]. Nevertheless, we focus here on the case which does not take into account the production of a nutrient due to temperature. In this work we introduce a new diffuse interface model for tumor growth, taking into account proliferation of cells, nutrient consumption and apoptosis and moreover temperature effects. Our aim consists in proving an existence result for weak entropy solutions (cnfr. Definition 2.1) to our model. We remark that a rigorous mathematical theory of well-posedness results has been addressed in multiple works, such as [56, 58, 102]. From the biological point of view, we assume that tumor cells only die by apoptosis, therefore we do not take into account the possibility of tumor necrosis (differently e.g. from [59]). We also suppose that the healthy cells surrounding the tumor do not interact with the tumor itself, neglecting the possible response of the immune system.

According to these considerations, we will derive the following PDE system, describing the behaviour of a two-component mixture consisting of healthy cells and tumor cells

$$\varphi_t = \Delta\mu + (\mathcal{P}\sigma - \mathcal{A})h(\varphi) \tag{2.1}$$

$$\mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta - \chi_\varphi\sigma \tag{2.2}$$

$$\theta_t + \theta\varphi_t - \operatorname{div}(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2 \tag{2.3}$$

$$\sigma_t = \Delta\sigma - \mathcal{C}\sigma h(\varphi) + \mathcal{B}(\sigma_B - \sigma). \tag{2.4}$$

We carry out our analysis in $\Omega \times (0, \infty)$, where $\Omega \subset \mathbb{R}^3$ is a smooth domain. According to the derivation of the model shown in Section 2.1, we suppose that the system is isolated from the exterior. This condition translates in no-flux boundary conditions (i.e. homogeneous Neumann) for all the unknowns.
The evolution of the tumor is described by the order parameter φ which represents the local concentration of tumor cells, $\varphi \in [-1, 1]$, with $\{\varphi = 1\}$ representing the tumor phase and $\{\varphi = -1\}$ the healthy one. Moreover μ denotes the chemical potential of phase transition from healthy to tumor cells, θ is the absolute temperature, $\kappa(\theta)$ represents the heat conductivity and ε is a small parameter related to the thickness of interfacial layers. We denote by σ the concentration of a nutrient consumed (only) by the tumor cells (e.g. oxygen and glucose). The parameter $\chi_\varphi \geq 0$ is linked to transport mechanisms such as chemotaxis and active uptake. Although we will show in Section 2.1 how this parameter is included in the model, for sake of simplicity we will neglect it throughout the mathematical analysis, with the aim of including it in future works. The positive constant parameters $\mathcal{P}, \mathcal{A}, \mathcal{C}$ and $\mathcal{B}$ indicate respectively the tumor proliferation rate, apoptosis rate, nutrient consuption rate and nutrient supply rate. The function h is chosen as monotone increasing, nonnegative in $[-1, 1]$ and such that $h(-1) = 0$ and $h(1) = 1$. The tumor growth is thus described by the term $\mathcal{P}\sigma h(\varphi)$, which reasonably increases proportionally to the concentration of tumor cells, while the death of tumor cells is modelled by the term $\mathcal{A}h(\varphi)$. Therefore, according to (2.1), if $\mathcal{P}\sigma - \mathcal{A} > 0$, then the tumor expands and it happens faster when the concentration of tumor cells is already high. If otherwise $\mathcal{P}\sigma - \mathcal{A} < 0$ then the tumor reduces and the tumor cells die faster when the concentration of tumor cells is large. The term $\mathcal{C}\sigma h(\varphi)$ represents the consumption of the nutrient by

the tumor cells. The term $\mathcal{B}(\sigma - \sigma_B)$ is due to the fact that we consider here the case where the tumor has its own vasculature (as in e.g. [16], [102]), where the threshold $\sigma_B \in (0, 1)$ is the constant nutrient concentration in the pre-existing vasculature. In particular, if $\sigma_B > \sigma$, $\mathcal{B}(\sigma_B - \sigma)$ models the supply of nutrient from the blood vessels, on the other hand if $\sigma_B < \sigma$, $\mathcal{B}(\sigma_B - \sigma)$ represents the transport of nutrient away from the domain. Eventually, the function $F(s)$ represents a polynomial potential having at least cubic growth at infinity, whose assumptions will be specified in Section 2.2. A simple choice might be a *double-well potential* with equal minima at $s = \pm 1$ penalizing the deviation of the length $|\varphi|$ from its natural value 1. This more general potential allows φ to take values also outside of the significance interval $[-1, 1]$, therefore we will carry out our analysis also in the case $|\varphi| > 1$ and correspondingly extend function h. We also remark that although among Cahn-Hilliard literature the singular potentials, such as logarithm type (see e.g. [20]), are very common, the growth conditions that the problem requires make them unsuitable for our case, as it will be clear in Section 2.3.1.

In this work we derive a new phase field model according to the laws of thermodynamics describing the tumor growth, published in [75]. The novelty of [75] with respect to the pre-existing literature is to include possible variations of temperature in the tumor growth model. The presence of nutrient concentration σ in the system implies that here the spatial mean of φ is not conserved in time (as we can see from equation (2.1)), therefore the derivation of the model cannot follow the techniques proposed in Section 1.2.1. However, we are able to gain enough regularity for the quadruple $(\varphi, \mu, \theta, \sigma)$ in order to prove the existence of weak (entropy) solutions to the initial-boundary value problem associated to (2.1)–(2.4).

The structure of this Chapter is the following. In Section 2.1 we derive system (2.1)–(2.4) according to the approach proposed by Gurtin in [69]. Then we proceed with the mathematical analysis of our problem in the case $\varepsilon = 1$, $\chi_\varphi = 0$. In particular, Section 2.3 is devoted to give the setting and to present the main result of this Chapter (which is Theorem 2.3.1) concerning the existence of weak entropy solutions to our problem. The proof is carried out in two steps. In Section 2.3.1 we gain a priori bounds for $(\varphi, \mu, \theta, \sigma)$. In Section 2.3.2 we use the weak sequential stability argument to prove the existence of weak entropy solutions. Namely, we exploit the a priori bounds obtained for a sequence of weak entropy solutions together with standard compactness results to pass to the limit.

2.1 Derivation of the model

We suppose that a two-component mixture consisting of healthy cells and tumor cells occupies an open spatial domain $\Omega \subset \mathbb{R}^3$. We denote by $\varphi(x,t)$ the tumor phase concentration, $\theta(x,t)$ is the absolute temperature and $\sigma(x,t)$ is the concentration of a nutrient for the tumor cells. According to the Ginzburg-Landau theory for phase transitions, we postulate the free energy density ψ in the form

$$\psi = \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}F(\varphi) + f(\theta) - \theta\varphi + N(\varphi,\sigma). \tag{2.5}$$

We observe that (2.5) differs from (1.11), because of the presence of the term N. This latter term describes both the chemical energy of the nutrient and the energy contributions given by the interactions between the tumor tissues and the nutrient.
One of the main difficulties we have to face in the derivation of our model is that, differently from standard Cahn-Hilliard models, such as the one studied in Chapter 1, the spatial mean of the tumor phase concentration φ is not conserved. Indeed the tumor may grow or shrink according to the right hand side of (2.1). Because of this issue, we cannot derive this model according to strategy proposed in Section 1.2.1. Indeed, we follow the approach presented in Section 1.2.2, proposed by Gurtin in [69], which relies on the balance law for internal microforces (1.39). The pivotal concepts behind Gurtin's strategy are described in Section 1.2.2. However, the main difference between this case and the one presented in Chapter 1 occurs in the mass balance equation.
Indeed, we are here considering the case where the tumor grows or reduces according to a source term. Therefore relation (1.40) is not suitable to describe this phenomenon. In this case the mass balance law thus reads

$$\varphi_t = -\operatorname{div} h + m, \tag{2.6}$$

where h is the mass flux and m is the external mass supply. The derivation procedure follows the same steps as the one in Section 1.2.2 is based on the first and second laws of thermodynamics.

Cahn-Hilliard system.
According to [69], we might write the first law in the form

$$\frac{d}{dt}\int_R e\mathrm{d}x = -\int_{\partial R} q\cdot\nu\mathrm{d}\eta + \mathcal{W}(R) + \mathcal{M}(R), \tag{2.7}$$

that coincides with (1.44), but we recall here for convenience. In particular

$$\mathcal{W}(R) = \int_{\partial R} (\zeta \cdot \nu) \frac{\partial \varphi}{\partial t} \mathrm{d}\eta, \tag{2.8}$$

$$\mathcal{M}(R) = -\int_{\partial R} \mu h \cdot \nu \mathrm{d}\eta + \int_R \mu m \mathrm{d}x \tag{2.9}$$

are the rate of working and the rate at which free energy is added to R (assuming no heat supply) respectively. We remark that (2.9) presents an extra term with respect to (1.43), given by the external mass supply m. From Green's formula, we can rewrite (2.7) as

$$\frac{\partial e}{\partial t} = -\operatorname{div} q + \frac{\partial \varphi}{\partial t} \operatorname{div} \zeta + \zeta \cdot \nabla \frac{\partial \varphi}{\partial t} - h \cdot \nabla \mu - \mu \operatorname{div} h + \mu m. \tag{2.10}$$

Because the control volume R is arbitary, exploiting the mass balance (2.6) and the microforce balance (1.39), we infer

$$\frac{\partial e}{\partial t} = -\operatorname{div} q + (\mu - \pi) \frac{\partial \varphi}{\partial t} + \zeta \cdot \nabla \frac{\partial \varphi}{\partial t} - h \nabla \mu. \tag{2.11}$$

Since we here consider the free energy density ψ in (2.5) to be possibly dependent also on the nutrients, the combination of (1.14) and (1.15) leads to

$$\frac{\partial e}{\partial t} = \frac{\partial \psi}{\partial t} + \theta \frac{\partial s}{\partial t} + s \frac{\partial \theta}{\partial t} = \frac{\partial \psi}{\partial \varphi} \frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial \nabla \varphi} \frac{\partial \nabla \varphi}{\partial t} + \frac{\partial \psi}{\partial \sigma} \frac{\partial \sigma}{\partial t} + \theta \frac{\partial s}{\partial t}, \tag{2.12}$$

hence

$$\frac{\partial \psi}{\partial t} + s \frac{\partial \theta}{\partial t} = \frac{\partial \psi}{\partial \varphi} \frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial \nabla \varphi} \frac{\partial \nabla \varphi}{\partial t} + \frac{\partial \psi}{\partial \sigma} \frac{\partial \sigma}{\partial t}. \tag{2.13}$$

We now impose the validity of the second law of thermodynamics in the form of the Clausius-Duhem inequality (1.18). We develop the left hand side of (1.18) as follows

$$\begin{aligned}
\theta \left(\frac{\partial s}{\partial t} + \operatorname{div} Q \right) &\overset{(1.14)}{=} \frac{\partial e}{\partial t} - \frac{\partial \psi}{\partial t} - s \frac{\partial \theta}{\partial t} + \theta \operatorname{div} Q \\
&\overset{(1.13)}{=} \frac{\partial e}{\partial t} - \frac{\partial \psi}{\partial t} - s \frac{\partial \theta}{\partial t} + \operatorname{div} q - Q \cdot \nabla \theta \\
&\overset{(1.45)}{=} (\mu - \pi) \frac{\partial \varphi}{\partial t} + \zeta \cdot \nabla \frac{\partial \varphi}{\partial t} - h \nabla \mu - \frac{\partial \psi}{\partial t} - s \frac{\partial \theta}{\partial t} - Q \cdot \nabla \theta \\
&\overset{(2.13)}{=} \left(\mu - \pi - \frac{\partial \psi}{\partial \varphi} \right) \frac{\partial \varphi}{\partial t} + \left(\zeta - \frac{\partial \psi}{\partial \nabla \varphi} \right) \frac{\partial \nabla \varphi}{\partial t} - \frac{\partial \psi}{\partial \sigma} \frac{\partial \sigma}{\partial t} - h \nabla \mu - Q \cdot \nabla \theta.
\end{aligned}$$

In order to satisfy relation (1.18), we impose

$$\mu - \pi - \frac{\partial \psi}{\partial \varphi} = 0, \tag{2.14}$$

$$\zeta = \frac{\partial \psi}{\partial \nabla \varphi}, \tag{2.15}$$

$$\frac{\partial \psi}{\partial \sigma} = 0, \tag{2.16}$$

$$h\nabla\mu + Q \cdot \nabla\theta \leq 0, \tag{2.17}$$

where in particular in order for (2.17) to hold, we exploited Fourier's law (1.49). The combination of (2.5) and (2.15) straightly gives

$$\zeta = \varepsilon \nabla \varphi, \tag{2.18}$$

which leads to, according to (2.5), (1.39) and (2.14),

$$\mu = -\varepsilon\Delta\varphi + \frac{1}{\varepsilon}F'(\varphi) - \theta + \frac{\partial N}{\partial \varphi}. \tag{2.19}$$

Eventually, inequality (2.17) can be satisfied choosing $h = -\nabla\mu$, which is a suitable assumption according to [69]. Therefore equation (2.6) reads

$$\varphi_t = \Delta\mu + m. \tag{2.20}$$

Temperature equation.
We start from the internal energy equation (2.11), taking advantage of (2.15) and of the expression for the chemical potential (2.14), therefore

$$\frac{\partial e}{\partial t} = -\operatorname{div} q + \frac{\partial \psi}{\partial \varphi}\frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial \nabla\varphi}\frac{\partial \nabla\varphi}{\partial t} - h\nabla\mu.$$

Now, exploiting the assumption $h = -\nabla\mu$ and Fourier's law (1.49), we infer

$$\frac{\partial e}{\partial t} = \operatorname{div}(\kappa(\theta)\nabla\theta) + \frac{\partial \psi}{\partial \varphi}\frac{\partial \varphi}{\partial t} + \frac{\partial \psi}{\partial \nabla\varphi}\frac{\partial \nabla\varphi}{\partial t} + |\nabla\mu|^2$$

and by identity (2.12), taking into account (2.16),

$$\theta\frac{\partial s}{\partial t} - \operatorname{div}(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2. \tag{2.21}$$

From (1.15), we might write

$$\theta\frac{\partial s}{\partial t} = \theta(-f'(\theta))_t + \theta\varphi_t.$$

On the other hand, according to the definition of Q, it holds $(Q(\theta))_t = (-f'(\theta))_t$, with in particular $(Q(\theta))_t = Q'(\theta)\theta_t$. Since we suppose, as in Chapter 1, the specific heat to be $c_V = 1$, it follows that $Q'(\theta) = 1$. This implies that

$$\theta s_t = \theta_t + \theta\varphi_t.$$

Thus, equation (2.21) reads

$$\theta_t + \theta\varphi_t - \operatorname{div}(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2. \tag{2.22}$$

We remark that, even though the temperature equation (2.22) reads exactly like (1.54), the expressions of φ_t and μ are different.

Nutrient equation.
We postulate the nutrient balance equation in the form

$$\sigma_t = -\operatorname{div} J - \mathcal{S}, \tag{2.23}$$

where J is the nutrient flux and $\mathcal{S}$ denotes a source/sink term for the nutrient. Choosing $J = -\nabla\sigma$, equation (2.23) reads

$$\sigma_t = \Delta\sigma - \mathcal{S}. \tag{2.24}$$

2.1.1 Constitutive relations

Owing to [16, 60, 102], we now make the following constitutive assumptions.

- $m = (\mathcal{P}\sigma - \mathcal{A})h(\varphi)$,
  where $h(\varphi)$ is a monotone increasing, nonnegative function in $[-1, 1]$ and such that $h(-1) = 0$ and $h(1) = 1$. Hence this relation states that on one hand the tumor growth is proportional to the nutrient supply in the tumoral region. This assumption reflects the fact that it often happens that tumors bring mutations which switch off certain growth inhibiting proteins. Therefore the tumor cells increasing is limited only by the supply of nutrients, despite of healthy cells where the mitotic cycle regulates the growth. On the other hand, when we are in the healthy region, this equation shows that the proliferation rate of the tumor is greater than the one of healthy cells.
- $\dfrac{\partial N}{\partial\varphi} = -\chi_\varphi\sigma$, in fact, we take $\chi_\varphi = 0$.
  Indeed, this equation is due to the mechanism of chemotaxis, which we exclude in our analysis.

- $\mathcal{S} = \mathcal{C}\sigma h(\varphi) - \mathcal{B}(\sigma_B - \sigma)$.
  We here assume that the sink/source of nutrient is regulated by consumption of nutrients and the term $\mathcal{B}(\sigma_B - \sigma)$ which models the fact that we here consider the case in which the tumor has its own vasculature. In particular the threshold σ_B indicates whether the nutrient is supplied to the tumor or transported away.

2.2 Assumptions

In order to carry out a mathematical analysis of our problem, let us introduce some notation we will use in the sequel. We recall that Ω is a smooth domain of $\mathbb{R}^3$ and we denote by $\partial\Omega$ its boundary. For sake of semplicity, let us assume $|\Omega| = 1$. We denote by $(0,T)$ an assigned but otherwise arbitrary time interval. We set $H := L^2(\Omega)$ and $V := H^1(\Omega)$ and we will use these symbols also referring to vector valued functions. The symbol $(\cdot,\cdot)$ will indicate the standard scalar product in H, while $\langle\cdot,\cdot\rangle$ will stand for the duality between V' and V. We denote by $\|\cdot\|_X$ the norm in the generic Banach space X. For brevity we will write $\|\cdot\|$ instead of $\|\cdot\|_H$. Still for brevity, we omit the variables of integration. We will specify them when there could be a misinterpretation.
For any function $v \in V$, we define

$$v_\Omega := \frac{1}{|\Omega|}\int_\Omega v = \int_\Omega v, \tag{2.25}$$

where the last equality holds since we assumed $|\Omega| = 1$.
We recall the Poincaré-Wirtinger inequality

$$\|v - v_\Omega\| \le c_\Omega \|\nabla v\| \quad \forall v \in V \tag{2.26}$$

and the non-linear Poincaré inequality

$$\|v^{\frac{p}{2}}\|_V^2 \le c_p \left(\|v\|_{L^1(\Omega)}^p + \|\nabla v^{\frac{p}{2}}\|^2 \right), \tag{2.27}$$

which holds $\forall v \in L^1(\Omega)$ s.t. $\nabla v^{\frac{p}{2}} \in L^2(\Omega)$ and $\forall p \in [2,\infty)$.

We assume the coefficients $\mathcal{P}, \mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ to be strictly positive and $\sigma_B \in (0,1)$. Next, we suppose that the derivative of potential $F \in C^1_{loc}(\mathbb{R},\mathbb{R})$ decomposes as a sum of a monotone increasing part β and a linear perturbation, namely

$$F'(r) = \beta(r) - \lambda r \quad \lambda \ge 0,\ r \in \mathbb{R}. \tag{2.28}$$

Moreover we normalize β s.t. $\beta(0) = 0$ and we require

$$\exists c_\beta > 0 \text{ s.t. } |\beta(r)| \leq c_\beta(1 + F(r)) \quad \forall r \in \mathbb{R}, \tag{2.29}$$

$$|\beta(r)| \geq k|r| \text{ for some } k > 0, \tag{2.30}$$

where (2.29) means that F has at most an exponential growth at infinity, while (2.30) states that β has superlinear growth. Moreover, we assume potential F to be strictly positive.

Next, we assume $h \in C^1(\mathbb{R})$ increasingly monotone s.t.

i) $h(-1) = 0,\ h(r) \equiv 1 \quad \forall r \geq 1.$

ii) $\exists\ \underline{h} \geq 0$ and $\underline{\varphi} \leq -1$ s.t. $h(r) \equiv -\underline{h} \quad \forall r \leq \underline{\varphi}$.

Therefore h is globally Lipschitz continuous and there exists a constant $c > 0$ s.t.

$$|h(r)| + |h'(r)| \leq c \quad \forall r \in \mathbb{R}. \tag{2.31}$$

Moreover we assume the thermal conductivity to depend on the absolute temperature θ as follows

$$\kappa(\theta) = 1 + \theta^q, \quad q \in [2, \infty), \quad \theta \geq 0. \tag{2.32}$$

Eventually, we require the initial data to be such that

$$\begin{aligned}
&\varphi|_{t=0} = \varphi_0, \quad \varphi_0 \in V, \quad F(\varphi_0) \in L^1(\Omega)\\
&\theta|_{t=0} = \theta_0, \quad \theta_0 \in L^1(\Omega),\ \theta_0 > 0 \text{ a.e. in } \Omega,\ \log\theta_0 \in L^1(\Omega)\\
&\sigma|_{t=0} = \sigma_0, \quad \sigma_0 \in L^\infty(\Omega),\ 0 \leq \sigma_0 \leq 1 \text{ a.e. in } \Omega
\end{aligned} \tag{2.33}$$

where the last assumption on σ_0 is due to the interpretation of σ as a nutrient concentration. We also recall that we couple our system with homogeneus Neumann boundary conditions for all the unknowns.

2.3 Existence of solutions

In this section we present the main result of this Chapter, concerning the existence of solutions for the tumor growth model (2.1)–(2.4) for $\chi_\varphi = 0$ and $\varepsilon = 1$. Namely, we work on system

$$\varphi_t = \Delta\mu + (\mathcal{P}\sigma - \mathcal{A})h(\varphi) \tag{2.34}$$

$$\mu = -\Delta\varphi + F'(\varphi) - \theta \tag{2.35}$$

$$\theta_t + \theta\varphi_t - \operatorname{div}(\kappa(\theta)\nabla\theta) = |\nabla\mu|^2 \tag{2.36}$$

$$\sigma_t = \Delta\sigma - \mathcal{C}\sigma h(\varphi) + \mathcal{B}(\sigma_B - \sigma). \tag{2.37}$$

We here present what will be called a *weak entropy solution*, already used for example in [97], which is in fact weaker than other corresponding notions appearing in related contexts. This is due to the fact that we do not get enough regularity to pass to the limit in some non-linear terms in the temperature equation (2.36). In particular, the definition stated here does not include (a weak formulation of) the conservation of internal energy, differently from e.g. [14, 46].
Multiplying (2.36) by $\frac{1}{\theta}$, we have

$$(\Lambda(\theta) + \varphi)_t - \operatorname{div}\left(\frac{\kappa(\theta)\nabla\theta}{\theta}\right) = \frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2 + \frac{|\nabla\mu|^2}{\theta}, \tag{2.38}$$

with

$$\Lambda(\theta) := \int_1^\theta \frac{1}{s}\mathrm{d}s = \log\theta. \tag{2.39}$$

We remark that in our case $\Lambda(\theta)$ is a very well-known function, but we stick with this notation in order to be coherent with the literature [46, 97], where $\Lambda(\theta)$ might be a more generic function. Testing (2.38) by $\xi \in C^\infty([0,T]\times\overline{\Omega})$, $\xi \geq 0$, $\xi(T,\cdot) = 0$ and integrating by parts we infer

$$\begin{aligned}
&\int_0^T\int_\Omega(\Lambda(\theta)+\varphi)\xi_t\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega\frac{\kappa(\theta)}{\theta}\nabla\theta\cdot\nabla\xi\mathrm{d}x\mathrm{d}t\\
&\quad= -\int_0^T\int_\Omega\frac{|\nabla\mu|^2}{\theta}\xi\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega\frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2\mathrm{d}x\mathrm{d}t - \int_\Omega(\Lambda(\theta_0)+\varphi_0)\xi(\cdot,0)\mathrm{d}x.
\end{aligned}$$

Setting $\delta(\theta) := \int_1^\theta \frac{\kappa(s)}{s}\mathrm{d}s = \ln\theta + \frac{1}{q}(\theta^q - 1)$ according to (2.32), we get

$$\begin{aligned}
&\int_0^T\int_\Omega(\Lambda(\theta)+\varphi)\xi_t\mathrm{d}x\mathrm{d}t + \int_=^T\int_\Omega\delta(\theta)\Delta\xi\mathrm{d}x\mathrm{d}t\\
&\quad= -\int_0^T\int_\Omega\frac{|\nabla\mu|^2}{\theta}\xi\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega\frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2\mathrm{d}x\mathrm{d}t - \int_\Omega(\Lambda(\theta_0)+\varphi_0)\xi(\cdot,0)\mathrm{d}x.
\end{aligned} \tag{2.40}$$

Definition 2.1. We say that $(\varphi,\mu,\theta,\sigma)$ is a *weak entropy solution* to our non-isothermal Cahn-Hilliard model if it sastisfies the following equations

$$\langle\varphi_t,\xi\rangle = -\int_\Omega\nabla\mu\cdot\nabla\xi\mathrm{d}x + \int_\Omega(\mathcal{P}\sigma - \mathcal{A})h(\varphi)\xi\mathrm{d}x \qquad \text{a.e. in } (0,T) \text{ and } \forall\xi\in V,$$

$$\mu = -\Delta\varphi + F'(\varphi) - \theta \qquad \text{a.e. in } (0,T)\times\Omega,$$

$$\langle \sigma_t, \xi \rangle = -\int_\Omega \nabla\sigma \cdot \nabla\xi \mathrm{d}x - \int_\Omega \mathcal{C}\sigma h(\varphi)\xi \mathrm{d}x + \int_\Omega \mathcal{B}(\sigma_B - \sigma)\xi \mathrm{d}x$$

a.e. in $(0, T)$ and $\forall \xi \in V$,

complying a.e. in Ω with the initial conditions (2.33), homogeneus Neumann boundary conditions and the entropy production inequality

$$\int_0^T \int_\Omega (\Lambda(\theta) + \varphi)\xi_t \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \delta(\theta)\Delta\xi \mathrm{d}x\mathrm{d}t \\ \leq -\int_0^T \int_\Omega \frac{|\nabla\mu|^2}{\theta}\xi \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \frac{\kappa(\theta)}{\theta^2}|\nabla\theta|^2 \mathrm{d}x\mathrm{d}t - \int_\Omega (\Lambda(\theta_0) + \varphi_0)\xi(\cdot, 0)\mathrm{d}x \tag{2.41}$$

$\forall \xi \in C^\infty([0, T] \times \overline{\Omega}),\ \xi \geq 0,\ \xi(T, \cdot) = 0.$

Theorem 2.3.1. *Suppose that the assumptions in Section 2.2 hold and let $T > 0$. Then there exists at least one weak solution to our model problem, namely a quadruple $(\varphi, \mu, \theta, \sigma)$ with regularity*

$$\varphi \in C([0, T]; V) \cap H^1(0, T; V') \cap L^2(0, T; H^2(\Omega))$$
$$\beta(\varphi) \in L^2(0, T; H)$$
$$\mu \in L^2(0, T; V)$$
$$\theta \in L^2(0, T; V) \cap L^\infty(0, T; L^1(\Omega)) \cap L^q(0, T; L^{3q}(\Omega)),\ q \geq 2, \quad \theta > 0 \ a.e. \ in \ \Omega$$
$$\sigma \in C([0, T]; H) \cap H^1(0, T; V') \cap L^\infty((0, T) \times \Omega) \cap L^2(0, T; V)$$

satisfying system (2.34)–(2.37) *in the sense of Definition 2.1.*

2.3.1 A priori estimates

This section is devoted to gain the suitable regularity for the quadruple $(\varphi, \mu, \theta, \sigma)$ to prove the existence of solutions in Section 2.3.2. These a priori bounds are obtained formally, working directly on our system (2.34)-(2.37). We remark that the existence (of weak entropy solutions) argument might be made rigorous by the Faedo-Galerkin method that we decided not to detail here.

Nutrient estimate

We first search for a priori bounds for the nutrient following [102]. Therefore we give here only a sketch of the main steps.
Testing (2.37) by $-\sigma_-$, where $\sigma_- \geq 0$ represents the negative part of the nutrient σ,

exploting the initial conditions on σ and applying the Gronwall lemma, we gain

$$\sigma(t,x) \geq 0 \quad \text{for a.e. } t \geq 0, x \in \Omega.$$

Now, testing (2.37) by $(\sigma - \bar{\sigma})_+$ (where $\bar{\sigma} \geq 1$ is a suitable constant) using the Gronwall lemma and our assuptions on h and σ_B, it is possible to obtain

$$\|\sigma\|_{L^\infty((0,T)\times\Omega)} \leq c_T, \tag{2.42}$$

where c_T is a constant depending on time.

Energy estimate

We test (2.34) by μ, (2.35) by φ_t and (2.36) by 1 and then sum up. This yields, taking into account the boundary conditions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega F(\varphi) + \int_\Omega \theta\right) = \int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\mu. \tag{2.43}$$

We take care of the right hand side, in particular

$$\begin{aligned}
&\int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\mu \\
\overset{(2.35)}{=}& -\int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\Delta\varphi + \int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)F'(\varphi) - \int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\theta \\
\overset{(2.28)}{=}& \int_\Omega (\mathcal{P}\sigma - \mathcal{A})h'(\varphi)|\nabla\varphi|^2 + \int_\Omega \mathcal{P}h(\varphi)\nabla\sigma\cdot\nabla\varphi + \int_\Omega \beta(\varphi)(\mathcal{P}\sigma - \mathcal{A})h(\varphi) \\
&+ \int_\Omega \lambda\varphi(\mathcal{A} - \mathcal{P}\sigma)h(\varphi) + \int_\Omega (\mathcal{A} - \mathcal{P}\sigma)h(\varphi)\theta.
\end{aligned}$$

Thus (2.43) reads

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega F(\varphi) + \int_\Omega \theta\right) =& \int_\Omega (\mathcal{P}\sigma - \mathcal{A})h'(\varphi)|\nabla\varphi|^2 \\
&+ \int_\Omega (\mathcal{P}\sigma - \mathcal{A})\beta(\varphi)h(\varphi) \\
&+ \int_\Omega \lambda(\mathcal{A} - \mathcal{P}\sigma)h(\varphi)\varphi + \mathcal{P}\int_\Omega h(\varphi)\nabla\sigma\cdot\nabla\varphi \\
&+ \int_\Omega (\mathcal{A} - \mathcal{P}\sigma)h(\varphi)\theta \\
:=& I + II + III + IV.
\end{aligned} \tag{2.44}$$

We now estimate each term on the right hand side separately. The estimate on the nutrient (2.42) is a key point for all these bounds. In particular this is where a time-dependent constant c_T comes from. Exploting the assumption (2.31), we infer

$$I \leq c_T \|\nabla\varphi\|^2. \tag{2.45}$$

According to (2.29) it is straightforward that

$$II \leq c_T \left(1 + \int_\Omega F(\varphi)\right). \tag{2.46}$$

Moreover, using once again the assumption (2.31) on h and Young's inequality, we get

$$III \leq \frac{1}{2}\|\nabla\sigma\|^2 + c_T \left(1 + \|\varphi\|_{L^1(\Omega)} + \|\nabla\varphi\|^2\right). \tag{2.47}$$

Eventually, by the same tools used to estimate III, it holds

$$IV \leq c_T \|\theta\|_{L^1(\Omega)}. \tag{2.48}$$

Combining estimates (2.45)–(2.48), (2.44) reads

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega F(\varphi) + \int_\Omega \theta\right) \\ &\leq \frac{1}{2}\|\nabla\sigma\|^2 + c_T\left(1 + \|\varphi\|_{L^1(\Omega)} + \|\nabla\varphi\|^2 + \int_\Omega F(\varphi) + \|\theta\|_{L^1(\Omega)}\right) \end{aligned} \tag{2.49}$$

Our aim is to apply Gronwall's lemma in order to gain the energy estimate. Therefore we estimate and reabsorb the term $\|\varphi\|_{L^1(\Omega)}$ according to (2.30). Moreover we test (2.37) by σ which yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma\|^2 + \|\nabla\sigma\|^2 \leq c(1 + \|\sigma\|^2). \tag{2.50}$$

Hence, summing this last estimate to (2.49) we finally get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla\varphi\|^2 + \int_\Omega F(\varphi) + \int_\Omega \theta + \frac{1}{2}\|\sigma\|^2\right) + \frac{1}{2}\|\nabla\sigma\|^2 \\ &\leq c_T\left(1 + \|\nabla\varphi\|^2 + \int_\Omega F(\varphi) + \|\theta\|_{L^1(\Omega)} + \|\sigma\|^2\right). \end{aligned} \tag{2.51}$$

We are now able to apply Gronwall's lemma to (2.49), therefore we obtain the following a priori estimates

$$\|\nabla\varphi\|_{L^\infty(0,T;H)} \leq c_T \tag{2.52}$$

$$\|F(\varphi)\|_{L^\infty(0,T;L^1(\Omega))} \leq c_T \tag{2.53}$$

$$\|\theta\|_{L^\infty(0,T;L^1(\Omega))} \leq c_T \tag{2.54}$$

$$\|\sigma\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq c_T. \tag{2.55}$$

In particular, combining (2.29) and (2.30) with (2.53), we gain

$$\|\varphi\|_{L^\infty(0,T;L^1(\Omega))} \leq c_T \tag{2.56}$$

Entropy estimate

We now derive the entropy estimate testing (2.36) by $-\dfrac{1}{\theta}$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (-\log\theta - \varphi) + \int_\Omega \frac{1}{\theta}|\nabla\mu|^2 + \int_\Omega \left(|\nabla\log\theta|^2 + k_q|\nabla\theta^{q/2}|^2\right) = 0, \tag{2.57}$$

where $k_q > 0$ is a suitable constant only depending on the exponent $q \geq 2$, introduced in (2.32).

Now, integrating in time, owing to (2.54) and (2.56) and recalling that $|\log r| \leq r - \log r \;\; \forall r > 0$, we infer

$$\|\log\theta\|_{L^\infty(0,T;L^1(\Omega))} + \|\log\theta\|_{L^2(0,T;V)} \leq c_T, \tag{2.58}$$

$$\|\nabla\theta^{q/2}\|_{L^2(0,T;H)} \leq c_T. \tag{2.59}$$

Then, combining (2.27) with (2.54) and (2.59), it holds

$$\|\theta^{\frac{q}{2}}\|_{L^2(0,T;V)} \leq c_T \tag{2.60}$$

which implies in particular, since $q \geq 2$

$$\|\theta\|_{L^2(0,T;V)} \leq c_T. \tag{2.61}$$

On the other hand, using Sobolev embedding theorems, (2.60) also implies

$$\|\theta^{\frac{q}{2}}\|_{L^2(0,T;L^6(\Omega))} \leq c_T$$

and hence

$$\|\theta\|_{L^q(0,T;L^{3q}(\Omega))} \leq c_T. \tag{2.62}$$

Chemical potential estimate

Integrating (2.36) over Ω and exploiting boundary conditions together with Gauss-Green formula, we infer

$$\|\nabla\mu\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta + \int_\Omega \theta\varphi_t. \tag{2.63}$$

We now rewrite the latter term according to (2.34), then using (2.31) and (2.42), it follows that (2.63) reads

$$\frac{1}{2}\|\nabla\mu\|^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \theta + \frac{1}{2}\|\nabla\theta\|^2 + c_T\|\theta\|_{L^1(\Omega)}. \tag{2.64}$$

Thus from (2.54) and (2.61), we obtain

$$\|\nabla\mu\|_{L^2(0,T;H)} \leq c_T. \tag{2.65}$$

Now we integrate (2.35) over Ω, then

$$|\mu_\Omega| \overset{(2.25),(2.28)}{=} \Big|\int_\Omega(\beta(\varphi)-\lambda\varphi) - \int_\Omega \theta\Big| \tag{2.66}$$

$$\leq \int_\Omega |\beta(\varphi)| + \int_\Omega |\lambda\varphi| + \|\theta\|_{L^1(\Omega)} \tag{2.67}$$

$$\overset{(2.29),(2.54),(2.56)}{\leq} c_\beta\left(1+\int_\Omega F(\varphi)\right) + c_T. \tag{2.68}$$

Using now the bound (2.53), we get

$$\|\mu_\Omega\|_{L^\infty(0,T)} \leq c_T. \tag{2.69}$$

Combining this last bound with the Poincaré inequality (2.26) and the previous estimate (2.65), we achieve

$$\|\mu\|_{L^2(0,T;V)} \leq c_T. \tag{2.70}$$

φ-dependent estimates

We start testing (2.34) by φ, which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\varphi|^2 = -\int_\Omega \nabla\mu\cdot\nabla\varphi + \int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)\varphi. \tag{2.71}$$

Exploiting Young's inequality, the uniform bounds on h and (2.42) we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2 \leq \frac{1}{2}\|\nabla\mu\|^2 + \frac{1}{2}\|\nabla\varphi\|^2 + c_T\|\varphi\|_{L^1(\Omega)}.$$

Thus, integrating in time and using (2.65), (2.52) and (2.56) we get

$$\|\varphi\|_{L^\infty(0,T;H)} \leq c_T,$$

whence estimate (2.52) gives

$$\|\varphi\|_{L^\infty(0,T;V)} \leq c_T. \tag{2.72}$$

Next we test (2.35) by $\beta(\varphi)$ and we obtain

$$\int_\Omega |\beta(\varphi)|^2 + \int_\Omega \beta'(\varphi)|\nabla\varphi|^2 = \int_\Omega \mu\beta(\varphi) + \int_\Omega \lambda\varphi\beta(\varphi) + \int_\Omega \theta\beta(\varphi)$$

Now, from (2.70), (2.72), (2.61) and the monotonicity of β, it follows

$$\|\beta(\varphi)\|_{L^2(0,T;H)} \leq c_T. \tag{2.73}$$

Taking advantage of this last estimate with (2.29) and again of (2.72) and (2.61), a direct comparison within equation (2.35) yields

$$\|\varphi\|_{L^2(0,T;H^2)} \leq c_T. \tag{2.74}$$

Further regularity

We start testing (2.34) by a nonzero test function $v \in V$ and we infer

$$\langle\varphi_t, v\rangle = -\int_\Omega \nabla\mu\cdot\nabla v + \int_\Omega (\mathcal{P}\sigma - \mathcal{A})h(\varphi)v.$$

Now, according to estimates (2.42), (2.65) and (2.72) it follows

$$\|\varphi_t\|_{L^2(0,T;V')} \leq c_T. \tag{2.75}$$

Taking advantage of this last estimate and exploting (2.74) together with (2.72), we infer (for example from [90])

$$\varphi \in C([0,T];V). \tag{2.76}$$

Similarly, multiplying equation (2.37) by a nonzero test function $v \in V$ and exploiting the bound (2.55), it holds

$$\|\sigma_t\|_{L^2(0,T;V')} \le c_T. \tag{2.77}$$

From standard embedding results (see e.g. [12]), combining (2.77) and (2.55), we gain the additional regularity for the nutrient

$$\sigma \in C([0,T];H). \tag{2.78}$$

2.3.2 Weak sequential stability

We assume to have a sequence of weak solutions $(\varphi_n, \mu_n, \theta_n, \sigma_n)$ which satisfies the a priori estimates obtained in Section 2.3.1 uniformly with respect to $n \in \mathbb{N}$.
We then show that, by weak compactness arguments, up to the extraction of a subsequence, $(\varphi_n, \mu_n, \theta_n, \sigma_n)$ converges in a suitable way to an entropy solution to our problem, i.e., to a limit quadruple $(\varphi, \mu, \theta, \sigma)$ solving (2.34)–(2.37) in the sense of Theorem 2.3.1.
Indeed, exploiting the above bounds (2.42), (2.54), (2.55), (2.61), (2.62), (2.70), (2.72), (2.74), (2.75) and (2.77), together with standard weak compactness results, it is possible to extract a nonrelabelled subsequence such that

$$\varphi_n \to \varphi \text{ weakly star in } L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;V') \tag{2.79}$$

$$\mu_n \to \mu \text{ weakly in } L^2(0,T;V) \tag{2.80}$$

$$\theta_n \to \theta \text{ weakly star in } L^2(0,T;V) \cap L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;L^{3q}(\Omega)) \tag{2.81}$$

$$\sigma_n \to \sigma \text{ weakly star in } L^\infty(0,T) \times \Omega)) \cap L^2(0,T;V) \cap H^1(0,T;V') \tag{2.82}$$

Moreover, combining (2.75) and (2.77) with (2.79) and (2.82) respectively and applying the Aubin-Lions lemma, we infer that

$$\varphi_n \to \varphi \text{ and } \sigma_n \to \sigma \text{ strongly in } L^2(0,T;H). \tag{2.83}$$

Furthermore, convergence (2.81) and interpolation theory for L^p spaces imply that

$$\theta_n \to \theta \text{ strongly in } L^p(0,T;L^p(\Omega)), \quad p \in \left[1, q + \frac{2}{3}\right). \tag{2.84}$$

Indeed, from Proposition A.2.1, with $s = \infty$ and $p = q$, it holds

$$\frac{1}{r} = \frac{\gamma}{q}. \tag{2.85}$$

We then apply the general interpolation result to the space-spaces L^1 and L^{3q}, from which it follows

$$\frac{1}{r} = \frac{\gamma}{3q} + \frac{1-\gamma}{1}. \tag{2.86}$$

Since (2.85) and (2.86) must hold simultaneously, we infer that $r = q + \frac{2}{3}$.
Now, according to Theorem A.2.6 with $s = r = 0$, it follows that

$$L^\infty(0,T;L^1(\Omega)) \cap L^q(0,T;L^{3q}(\Omega)) \hookrightarrow\hookrightarrow L^p(0,T;L^p(\Omega)) \quad p \in \left[1, q+\frac{2}{3}\right).$$

Therefore it is possible to pass to the limit also in the nonlinear terms, according to the continuity of κ, β and h. Indeed, by a generalized version of Lebesgue's dominated convergence theorem it holds

$$\kappa(\theta_n) \to \kappa(\theta) \text{ strongly in } L^p(0,T;L^p(\Omega)), \quad p \in \left[1, 1+\frac{2}{3q}\right) \tag{2.87}$$

$$\beta(\varphi_n) \to \beta(\varphi) \text{ weakly in } L^2(0,T;H). \tag{2.88}$$

We now want to pass to the limit in the balance of entropy. Namely let us assume that (2.38) is satisfied by the approximate solution $(\varphi_n, \mu_n, \theta_n, \sigma_n), \forall n \in \mathbb{N}$. Testing it by $\xi \in C^\infty([0,T] \times \overline{\Omega}),\ \xi \geq 0,\ \xi(T,\cdot) = 0$ and integrating by parts we infer

$$\begin{aligned}
&\int_0^T \int_\Omega (\Lambda(\theta_n) + \varphi_n)\xi_t \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \delta(\theta_n)\Delta\xi \mathrm{d}x\mathrm{d}t \\
&= -\int_0^T \int_\Omega \frac{|\nabla\mu_n|^2}{\theta_n}\xi \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Omega \frac{\kappa(\theta_n)}{\theta_n^2}|\nabla\theta_n|^2 \mathrm{d}x\mathrm{d}t - \int_\Omega (\Lambda(\theta_0) + \varphi_0)\xi(\cdot,0)\mathrm{d}x,
\end{aligned} \tag{2.89}$$

where $\delta(\theta_n) := \int_1^{\theta_n} \frac{\kappa(s)}{s} \mathrm{d}s$.
We first take care of the terms on the left hand side. According to (2.39), by (2.58) and (2.84),

$$\Lambda(\theta_n) \to \Lambda(\theta_n) \quad \text{strongly in } L^{1+}(0,T) \times \Omega). \tag{2.90}$$

Moreover, from (2.58) and (2.84), it follows that

$$\delta(\theta_n) \to \delta(\theta) \quad \text{strongly in } L^{1+}((0,T) \times \Omega), \tag{2.91}$$

hence in particular

$$\int_0^T \int_\Omega \delta(\theta_n)\Delta\xi \to \int_0^T \int_\Omega \delta(\theta)\Delta\xi.$$

Then the first row of (2.89) passes to the desired limit not only as a supremum limit, but as a true limit. In order to deal with the first two terms in the right hand side we recall a useful lower semicontinuity result by Ioffe.

Theorem 2.3.2 (Ioffe). *Let $\mathcal{O} \subset \mathbb{R}^d$ be a smooth bounded open set and $f : \mathcal{O} \times \mathbb{R}^n \times \mathbb{R}^m \to [0, +\infty], d, n, m \geq 1,$ be a measurable non-negative function such that*

$f(x, \cdot, \cdot)$ *is lower semicontinuous on* $\mathbb{R}^n \times \mathbb{R}^m$ *for every* $x \in \mathcal{O}$,

$f(x, u, \cdot)$ *is convex on* $\mathbb{R}^m$ *for every* $(x, u) \in \mathcal{O} \times \mathbb{R}^n$.

Let also $(u_k, v_k), (u, v) : \mathcal{O} \to \mathbb{R}^n \times \mathbb{R}^m$ *be measurable functions s. t.*
$u_k(x) \to u(x)$ *in measure in* $\mathcal{O}$, $\qquad v_k \rightharpoonup v$ *weakly in* $L^1(\mathcal{O}; \mathbb{R}^m)$.
Then,

$$\liminf_{k\to+\infty} \int_{\mathcal{O}} f(x, u_k(x), v_k(x)) \geq \int_{\mathcal{O}} f(x, u(x), v(x)).$$

We start considering the first term in the right hand side. We exploit this result setting $\mathcal{O} = \Omega \times (0, T), f : \mathcal{O} \times \mathbb{R}^+ \times \mathbb{R}^3 \to [0, \infty]$ s.t $(x, t) \times w \times v \mapsto w|v|^2$. Such f satisfies Ioffe's assumptions. putting $w_n = \frac{\xi}{\theta},\ v_n = \nabla\mu_n,\ \forall n \in \mathbb{N}$. Hence, by (2.80) it holds $\{\nabla\mu_n\}_n \rightharpoonup \nabla\mu$ in $L^1(\mathcal{O})$. Therefore by Ioffe's theorem,

$$\liminf_{n\to+\infty} \int_0^T \int_\Omega |\nabla\mu_n|^2 \xi \geq \int_0^T \int_\Omega |\nabla\mu|^2 \xi. \tag{2.92}$$

In a similar way, from (2.84) and (2.87),

$$\liminf_{n\to+\infty} \int_0^T \int_\Omega \frac{\xi}{\theta_n} \frac{\kappa(\theta_n)}{\theta_n} |\nabla\theta_n|^2 \geq \int_0^T \int_\Omega \frac{\xi}{\theta} \frac{\kappa(\theta)}{\theta} |\nabla\theta|^2. \tag{2.93}$$

Furthermore, assuming that $\theta_n(0, \cdot)$ converges in a suitable way to θ_0, putting together (2.83), (2.90), (2.91), (2.92) and (2.93), it follows that we eventually recover (2.40). It is worth noting that the inequality sign is due to the application of Ioffe's theorem. This concludes the procedure and so the proof of existence of weak entropy solutions.

Remark 1. We notice that we have assumed throughout the proof that the absolute temperature is a.e. positive. This is crucial in order for estimates in Section 2.3.1 to make sense. In particular it should be shown that the solution θ_n of the discretized problem (for instance in a Faedo-Galerkin scheme) is positive. Anyway, taking the initial datum $\theta_0 > 0$, according to (2.58) the strict positivity of θ_n will be preserved a.e. in $(0, T) \times \Omega$ also in the limit.

Part II

A model for assisted periodic breathing

We represent the lungs as a nonhomogeneous deformable viscoelastic porous body and breathing as an isothermal time-periodic process modelled by a PDE system with hysteresis. In particular, Part III contains the results obtained in [76], where we extended the model proposed in [40] taking into account nonconstant and possibly degenerate permeability of the lungs.

As pointed out in [99], the first measurements which showed a hysteretic pressure-volume characteristic in mammalian lungs were obtained in [23] in 1913. There exist different hypotheses about the nature of the forces which originate the hysteresis behavior by opposing the lung distension, but up to now, there is no theory which could explain both small and large volume excursions, as reported in [117].
The process of breathing has been studied from a mathematical point of view first for cats [73] and dogs [91] and later also for humans [86]. Understanding the relationship between pressure and volume in lungs is important for controlling the ventilation of patients with lung diseases [4, 33, 71]. In order to set up the ventilator correctly and so to prevent the possibility of damaging the patient's lungs, it is crucial to test how the healthy lungs respond to mechanical ventilation. In [103] such a ventilation study simulating hysteretic pressure-volume relationships in the lungs of a healthy individual is shown. Moreover, the knowledge of the pressure-volume hysteresis loops were used to develop a new technique for treating patients with Acute Lung Injury (ALI). This procedure gave the same levels of oxygen but with lower airway pressures and less over-inflation than other techniques [3]. Furthermore, [3] also shows that this kind of strategy reduces trauma and increases the probability of survival with respect to methods which do not take into account pressure-volume curves. We also remark that such studies are useful not only for treating damaged lungs; indeed, e. g., cerebrospinal pressure-volume curve might be of interest for hydrocephalus symptoms [79].

Our aim is to further develop the model proposed in [40], motivated by the analysis carried out in [52], where the Preisach operator is shown to be an appropriate model for the pressure-volume hysteresis relationship in lungs. The process of breathing is assumed to be isothermal and driven by a periodic muscle activity and, possibly, by a periodic external ventilation. Our model is represented by a PDE system consisting of a momentum balance equation and a mass balance equation coupled with boundary conditions prescribing the mechanical reaction between lungs and their surroundings. The mathematical problem consists in proving that the PDE system with a degenerating pressure-mass content term under the time derivative admits a periodic solution for every periodic boundary forcing with a given regularity. The novelty of [76] with respect to [40] is that we expand the previous model considering the permeability of the lungs depending also on the pressure and possibly degenerate. This is, on the one hand, a realistic hypothesis from the physical and biological point of view. On the other hand, this condition produces considerable difficulties in the mathematical problem. Indeed, the additional degenerate nonlinearity requires to choose a multi-parameter approximation technique involving several steps with a series of regularizing terms in the system which have to vanish in the limit. The main ingredient of the proof is a time-periodic parameter-dependent variant of the Moser iterations.

The structure of this Part is the following. In Section 3.1 we present our model and state the main result of this work, which is Theorem 3.1.3 stating the existence and regularity of the resulting PDE system with hysteresis. In Section 3.3 we recall the main properties of the Preisach operator which are needed in the analysis of the problem. The proof of Theorem 3.1.3 is given in several steps in Sections 4.1–4.4. In order to get the nonlinearities under control, we introduce in Section 4.1 a regularized problem with a large cut-off parameter R and with some higher order regularizing terms depending on small parameters δ and ε. For the regularized system, we construct a periodic solution by Galerkin approximations and a fixed point argument. In Sections 4.2 and 4.3, we derive suitable a priori bounds independent of the parameters δ and ε. These estimates are used in Section 4.4, where according to a Moser-type technique we obtain L^∞-estimates of the pressure which allow us to remove the cut-off parameter and eventually let δ and ε tend to zero and prove Theorem 3.1.3.

Chapter 3

Viscoelastic porous medium model

3.1 Statement of the problem

As proposed in [40], we represent the lungs as a deformable viscoelastic porous medium, and breathing as a periodic isothermal process of gas exchange between the inside and the outside of the body. The mathematical model is based on a momentum balance equation

$$\rho u_{tt} = \operatorname{div} \sigma, \tag{3.1}$$

where ρ is the solid mass density, u is the displacement vector in the solid and σ is the stress tensor, and a gas mass balance equation

$$s_t + \operatorname{div} q = 0, \tag{3.2}$$

where s is the gas mass content in the pores and q is the mass flux.
We also consider three constitutive relations. The first one represents the mechanical interaction, namely

$$\sigma = \mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u - p\mathbf{1}, \tag{3.3}$$

where $\mathbf{B}(x)$ (viscosity), $\mathbf{A}(x)$ (elasticity) are symmetric positive definite space-dependent tensors of order 4, the symbol ∇_s denotes the symmetric gradient, p is the air pressure, and $\mathbf{1}$ in the Kronecker tensor.
The second one assumes the pressure-volume relation in the form

$$f(x,p) + G[p] = \frac{1}{\rho_a} s - \operatorname{div} u, \tag{3.4}$$

where $\rho_a > 0$ is the referential air mass density at standard pressure, $f : \Omega \times \mathbb{R} \to (0, \infty)$ is a function increasing in p, and G is a Preisach operator defined in Section 3.3.

We can interpret (3.4) as follows. Assuming that the volume of the solid phase remains constant during the process, the term $\operatorname{div} u$ represents the void volume difference with respect to the reference state. At constant pressure p, if $\operatorname{div} u$ increases, then s/ρ_a increases at the same rate. At constant void volume, the mass content s is an increasing function (with different inflation and deflation curves) of the pressure. At constant gas mass content, the pressure increases if the void volume decreases.
Note that the mass content s cannot be negative, so that, e. g., at constant volume, the pressure-dependent term $f(x,p) + G[p]$ must be bounded from below. In particular, the partial derivative $f_p(x,p)$ of f with respect to p has to degenerate as $p \to -\infty$.

As the third constitutive relation, we assume the validity of Darcy's law for the mass flux q in (3.2)

$$q = -\rho_a \mu(x,p) \nabla p \tag{3.5}$$

where we underline that, as a novelty with respect to [40], the permeability $\mu(x,p)$ is allowed to depend both on the space variable and on the pressure, and degenerate as $p \to \pm\infty$.
The combination of (3.1)–(3.5) leads to the following PDE system

$$\begin{aligned} \rho u_{tt} &= \operatorname{div}\left(\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u\right) - \nabla p\,, && (3.6)\\ (f(x,p) + G[p])_t &= -\operatorname{div} u_t + \operatorname{div} \mu(x,p)\nabla p\,, && (3.7) \end{aligned}$$

for x in a bounded connected Lipschitzian domain $\Omega \subset \mathbb{R}^3$ and for $t \geq 0$.
We couple system (3.6)–(3.7) with the following boundary conditions on $\partial\Omega$

$$\begin{aligned} -\sigma \cdot n\big|_{\partial\Omega} &= \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g) + pn\,, && (3.8)\\ \frac{1}{\rho_a} q \cdot n\big|_{\partial\Omega} &= \alpha(x)(p - \bar{p}) - u_t \cdot n\,, && (3.9) \end{aligned}$$

where n is the unit outward normal vector, $g = g(x,t)$ is a given external force acting on the body Ω, $\bar{p} = \bar{p}(x,t)$ is the given outer air pressure possibly supplied by the ventilator, $\mathbf{C}(x), \mathbf{D}(x)$ are symmetric positive definite 3×3 matrices, $\beta(x) \geq 0$ is the relative elasticity modulus of the boundary at the point $x \in \partial\Omega$, and $\alpha(x) \geq 0$ is the boundary permeability at the point $x \in \partial\Omega$.
The first boundary condition (3.8) describes the mechanical interaction between the inside and the exterior of the body on the boundary, which is controlled by β. The second boundary condition (3.9), on the other hand, describes the gas exchange between the inside and the exterior of the body on the boundary, which is controlled by α (see [40]).

We denote

$$X_3 = W^{1,2}(\Omega;\mathbb{R}^3)\,, \quad X = W^{1,2}(\Omega)\,. \tag{3.10}$$

Then, the full PDE system in variational form for all test functions $\phi \in X_3$ and $\psi \in X$ reads as follows:

$$\int_\Omega \big(\rho u_{tt}\phi + (\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u) : \nabla_s\phi + \nabla p\ \phi\big)\mathrm{d}x + \int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g)\phi \mathrm{d}s(x) = 0\,, \tag{3.11}$$

$$\int_\Omega \big((f(x,p) + G[p])_t\psi + (\mu(x,p)\nabla p - u_t)\nabla\psi\big)\mathrm{d}x + \int_{\partial\Omega} \alpha(x)(p-\bar{p})\psi\mathrm{d}s(x) = 0\,. \tag{3.12}$$

We fix a period $T > 0$ and denote by L^q_T the L^q-space of T-periodic functions $v : \mathbb{R} \to \mathbb{R}$ for $q \geq 1$, by $W^{1,q}_T$ the associated Sobolev space, and by C_T the space of continuous real T-periodic functions on $\mathbb{R}$. Similarly, we deal with the spaces $L^q_T(Y)$ of T-periodic L^q-functions $v : \mathbb{R} \to Y$ with values in a Banach space Y, as well as with the spaces $L^q(\Omega; C_T)$ and $L^q(\Omega; W^{1,q}_T)$.

Hypothesis 3.1.1. *We assume that*

(i) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is such that* $f(\cdot,p)$ *is bounded and measurable for all* $p \in \mathbb{R}$, $f(x,\cdot)$ *is continuously differentiable in* $\mathbb{R}$ *for a. e.* $x \in \Omega$, *and there exist constants* $0 < f^\flat < f^\sharp$ *and* $\frac{1}{2} < \eta < 1$ *with the property*

$$f^\flat(1+p^2)^{-\eta} \leq \partial_p f(x,p) \leq f^\sharp \quad \text{a. e.} \quad \forall p \in \mathbb{R}\,.$$

(ii) *The permeability coefficient* $\mu(x,p)$ *is positive and continuous. Moreover, putting* $\mu_0(x) = \mu(x,0)$, $\bar{\mu}(x,p) = \frac{\mu(x,p)}{\mu_0(x)}$, *and* $M(x,p) = \int_0^p \bar{\mu}(x,p')\,dp'$ *we assume that there exist constants* $\mu^\flat, \mu^\sharp, \gamma$ *such that*

$$0 \leq \gamma < \frac{1}{6}\,\mathrm{e}^{-1}, \quad \gamma + \eta \leq 1, \tag{3.13}$$

and for a. e. $(x,p) \in \Omega \times \mathbb{R}$ *we have*

$$\mu^\sharp \geq \mu(x,p) \geq \mu^\flat(1+p^2)^{-\gamma}, \tag{3.14}$$

$$|\nabla_x\bar{\mu}(x,p)| \leq \mu^\sharp(1+p^2)^{-\eta/2}, \tag{3.15}$$

$$|\nabla_x M(x,p)| \leq \mu^\sharp, \tag{3.16}$$

where the symbol ∇_x *denotes the partial gradient with respect to* x.

(iii) *The nonnegative functions* α *and* β *belong to* $L^\infty(\partial\Omega)$ *and we have* $\int_{\partial\Omega}\beta(x)ds(x) >$

0, $\int_{\partial\Omega} \alpha(x)\, ds(x) > 0$.

(iv) *The functions g, g_t belong to $L^2_T(L^2(\partial\Omega;\mathbb{R}^3))$, $\bar{p}, \bar{p}_t$ belong to $L^2_T(L^2(\partial\Omega))$, and there exists a constant $p^\sharp$ such that*

$$|\bar{p}(x,t)| \le p^\sharp \quad \text{for a. e.} \quad (x,t) \in \partial\Omega \times (0,T).$$

(v) *The symmetric positive definite tensors $\mathbf{A}, \mathbf{B} \in L^\infty(\Omega; \mathbb{R}^{3\times 3}_s \times \mathbb{R}^{3\times 3}_s)$, where $\mathbb{R}^{3\times 3}_s$ is the space of real symmetric tensors of order 3×3, and symmetric definite matrices $\mathbf{C}, \mathbf{D} \in L^\infty(\partial\Omega; \mathbb{R}^3 \times \mathbb{R}^3)$ are given and there exists a constant $\bar{c}$, independent of x, such that*

$$\begin{cases} \mathbf{A}(x)\xi : \xi \ \ge \ \bar{c}(\xi : \xi), & \mathbf{B}(x)\xi : \xi \ \ge \ \bar{c}(\xi : \xi) \quad \textit{a. e. } \forall \xi \in \mathbb{R}^{3\times 3}_s \\ \mathbf{C}(x)v \cdot v \ \ge \ \bar{c}|v|^2, & \mathbf{D}(x)v \cdot v \ \ge \ \bar{c}|v|^2 \qquad \textit{a. e. } \forall v \in \mathbb{R}^3. \end{cases}$$

Note that in the case $\mu(x,p) = \mu_1(x)\mu_2(p)$ we have $\bar{\mu}(x,p) = \mu_2(p)/\mu_2(0)$, hence $\nabla_x \bar{\mu}(x,p) = 0$ and the conditions (3.15)–(3.16) are trivially satisfied.

The Preisach operator G is characterized by its *density function* ψ, see Definition 3.3.1 below. We suppose that it has the following properties.

Hypothesis 3.1.2. *A function $\psi \in L^\infty(\Omega \times (0,\infty) \times \mathbb{R})$ is given, and there exist constants $\Psi^\sharp > 0$, $B > 0$ such that*

$$0 \le \psi(x,r,v) \le \Psi^\sharp \quad \textit{a. e.}, \qquad \psi(x,r,v) = 0 \ \ \textit{for} \ \ r + |v| \ge B. \tag{3.17}$$

We are then ready to state the main result of this Part, concerning the existence of a time-periodic solution to our model.

Theorem 3.1.3. *Let Hypotheses 3.1.1, 3.1.2 hold. Then there exists a solution (u,p) to (3.11)–(3.12) such that $u, u_t, \nabla p \in L^2_T(L^2(\Omega;\mathbb{R}^3)) \cap L^\infty_T(L^2(\Omega;\mathbb{R}^3))$, $u_{tt} \in L^2_T(L^2(\Omega;\mathbb{R}^3))$, $p_t \in L^2_T(L^2(\Omega))$, $p \in L^\infty(\Omega \times (T,2T))$, $\nabla_s u, \nabla_s u_t \in L^2_T(L^2(\Omega;\mathbb{R}^{3\times 3})) \cap L^\infty_T(L^2(\Omega;\mathbb{R}^{3\times 3}))$.*

We remark that the reason why $[T, 2T]$ is chosen to be the referential interval for T-periodic functions is related to the delayed periodicity of the operator G stated in Proposition 3.3.2.

The main issue to face in order to prove Theorem 3.1.3 is the degenerate character of the functions f and μ and of the operator G, which makes the analysis difficult. The proof will therefore be carried out in several steps. In Section 4.1 we regularize the critical terms by means of small parameters ε and δ and a large cut-off parameter R, propose a

Galerkin discretization scheme in dimension $m \in \mathbb{N}$, and pass to the limit as $m \to \infty$ to obtain a solution to the regularized problem. In Sections 4.2–4.4 we show the existence of estimates independent of ε, δ, and R which enable us to pass to the limit as $\varepsilon \to 0$, $\delta \to 0$, and $R \to \infty$. In particular, the estimates independent of R follow from a time-periodic variant of the Moser iteration scheme.

3.2 Comparison with previous model

As already mentioned, the system presented in Section 3.1 is a further development from the modellistic point of view of the one in [40], namely

$$\rho u_{tt} = \operatorname{div}\left(\mathbf{B}(x)(x)\nabla_s u_t + \mathbf{A}(x)(x)\nabla_s u\right) - \nabla p\,, \tag{3.18}$$

$$(f(p) + G[p])_t = -\operatorname{div} u_t + \operatorname{div} \mu(x)\nabla p\,, \tag{3.19}$$

where we considered a permeability μ only dependent on the position.
Although in [40] we were able to prove the existence of periodic solutions (u, p) with the same regularity stated in Theorem 3.1.3, the assumptions needed to be refined and so were several steps of the proof.
Concerning the hypotheses, the main focus is of course on μ. Indeed, even though in [40] we only asked for the permeability coefficient to belong to $L^\infty(\Omega)$ and to be positively bounded from below, the dependence on p in [76] introduces a nonlinearity. Therefore, we need stronger assumptions, namely Hypotheses 3.1.1 (ii). Moreover, as a consequence of this new nonlinearity, we cannot follow the Galerkin approximations proposed in [40]. As a matter of fact, we here want to test our PDE system by a nonlinear expression v of pressure p, which is its Kirchhoff transform introduced in Hypothesis 3.1.1. This leads to considering the approximations of u and v, since the Galerkin method allows testing only by linear functions and their derivatives. Furthermore we need to regularize the critical terms by means of small parameters ε and δ. This adds an ulterior complexity to our proof, since the estimates must be independent of these parameters. According to these technicalities, also the Moser-type iteration scheme had to be revised.

3.3 Preisach operator

We recall here the basic theory of hysteresis operators, in particular of the Preisach operator G, which is needed in the sequel.
The detailed proofs of the statements of this Section can be found in [84].

We fix some $\hat{T} > 2T$ and construct the Preisach operator G in terms of the play operator, which is defined as a solution to the following variational inequality

$$\begin{cases} |p(t) - \xi_r(t)| \leq r & \forall t \in [0, \hat{T}], \\ (\xi_r(t))_t (p(t) - \xi_r(t) - z) \geq 0 & \text{a.e. } \forall z \in [-r, r], \\ p(0) - \xi_r(0) = \max\{-r, \min\{p(0), r\}\}. \end{cases} \tag{3.20}$$

It is well known (see [83, 116]) that for any given input function $p \in W^{1,1}(0, \hat{T})$ and each parameter $r > 0$, there exists a unique solution $\xi_r \in W^{1,1}(0, \hat{T})$ of the variational inequality (3.20). The mapping $\mathfrak{f}_r : W^{1,1}(0, \hat{T}) \to W^{1,1}(0, \hat{T})$ which with each $p \in W^{1,1}(0, \hat{T})$ associates the solution $\xi_r = \mathfrak{f}_r[p] \in W^{1,1}(0, \hat{T})$ of (3.20) is called the *play operator* and the parameter $r > 0$ can be interpreted as a *memory parameter.*

Proposition 3.3.1. *For each $r > 0$, the mapping $\mathfrak{f}_r : W^{1,1}(0, \hat{T}) \to W^{1,1}(0, \hat{T})$ is Lipschitz continuous and admits a Lipschitz continuous extension to $\mathfrak{f}_r : C[0, \hat{T}] \to C[0, \hat{T}]$, in the sense that for every $p_1, p_2 \in C[0, \hat{T}]$ and for every $t \in [0, \hat{T}]$ we have*

$$|\mathfrak{f}_r[p_1](t) - \mathfrak{f}_r[p_2](t)| \leq \|p_1 - p_2\|_{[0,t]} := \max_{\tau \in [0,t]} |p_1(\tau) - p_2(\tau)|. \tag{3.21}$$

Besides, for every $p \in W^{1,1}(0, \hat{T})$, the energy balance equation

$$\mathfrak{f}_r[p]_t p - \frac{1}{2}(\mathfrak{f}_r^2[p])_t = |r \mathfrak{f}_r[p]_t| \tag{3.22}$$

and the identity

$$\mathfrak{f}_r[p]_t p_t = (\mathfrak{f}_r[p]_t)^2 \geq 0 \tag{3.23}$$

hold almost everywhere in $(0, \hat{T})$.

Proposition 3.3.2. *Let $p \in W^{1,1}_{loc}(0, \infty)$ be periodic with period $T > 0$. Then $\mathfrak{f}_r[p](t+T) = \mathfrak{f}_r[p](t)$ for all $t \geq T$, namely $\mathfrak{f}_r[p]$ is periodic for $t \geq T$ for all $r > 0$.*

In what follows, we consider input functions p which depend on $x \in \Omega$ and $t > 0$. For $r > 0, q \geq 1$ and $p \in L^q(\Omega; C_T)$, we interpret the play $\mathfrak{f}_r[p](x,t)$ as

$$\mathfrak{f}_r[p](x,t) = \mathfrak{f}_r[p(x, \cdot)](t).$$

Definition 3.3.1. Let ψ be given and satisfy Hypothesis 3.1.2. We define the Preisach operator

$$G[p](x,t) = \int_0^\infty \int_0^{\xi_r(x,t)} \psi(x, r, v) \mathrm{d}v \mathrm{d}r, \tag{3.24}$$

where $\xi_r(x,t) = \mathfrak{f}_r[p](x,t)$ is the output of the play operator applied to $p(x, \cdot)$.

From the properties of the play operator in Proposition 3.3.1, namely (3.21) and (3.23), we can prove the Lipschitz continuity and local monotonicity of the Preisach operator.

Proposition 3.3.3. *Let ψ be a function fulfilling Hypothesis 3.1.2. Then, there exists a constant $\Psi^\sharp > 0$ such that for $q \in [1, \infty]$ and for all $p_1, p_2 \in L^q(\Omega; C_T)$ and $t \geq 0$, it holds*

$$|G[p_1](x,t) - G[p_2](x,t)| \leq \Psi^\sharp B \max_{\tau \in [0,t]} |p_1(x,\tau) - p_2(x,\tau)| \quad a.\,e. \ in \ \ \Omega. \tag{3.25}$$

In particular, the mapping G is Lipschitz continuous in $L^q(\Omega; C_T)$ for every $q \in [1, \infty]$.

Proposition 3.3.4. *Let the Preisach operator G from Definition 3.3.1 satisfy Hypothesis 3.1.2. Then G is locally monotone, i. e. $\forall p \in L^q(\Omega; W_T^{1,1})$, $G[p]$ belongs to $L^q(\Omega; W_T^{1,1})$, and*

$$G[p]_t(x,t) p_t(x,t) \geq 0 \ a.\,e. \tag{3.26}$$

Moreover, as a consequence of the energy identity (3.22), we have $\mathfrak{f}_r[p]_t(p - \mathfrak{f}_r[p]) \geq 0$ a. e. Hence the inequality $\mathfrak{f}_r[p]_t(h(p) - h(\mathfrak{f}_r[p])) \geq 0$ holds almost everywhere for every nondecreasing function $h : \mathbb{R} \to \mathbb{R}$. We then easily conclude that the following Preisach energy inequality holds.

Proposition 3.3.5. *Let $h : \mathbb{R} \to \mathbb{R}$ be nondecreasing, $h(0) = 0$. Then for every $p \in L^q(\Omega; W_T^{1,1})$ and a. e. $(x,t) \in \Omega \times (0, \infty)$ it holds*

$$\left(\frac{d}{dt} G[p]\right) h(p) - \frac{d}{dt} V_h[p] \geq 0 \tag{3.27}$$

where $V_h[p](x,t) = \int_0^\infty \int_0^{\xi_r} \psi(x,r,v) h(v)\, dv\, dr$ is the h-energy potential. If moreover Hypothesis 3.1.2 holds, then

$$0 \leq V_h[p](x,t) \leq \Psi^\sharp B^2 \max\{h(B), -h(-B)\} \quad a.\,e. \tag{3.28}$$

Chapter 4

Proof of Theorem 3.1.3

4.1 Approximation scheme

We choose a cut-off parameter $R > 1$ and consider the truncations

$$
\begin{aligned}
f_R(x,p) &= \begin{cases} f(x,-R) + \partial_p f(x,-R)(p+R) & \text{for } p \le -R, \\ f(x,p) & \text{for } p \in (-R,R), \\ f(x,R) + \partial_p f(x,R)(p-R) & \text{for } p \ge R, \end{cases} \\
Q_R(p) &= \max\{-R, \min\{p,R\}\}, \\
\mu_R(x,p) &= \mu(x, Q_R(p)), \\
\bar{\mu}_R(x,p) &= \bar{\mu}(x, Q_R(p)) = \frac{\mu(x,Q_R(p))}{\mu_0(x)}.
\end{aligned}
\tag{4.1}
$$

Instead of (3.11)–(3.12), we further choose (small) parameters $\delta > 0$, $\varepsilon > 0$ and for all test functions $\phi \in X_3$, $\psi \in X$, and $\lambda \in L^\infty_T$ we consider the approximate system

$$
\begin{aligned}
&\int_T^{2T}\int_\Omega \big((\rho u_{tt} + \delta|u_t|u_t)\phi + (\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u + \varepsilon\mathbf{B}(x)\nabla_s u_{tt}) : \nabla_s\phi + \nabla p\ \phi\big)\lambda(t)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_T^{2T}\int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g)\phi\lambda(t)\mathrm{d}s(x)\mathrm{d}t = 0,
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
&\int_T^{2T}\int_\Omega \Big(\big((f_R(x,p) + G[p])_t + \delta|p|p\big)\psi + (\mu_R(x,p)\nabla p - u_t)\nabla\psi\Big)\lambda(t)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_T^{2T}\int_{\partial\Omega} \alpha(x)(p-\bar{p})\psi\lambda(t)\mathrm{d}s(x)\mathrm{d}t = 0.
\end{aligned}
\tag{4.3}
$$

The strategy is the following. We first prove the solvability of (4.2)–(4.3) for fixed $\delta > 0$, $\varepsilon > 0$, $R > 0$ via a Galerkin scheme. Then, we derive estimates for the solution to (4.2)–(4.3) independent of the regularizing parameters δ, ε, R. Further, we apply a time-periodic Moser-Alikakos iteration scheme (see [1]) which yields an upper bound for $\sup_{(x,t)\in\Omega\times\mathbb{R}} |p(x,t)|$ and will allow us to remove the R-truncation of p by choosing R sufficiently large. Finally, we prove that passing to the limit as $\delta \to 0$ and $\varepsilon \to 0$ we obtain a solution to the original problem (3.11)–(3.12).

The remaining part of this section is devoted to the proof of the following result.

Proposition 4.1.1. *Let Hypotheses 3.1.1, 3.1.2 hold. Then there exists a solution* (u,p) *to* (4.2)–(4.3) *such that* $u, u_t, \nabla p \in L^\infty_T(L^2(\Omega;\mathbb{R}^3))$, $u, u_t, u_{tt} \in L^2_T(W^{1,2}(\Omega;\mathbb{R}^3))$, $p_t \in L^2_T(L^2(\Omega))$.

4.1.1 Galerkin approximations

We choose orthonormal bases $\{\phi_l\}_{l=0}^\infty$ in $L^2(\Omega)$ and $\{e_k\}_{k=-\infty}^\infty$ in L^2_T as

$$\begin{aligned} -\Delta\phi_l &= \lambda_l\phi_l \ \text{ in } \ \Omega\,,\ \nabla\phi_l\cdot n = 0 \ \text{ on } \ \partial\Omega, \\ e_k(t) &= \begin{cases} \frac{2}{T}\sin\frac{2\pi k}{T}t & \text{for } k\ge 1\,, \\ \frac{1}{T} & \text{for } k = 0\,, \\ \frac{2}{T}\cos\frac{2\pi k}{T}t & \text{for } k \le -1\,. \end{cases} \end{aligned} \tag{4.4}$$

Note that for every $k \in \mathbb{Z}$ we have

$$\dot{e}_k(t) = \frac{2\pi k}{T} e_{-k}(t), \tag{4.5}$$

where the dot denotes here and in the sequel the derivative with respect to t.

For a fixed $m \in \mathbb{N}$ we consider Galerkin approximations

$$\begin{aligned} u_j^{(m)}(x,t) &= \sum_{k=-m}^{m}\sum_{l=0}^{m} u_{jkl}\phi_l(x)e_k(t)\,, \\ v^{(m)}(x,t) &= \sum_{k=-m}^{m}\sum_{l=0}^{m} v_{kl}\phi_l(x)e_k(t)\,. \end{aligned} \tag{4.6}$$

We denote by

$$M_R(x,p) = \int_0^p \bar{\mu}_R(x,p')\,\mathrm{d}p' \tag{4.7}$$

the Kirchhoff transform associated with $\bar{\mu}_R$, and by M_R^{-1} its partial inverse defined by the identity $M_R^{-1}(x, M_R(x,p)) = p$ for $(x,p) \in \Omega\times\mathbb{R}$. We put

$$p^{(m)} = M_R^{-1}(x, v^{(m)}) \tag{4.8}$$

and consider the scalars u_{jkl}, v_{kl} the unknowns of the problem

$$\int_T^{2T}\int_\Omega \big((\rho u_{tt}^{(m)} + \delta|u_t^{(m)}|u_t^{(m)})\phi_l\eta_j + (\mathbf{B}(x)\nabla_s u_t^{(m)} + \mathbf{A}(x)\nabla_s u^{(m)} + \varepsilon\mathbf{B}(x)\nabla_s u_{tt}^{(m)}) : \nabla_s(\phi_l\eta_j)$$
$$+ \nabla p^{(m)}\ \phi_l\eta_j\big)e_{-k}(t)\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u^{(m)} + \mathbf{D}(x)u_t^{(m)} - g)\phi_l\eta_j e_{-k}(t)\mathrm{d}s(x)\mathrm{d}t = 0, \tag{4.9}$$

$$\int_T^{2T}\int_\Omega \Bigg(\big((f_R(x,p^{(m)}) + G[p^{(m)}]\big)_t + \delta|p^{(m)}|p^{(m)}\big)\phi_l$$
$$+ \Big(\mu_0(x)\left(\nabla v^{(m)} - \nabla_x M_R(x,p^{(m)})\right) - u_t^{(m)}\Big)\nabla\phi_l\Bigg)e_k(t)\mathrm{d}x\mathrm{d}t$$
$$+ \int_T^{2T}\int_{\partial\Omega} \alpha(x)(p^{(m)} - \bar{p})\phi_l e_k(t)\mathrm{d}s(x)\mathrm{d}t = 0, \tag{4.10}$$

for $j = 1,2,3$, $l = 0,\dots,m$, and $k = -m,\dots,m$, where η_j are the vectors

$$\eta_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \eta_2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\ \eta_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Notice that we used in (4.10) the identity $\bar{\mu}_R(x,p^{(m)})\nabla p^{(m)} = \nabla v^{(m)} - \nabla_x M_R(x,p^{(m)})$ according to (4.8).

The existence of a solution to (4.9)–(4.10) will be proved by a topological degree argument (see Appendix B). The left-hand side of (4.9)–(4.10) defines a mapping $H : \mathbb{R}^{4(m+1)(2m+1)} \to \mathbb{R}^{4(m+1)(2m+1)}$, and we look for a solution $z \in \mathbb{R}^{4(m+1)(2m+1)}$ to the equation $H(z) = 0$. We define a homotopy H_χ for $\chi \in [0,1]$ such that $H_1 = H$ and H_0 is an odd mapping, and find a sufficiently large ball $\mathcal{B} \subset \mathbb{R}^{4(m+1)(2m+1)}$ such that the equation $H_\chi(z) = 0$ has no solution on $\partial\mathcal{B}$ for any $\chi \in [0,1]$. Since the topological degree of H_0 with respect to $\mathcal{B}$ is odd, we conclude that also the degree of H_1 is odd, and this will enable us to conclude that the equation $H_1(z) = 0$ has a solution.

The mapping H_χ is defined as the left-hand side of the system

$$\int_T^{2T}\int_\Omega \big((\rho u_{tt}^{(m)} + \delta|u_t^{(m)}|u_t^{(m)})\phi_l\eta_j + (\mathbf{B}(x)\nabla_s u_t^{(m)} + \mathbf{A}(x)\nabla_s u^{(m)} + \varepsilon\mathbf{B}(x)\nabla_s u_{tt}^{(m)}) : \nabla_s(\phi_l\eta_j)$$
$$+ \chi\nabla p^{(m)}\ \phi_l\eta_j\big)e_{-k}(t)\mathrm{d}x\mathrm{d}t$$
$$+ \int_T^{2T}\int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u^{(m)} + \mathbf{D}(x)u_t^{(m)} - \chi g)\phi_l\eta_j e_{-k}(t)\mathrm{d}s(x)\mathrm{d}t = 0, \tag{4.11}$$

$$\int_T^{2T}\int_\Omega \Big(\big((\chi(f_R(x,p^{(m)}) + G[p^{(m)}]) + (1-\chi)v^{(m)}\big)\big)_t + \delta|p^{(m)}|p^{(m)}\Big)\phi_l$$

$$+\Big(\mu_0(x)(\nabla v^{(m)} - \chi\nabla_x M_R(x,p^{(m)})) - u_t^{(m)}\Big)\nabla\phi_l\Big)e_k(t)\mathrm{d}x\mathrm{d}t$$
$$+\int_T^{2T}\int_{\partial\Omega}\alpha(x)\Big(\chi(p^{(m)}-\bar{p}) + (1-\chi)v^{(m)}\Big)\phi_l e_k(t)\mathrm{d}s(x)\mathrm{d}t = 0, \tag{4.12}$$

for $j = 1,2,3$, $l = 0,\dots,m$, and $k = -m,\dots,m$.

Testing (4.11) by $\frac{2k\pi}{T}u_{jkl}$ and (4.10) by v_{kl}, and using the fact that the integral over the period of the time derivative of a periodic function vanishes, we obtain

$$\int_T^{2T}\int_\Omega\Big(\delta|u_t^{(m)}|^3 + \mathbf{B}(x)\nabla_s u_t^{(m)} : \nabla_s u_t^{(m)} + \chi\nabla p^{(m)} u_t^{(m)}\Big)\mathrm{d}x\mathrm{d}t$$
$$+\int_T^{2T}\int_{\partial\Omega}\big(\beta(x)(\mathbf{D}(x)u_t^{(m)}\cdot u_t^{(m)} - \chi g u_t^{(m)}\big)\mathrm{d}s(x)\mathrm{d}t = 0, \tag{4.13}$$

$$\int_T^{2T}\int_\Omega\Big(G[p^{(m)}])_t v^{(m)} + \delta|p^{(m)}|p^{(m)}v^{(m)} + \big(\mu_0(x)(\nabla v^{(m)} - \chi\nabla_x M_R(x,p^{(m)})) + u_t^{(m)}\big)\cdot\nabla v^{(m)}\Big)\mathrm{d}x\mathrm{d}t$$
$$+\int_T^{2T}\int_{\partial\Omega}(\alpha(x)\Big(\chi(p^{(m)}-\bar{p}) + (1-\chi)v^{(m)}\Big)v^{(m)})\mathrm{d}s(x)\mathrm{d}t = 0. \tag{4.14}$$

In order to deal with the term $G[p^{(m)}]_t v^{(m)}$, we introduce the modified Preisach potential

$$V_{M,R}[p] := \int_0^\infty\int_0^{\mathfrak{f}_r[p]} M_R(x,v)\psi(x,r,v)\mathrm{d}v\mathrm{d}r \geq 0, \tag{4.15}$$

which satisfies the inequality

$$G[p]_t M_R(p) - V_{M,R}[p]_t \geq 0 \quad a.\,e. \tag{4.16}$$

in agreement with Proposition 3.3.5. We also have

$$\frac{1}{C(R)}(|v^{(m)}|^2 + |p^{(m)}|^2) \leq v^{(m)}p^{(m)} \leq C(R)(|v^{(m)}|^2 + |p^{(m)}|^2)$$

with a constant $C(R)$ depending only on R. More generally, in all the estimates below, c and C denote some non-specified constants, possibly different in different estimates, independent of δ, ε, m and R. Similarly, $C(R)$ and $C(R,\delta)$ denote constants which may possibly depend only on R, or only on R and δ, respectively.

Summing up (4.13)–(4.14) and exploiting Hypothesis 3.1.1 (ii) we thus get

$$\int_T^{2T}\int_\Omega\Big(|\nabla v^{(m)}|^2 + \mathbf{B}(x)\nabla_s u_t^{(m)} : \nabla_s u_t^{(m)} + \delta(|u_t^{(m)}|^3 + |p^{(m)}|^3)\Big)\mathrm{d}x\mathrm{d}t$$
$$+\int_T^{2T}\int_{\partial\Omega}(\alpha(x)|v^{(m)}|^2 + \beta(x)\mathbf{D}(x)u_t^{(m)}u_t^{(m)})\mathrm{d}s(x)\mathrm{d}t$$

$$\leq C(R)\bigg(1+\int_T^{2T}\int_\Omega(|u_t^{(m)}|+|p^{(m)}|)|\nabla v^{(m)}|\bigg)\mathrm{d}x\mathrm{d}t. \tag{4.17}$$

Using Hölder's inequality with exponents $3, 2$ and 6 and then Young's inequality with exponents 3 and 3/2 we have

$$\begin{aligned}
&\int_T^{2T}\int_\Omega|u_t^{(m)}||\nabla v^{(m)}|\mathrm{d}x\mathrm{d}t\\
&\leq\delta^{\frac{1}{3}}\bigg(\int_T^{2T}\int_\Omega|u_t^{(m)}|^3\mathrm{d}x\mathrm{d}t\bigg)^{1/3}\delta^{-\frac{1}{3}}\bigg(\int_T^{2T}\int_\Omega|\nabla v^{(m)}|^2\mathrm{d}x\mathrm{d}t\bigg)^{1/2}\bigg(\int_T^{2T}\int_\Omega 1\mathrm{d}x\mathrm{d}t\bigg)^{1/6}\\
&\leq\frac{1}{3}\delta\int_T^{2T}\int_\Omega|u_t^{(m)}|^3\mathrm{d}x\mathrm{d}t+\frac{2}{3}C^{3/2}\delta^{-\frac{1}{2}}\bigg(\int_T^{2T}\int_\Omega|\nabla v^{(m)}|^2\mathrm{d}x\mathrm{d}t\bigg)^{3/4}
\end{aligned}$$

and similarly for the term $\int_T^{2T}\int_\Omega|p^{(m)}||\nabla v^{(m)}|\mathrm{d}x\mathrm{d}t$. From (4.17) we thus deduce the inequality

$$\int_T^{2T}\int_\Omega|\nabla v^{(m)}|^2\mathrm{d}x\mathrm{d}t\leq C\bigg(1+\delta^{-\frac{1}{2}}\bigg(\int_T^{2T}\int_\Omega|\nabla v^{(m)}|^2\mathrm{d}x\mathrm{d}t\bigg)^{3/4}\bigg),$$

which is an inequality of the form

$$V\leq C(R)\left(1+\frac{1}{\sqrt{\delta}}V^{\frac{3}{4}}\right) \tag{4.18}$$

for $V=\int_T^{2T}\int_\Omega|\nabla v^{(m)}|^2\mathrm{d}x\mathrm{d}t$. From Young's inequality (with exponents 4 and $\frac{4}{3}$) it follows that

$$\frac{C(R)}{\sqrt{\delta}}V^{3/4}\leq\frac{1}{4}\frac{C(R)^4}{\delta^2}+\frac{3}{4}V.$$

Then from (4.18), we infer

$$\frac{1}{4}V\leq C(R)+\frac{1}{4}\frac{C(R)^4}{\delta^2},$$

therefore

$$V\leq 4C(R)+\frac{C(R)^4}{\delta^2}=:C(R,\delta).$$

Putting everything together, (4.17) gives the following estimate

$$\begin{aligned}
&\int_T^{2T}\int_\Omega|\nabla v^{(m)}|^2+\mathbf{B}(x)\nabla_s u_t^{(m)}:\nabla_s u_t^{(m)}+\delta\big(|u_t^{(m)}|^3+|p^{(m)}|^3\big)\mathrm{d}x\mathrm{d}t\\
&\quad+\int_T^{2T}\int_{\partial\Omega}\big(\alpha(x)|v^{(m)}|^2+\beta(x)|u_t^{(m)}|^2\big)\mathrm{d}s(x)\mathrm{d}t\leq C(R,\delta).
\end{aligned} \tag{4.19}$$

We see that all possible solutions of the equation $H_\chi(z) = 0$ remain bounded independently of $\chi \in [0,1]$. Hence, the topological degree of H_1 with respect to a sufficiently large ball is the same as the topological degree of H_0, which is odd. We conclude that a solution to (4.9)–(4.10) exists, see [119, § 13.6]. In the next subsection, we derive further estimates which will enable us to pass to the limit as $m \to \infty$.

4.1.2 Limit as $m \to \infty$

We now test (4.9) by $-\left(\frac{2\pi k}{T}\right)^2 u_{j(-k)l}$ and get

$$\begin{aligned}
&\int_T^{2T}\int_\Omega \big(\rho|u_{tt}^{(m)}|^2 + \varepsilon\mathbf{B}(x)|\nabla_s u_{tt}^{(m)}|^2 \mathrm{d}x\mathrm{d}t \\
&\quad= \int_T^{2T}\int_\Omega (\mathbf{A}(x)\nabla_s u_t^{(m)} : \nabla_s u_t^{(m)} - \nabla p^{(m)}\ u_{tt}^{(m)})\mathrm{d}x\mathrm{d}t \\
&\quad+ \int_T^{2T}\int_{\partial\Omega} \beta(x)\big(\mathbf{C}(x)u_t^{(m)}\cdot u_t^{(m)} + g_t u_t^{(m)}\big)\mathrm{d}s(x)\mathrm{d}t
\end{aligned}$$

which leads to

$$\int_T^{2T}\int_\Omega \big(\rho|u_{tt}^{(m)}|^2 + \varepsilon\mathbf{B}(x)\nabla_s u_{tt}^{(m)} : \nabla_s u_{tt}^{(m)}\mathrm{d}x\mathrm{d}t \le C(R,\delta), \tag{4.20}$$

where we used Hypothesis 3.1.1 and the estimate (4.19). Testing (4.9) by $u_{j(-k)l}$ we infer

$$\begin{aligned}
&\int_T^{2T}\int_\Omega \Big(\delta|u_t^{(m)}|^2 u^{(m)} + \mathbf{A}(x)\nabla_s u^{(m)} : \nabla_s u^{(m)} + u^{(m)}\nabla p^{(m)}\Big)\,\mathrm{d}x\mathrm{d}t \\
&- \int_T^{2T}\int_\Omega \Big(\rho|u_t^{(m)}|^2 + \varepsilon\mathbf{B}(x)\nabla_s u_t^{(m)} : \nabla_s u_t^{(m)}\Big)\,\mathrm{d}x\mathrm{d}t \\
&+ \int_T^{2T}\int_{\partial\Omega} \beta(x)\,\Big(\mathbf{C}(x)u^{(m)}\cdot u^{(m)} - g\ u_t^{(m)}\Big)\,\mathrm{d}s\mathrm{d}t = 0.
\end{aligned}$$

The terms $\rho|u_t^{(m)}|^2$ and $\varepsilon B\nabla_s u_t^{(m)} : \nabla_s u_t^{(m)}$ are under control because of (4.19), Hypothesis 3.1.1 and the Poincaré inequality. Moreover, we can handle the term $\delta|u_t^{(m)}|^2 u^{(m)}$ thanks to the embedding inequality

$$\begin{aligned}
&\left(\int_T^{2T}\int_\Omega |u_t^{(m)}|^4\mathrm{d}x\mathrm{d}t\right)^{1/4} \\
&\quad\le C\left[\left(\int_T^{2T}\int_\Omega |u_{tt}|^2\mathrm{d}x\mathrm{d}t\right)^{1/2} + \left(\int_T^{2T}\int_\Omega |\nabla_s u_t^{(m)}|^2\mathrm{d}x\mathrm{d}t\right)^{1/2}\right]
\end{aligned} \tag{4.21}$$

Thus it holds

$$\int_T^{2T}\int_\Omega \mathbf{A}(x)\nabla_s u^{(m)} : \nabla_s u^{(m)} \mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega} \beta(x)\mathbf{C}(x)u^{(m)} \cdot u^{(m)} \mathrm{d}s(x)\mathrm{d}t \leq C(R,\delta). \tag{4.22}$$

Finally, we test (4.10) by $-\frac{2\pi k}{T} v_{-kl}$ and obtain

$$\begin{aligned}
&\int_T^{2T}\int_\Omega f_R(p^{(m)})_t v_t^{(m)} \mathrm{d}x\mathrm{d}t \\
&\leq C(R,\delta)\left(\int_T^{2T}\int_\Omega (|p_t^{(m)}| + |u_{tt}^{(m)}|)|\nabla v^{(m)}|\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega} \alpha(x)|\bar{p}_t|\,|v^{(m)}|\mathrm{d}s(x)\mathrm{d}t\right).
\end{aligned}$$

From the previous estimates we get

$$\int_T^{2T}\int_\Omega |v_t^{(m)}|^2 \mathrm{d}x\mathrm{d}t \leq C(R,\delta). \tag{4.23}$$

The estimates (4.19), (4.20), (4.22), and (4.23) are sufficient for using the compactness argument and pass to the limit in (4.9)–(4.10) as $m \to \infty$, and get a solution to the regularized system (4.2)–(4.3), completing thus the proof of Proposition 4.1.1.

4.2 Estimates independent of δ and ε

System (4.2)–(4.3) can be equivalently written as

$$\begin{aligned}
&\int_\Omega \big((\rho u_{tt} + \delta|u_t|u_t)\phi + (\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u + \varepsilon\mathbf{B}(x)\nabla_s u_{tt}) : \nabla_s\phi + \nabla p\ \phi\big)\mathrm{d}x \\
&\quad + \int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g)\phi\mathrm{d}s(x) = 0, \qquad (4.24)\\
&\int_\Omega \Big(\big((f_R(x,p) + G[p])_t + \delta|p|p\big)\psi + (\mu_R(x,p)\nabla p - u_t)\nabla\psi\Big)\mathrm{d}x \\
&\quad + \int_{\partial\Omega} \alpha(x)(p - \bar{p})\psi\mathrm{d}s(x) = 0 \qquad (4.25)
\end{aligned}$$

for almost all $t \in (T, 2T)$, and therefore admits a solution with the regularity stated in Proposition 4.1.1. In order to derive estimates independent of δ and ε, We test (4.24) by $\phi = u_t$ and (4.25) by $\psi = p$ and obtain, using (3.27) with $h(p) = p$, that

$$\int_T^{2T}\int_\Omega \left(\delta|u_t|^3 + \mathbf{B}(x)\nabla_s u_t : \nabla_s u_t + \nabla p \cdot u_t\right) \mathrm{d}x\mathrm{d}t$$

$$+\int_T^{2T}\int_{\partial\Omega}\beta(x)\left(\mathbf{D}(x)u_t\cdot u_t - gu_t\right)\mathrm{d}s(x)\mathrm{d}t = 0,$$

$$\int_T^{2T}\int_\Omega\big(\mu_R(x,p)|\nabla p|^2 + \delta|p|^3 - \nabla p\cdot u_t\big)\,\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega}\big(\alpha(x)p^2 - \alpha(x)\bar{p}p\big)\,\mathrm{d}s(x)\mathrm{d}t \le 0.$$

Summing up the two above relations we get

$$\int_T^{2T}\int_\Omega\big(\delta(|u_t|^3+|p|^3) + \mathbf{B}(x)\nabla_s u_t : \nabla_s u_t + \mu_R(x,p)|\nabla p|^2\big)\,\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega}\big(\beta(x)\mathbf{D}(x)u_t\cdot u_t + \alpha(x)p^2\big)\,\mathrm{d}s(x)\mathrm{d}t \le C \tag{4.26}$$

as a consequence of Hypothesis 3.1.1 and Hölder's inequality. As a next step, we test (4.24) by $\phi = u_{tt}$. Integrating by parts we infer

$$\begin{aligned}&\int_T^{2T}\int_\Omega\big(\rho|u_{tt}|^2 + \varepsilon\mathbf{B}(x)\nabla_s u_{tt} : \nabla_s u_{tt}\big)\,\mathrm{d}x\mathrm{d}t \qquad (4.27)\\ =\ & \int_T^{2T}\int_\Omega\left(\mathbf{A}(x)\nabla_s u_t : \nabla_s u_t - u_{tt}\nabla p\right)\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega}\left(\beta(x)\mathbf{C}(x)u_t\cdot u_t - \beta(x)g_t\, u_t\right)\mathrm{d}s(x)\mathrm{d}t.\end{aligned}$$

Put

$$\Gamma_R := (1+R^2)^\gamma. \tag{4.28}$$

We have by (4.1), (4.26), and Hypothesis 3.1.1 (ii) that

$$\int_T^{2T}\int_\Omega|\nabla p|^2\mathrm{d}x\mathrm{d}t \le \frac{1}{\mu^\flat}\Gamma_R\int_T^{2T}\int_\Omega\mu_R(x,p)|\nabla p|^2\mathrm{d}x\mathrm{d}t \le C\Gamma_R, \tag{4.29}$$

and (4.26)–(4.27) yield that

$$\int_T^{2T}\int_\Omega\big(\rho|u_{tt}|^2 + \varepsilon\mathbf{B}(x)\nabla_s u_{tt} : \nabla_s u_{tt}\big)\,\mathrm{d}x\mathrm{d}t \le C\Gamma_R\,. \tag{4.30}$$

The next estimate is obtained by testing (4.24) by $\phi = u$ and integrating by parts in time:

$$\begin{aligned}&\int_T^{2T}\int_\Omega\mathbf{A}(x)\nabla_s u : \nabla_s u\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega}\beta(x)\mathbf{C}(x)u\cdot u\mathrm{d}s(x)\mathrm{d}t\\ &\quad\le \int_T^{2T}\int_\Omega\big(\delta|u_t|^2|u| + |u|\,|\nabla p| + \rho|u_t|^2 + \varepsilon\mathbf{B}(x)\nabla_s u_t : \nabla_s u_t\big)\,\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega}\beta(x)gu\mathrm{d}s(x)\mathrm{d}t,\end{aligned} \tag{4.31}$$

and from (4.26) and Hölder's inequality we get

$$\int_T^{2T}\int_\Omega \mathbf{A}(x)\nabla_s u : \nabla_s u \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_T^{2T}\int_{\partial\Omega} \beta(x)\mathbf{C}(x)u\cdot u\mathrm{d}s(x)\mathrm{d}t \\ \leq \int_T^{2T}\int_\Omega \left(\delta|u_t|^2|u| + |u|\,|\nabla p|\right)\mathrm{d}x\mathrm{d}t + C. \tag{4.32}$$

To estimate the right hand side of (4.32), we rewrite $\delta|u_t|^2|u|$ as $\delta^{2/3}|u_t|^2\delta^{1/3}|u|$, and use Hölder's inequality to obtain

$$\int_T^{2T}\int_\Omega \delta|u_t|^2|u|\mathrm{d}x\mathrm{d}t \leq \left(\int_T^{2T}\int_\Omega \delta|u_t|^3\mathrm{d}x\mathrm{d}t\right)^{2/3}\left(\int_T^{2T}\int_\Omega \delta|u|^3\mathrm{d}x\mathrm{d}t\right)^{1/3} \\ \leq C\left(\int_T^{2T}\int_\Omega \delta|u|^3\mathrm{d}x\mathrm{d}t\right)^{1/3} \tag{4.33}$$

by virtue of (4.26). In order to estimate the remaining term on the right-hand side of (4.32), we exploit Hypothesis 3.1.1 and Hölder's inequality, namely

$$\int_T^{2T}\int_\Omega |u|\,|\nabla p|\mathrm{d}x\mathrm{d}t \leq C\int_T^{2T}\int_\Omega \sqrt{\mu_R(x,p)}|\nabla p||u|(1+p^2)^{\gamma/2}\mathrm{d}x\mathrm{d}t \\ \leq C\left(\int_T^{2T}\int_\Omega \mu_R(x,p)|\nabla p|^2\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_T^{2T}\int_\Omega |u|^2(1+p^2)^\gamma\mathrm{d}x\mathrm{d}t\right)^{1/2} \\ \leq C\left(\int_T^{2T}\int_\Omega |u|^4\mathrm{d}x\mathrm{d}t\right)^{1/4}\left(\int_T^{2T}\int_\Omega (1+p^2)^{2\gamma}\mathrm{d}x\mathrm{d}t\right)^{1/4}$$

where the term $\int_T^{2T}\int_\Omega \mu_R(x,p)|\nabla p|^2\mathrm{d}x\mathrm{d}t$ was estimated according to (4.26). By the Sobolev embedding theorem, we have

$$\left(\int_T^{2T}\int_\Omega |u|^4\mathrm{d}x\mathrm{d}t\right)^{1/4} \leq \left(\int_T^{2T}\int_\Omega \mathbf{A}(x)\nabla_s u : \nabla_s u\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_T^{2T}\int_{\partial\Omega}\beta(x)\mathbf{C}(x)u\cdot u\mathrm{d}s(x)\mathrm{d}t\right)^{1/2} \\ + \left(\int_T^{2T}\int_\Omega |u_t|^2\mathrm{d}x\mathrm{d}t\right)^{1/2},$$

and a similar estimate holds for the right-hand side of (4.33). From the above computations, Young's inequality, and (4.32) we thus get

$$\int_T^{2T}\int_\Omega \mathbf{A}(x)\nabla_s u : \nabla_s u\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega}\beta\mathbf{C}(x)u\cdot u\mathrm{d}s\mathrm{d}t \leq C\left(1 + \int_T^{2T}\int_\Omega (1+p^2)^{2\gamma}\mathrm{d}x\mathrm{d}t\right)^{1/2}. \tag{4.34}$$

To estimate the right-hand side of (4.34), we define an auxiliary function

$$h(p) := p(1+p^2)^{-\gamma/2}. \tag{4.35}$$

We have $0 \le h'(p) \le (1+\gamma)(1+p^2)^{-\gamma/2} \le C\sqrt{\mu_R(x,p)}$ for all $p \in \mathbb{R}$. Hence, by Sobolev embedding and by (4.26) we have

$$\begin{aligned}\int_T^{2T}\!\!\int_\Omega |h(p)|^2 \mathrm{d}x\mathrm{d}t &\le C\left(\int_T^{2T}\!\!\int_\Omega |\nabla h(p)|^2 \mathrm{d}x\mathrm{d}t + \int_T^{2T}\!\!\int_{\partial\Omega} \alpha(x)|h(p)|^2 \mathrm{d}s(x)\mathrm{d}t\right)\\ &\le C\left(\int_T^{2T}\!\!\int_\Omega \mu_R(x,p)|\nabla p|^2 \mathrm{d}x\mathrm{d}t + \int_T^{2T}\!\!\int_{\partial\Omega} \alpha(x)|p|^2 \mathrm{d}s(x)\mathrm{d}t\right) \le C.\end{aligned}$$

We thus have

$$\int_T^{2T}\!\!\int_\Omega (1+p^2)^{1-\gamma} \mathrm{d}x\mathrm{d}t \le \int_T^{2T}\!\!\int_\Omega (1+|h(p)|^2) \mathrm{d}x\mathrm{d}t \le C.$$

Since $(1-\gamma) > 2\gamma$ by (3.13), we conclude from (4.34) that

$$\int_T^{2T}\!\!\int_\Omega \mathbf{A}(x)\nabla_s u : \nabla_s u \mathrm{d}x\mathrm{d}t + \int_T^{2T}\!\!\int_{\partial\Omega} \beta\mathbf{C}(x)u\cdot u \mathrm{d}s\mathrm{d}t \le C. \tag{4.36}$$

4.3 L^∞-bounds

Testing (4.24) again by $\phi = u_t$ we get

$$\begin{aligned}&\int_\Omega \big(\rho u_{tt}\cdot u_t + \delta|u_t|^3 + \mathbf{B}(x)\nabla_s u_t : \nabla_s u_t + \mathbf{A}(x)\nabla_s u : \nabla_s u_t + \varepsilon\mathbf{B}(x)\nabla_s u_{tt} : \nabla_s u_t + \nabla p\cdot\ u_t\big)\,\mathrm{d}x\\ &\quad + \int_{\partial\Omega} (\beta(x)\mathbf{C}(x)u\cdot u_t + \beta(x)\mathbf{D}(x)u_t\cdot u_t - \beta(x)g\cdot u_t)\,\mathrm{d}s(x) = 0\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \big(\rho|u_t|^2 + \mathbf{A}(x)\nabla_s u : \nabla_s u + \varepsilon\mathbf{B}(x)\nabla_s u_t : \nabla_s u_t\big)\,\mathrm{d}x + \int_{\partial\Omega} \beta(x)\mathbf{C}(x)u\cdot u\mathrm{d}s(x)\right)\\ &\quad + \int_\Omega \big(\delta|u_t|^3 + \mathbf{B}(x)\nabla_s u_t : \nabla_s u_t\big)\,\mathrm{d}x + \int_{\partial\Omega} \beta(x)\,(\mathbf{D}(x)u_t\cdot u_t)\,\mathrm{d}s(x)\\ &\quad = \int_\Omega -\nabla p\cdot u_t \mathrm{d}x + \int_{\partial\Omega} \beta(x)g\cdot u_t \mathrm{d}s(x) \qquad \text{a.e. in } (T,2T).\end{aligned} \tag{4.37}$$

Let $y(t)$ denote the expression under the time derivative in (4.37), and let $z(t)$ denote the remaining terms. According to (4.26), (4.36), and Hypothesis 3.1.1, Eq. (4.37) is of the type $\dot{y}(t) = z(t)$ with $y, z \in L^1_T$, and

$$\int_T^{2T} |z|\mathrm{d}t \le C + \left(\int_T^{2T}\int_\Omega |\nabla p|^2 \mathrm{d}x\mathrm{d}t\right)^{1/2} \left(\int_T^{2T}\int_\Omega |u_t|^2 \mathrm{d}x\mathrm{d}t\right)^{1/2} \le C\sqrt{\Gamma_R}.$$

by virtue of (4.26) and (4.29). From the identity

$$y(t) = \frac{1}{T}\int_{t-T}^{t} (y(s) + (T+s-t)z(s))\mathrm{d}s$$

it follows that the L^1-norm of y and z control the L^∞-norm of y and so it follows in particular that

$$\sup_{t\in(T,2T)}\mathrm{ess} \int_\Omega \left(|u_t|^2 + |\nabla_s u|^2 + \varepsilon|\nabla_s u_t|^2\right)\mathrm{d}x + \int_{\partial\Omega} \beta(x)|u|^2 \mathrm{d}s(x) \le C\sqrt{\Gamma_R}. \tag{4.38}$$

Then we test (4.24) by $\phi = u_{tt}$. The term $\sqrt{\varepsilon}u_{tt}$ belongs to $L^2_T(W^{1,2}(\Omega))$, so that it is an admissible choice. This yields for a. e. $t \in (T, 2T)$ that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\mathbf{B}(x)\nabla_s u_t : \nabla_s u_t + \mathbf{A}(x)\nabla_s u : \nabla_s u_t + \frac{\delta}{3}|u_t|^3\right)\mathrm{d}x \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega} \beta(x)\left(\mathbf{C}(x)u\cdot u_t + \frac{1}{2}\mathbf{D}(x)u_t\cdot u_t\right)\mathrm{d}s(x) + \int_\Omega \left(\rho u_{tt}^2 + \varepsilon\mathbf{B}(x)\nabla_s u_{tt} : \nabla_s u_{tt}\right)\mathrm{d}x \\
&= \int_\Omega \left(-\nabla p\cdot u_{tt} - \mathbf{A}(x)\nabla_s u_t : \nabla_s u_t\right)\mathrm{d}x + \int_{\partial\Omega} \beta(x)\left(-g_t\cdot u_t + \mathbf{C}(x)u_t\cdot u_t\right)\mathrm{d}s(x)
\end{aligned}$$

Arguing similarly as above and using (4.26), (4.29)–(4.30) and (4.38), we get

$$\sup_{t\in(T,2T)}\mathrm{ess} \int_\Omega |\nabla_s u_t|^2 \mathrm{d}x + \int_{\partial\Omega} \beta(x)|u_t|^2 \mathrm{d}s(x) \le C\Gamma_R. \tag{4.39}$$

By Sobolev embedding of $W^{1,2}(\Omega)$ into $L^6(\Omega)$ we obtain from (4.39) that

$$\sup_{t\in(0,T)}\mathrm{ess}\, |u_t(t)|_6^2 \le C\Gamma_R. \tag{4.40}$$

The next strategy is to test (4.25) by v_t with

$$v = M_R(x,p), \tag{4.41}$$

see (4.7). However, there is no evidence that $v_t(\cdot,t) \in W^{1,2}(\Omega)$, so that this is not an admissible choice. Instead, we therefore test (4.25) by $\hat{v}(x,t) = \frac{1}{\tau}(v(x,t) - v(x,t-\tau))\lambda(t)$, where $\tau > 0$ is a small time shift, and λ is a positive Lipschitz continuous T-periodic function. We get

$$\begin{aligned}&\int_\Omega \big((f_R(x,p)+G[p])_t+\delta|p|p\big)\frac{1}{\tau}(v(x,t){-}v(x,t{-}\tau)\lambda(t)\mathrm{d}x\\&+\int_\Omega \mu_R(x,p)\nabla p\cdot\nabla\left(\frac{v(x,t)-v(x,t-\tau)}{\tau}\lambda(t)\right)\mathrm{d}x\\&-\int_\Omega u_t\cdot\nabla\left(\frac{v(x,t)-v(x,t-\tau)}{\tau}\lambda(t)\right)\mathrm{d}x+\int_{\partial\Omega}\alpha(x)(p-\bar p)\frac{v(x,t)-v(x,t-\tau)}{\tau}\lambda(t)=0\end{aligned} \tag{4.42}$$

We now treat separately each of the four integrals on the left hand side of (4.42), which we write for simplicity as $I+II+III+IV$. In order to evaluate the term I, we note that for a general function $v_0\in W^{1,2}_T$ we have

$$\begin{aligned}\int_T^{2T}\left|\dot v_0(t)-\frac{1}{\tau}(v_0(t)-v_0(t-\tau))\right|^2\mathrm{d}t&=\int_T^{2T}\left|\dot v_0(t)-\frac{1}{\tau}\int_{t-\tau}^t\dot v_0(s)\mathrm{d}s\right|^2\mathrm{d}t\\&=\frac{1}{\tau^2}\int_T^{2T}\int_{t-\tau}^t\left(|\dot v_0(t)-\dot v_0(s)|\mathrm{d}s\right)^2\mathrm{d}t\le\frac{1}{\tau}\int_T^{2T}\int_{t-\tau}^t|\dot v_0(t)-\dot v_0(s)|^2\mathrm{d}s\mathrm{d}t\\&\le\frac{1}{\tau}\int_T^{2T}\int_0^\tau|\dot v_0(t)-\dot v_0(t-h)|^2\mathrm{d}h\mathrm{d}t=\frac{1}{\tau}\int_0^\tau\int_T^{2T}|\dot v_0(t)-\dot v_0(t-h)|^2\mathrm{d}t\mathrm{d}h\end{aligned}$$

where $\dot v_0(t)$ is defined as the time derivative of $v_0(t)$. From Lusin's Theorem it follows that $\frac{1}{\tau}(v_0(t)-v_0(t-\tau))$ converge strongly to $\dot v_0$ in L^2_T as $\tau\to 0$. Hence, $\hat v$ converge strongly in $L^2_T(L^2(\Omega))$ to $v_t\lambda(t)$, with $v_t=\bar\mu(x,p)p_t$, and we get

$$\begin{aligned}\lim_{\tau\to0^+}\int_T^{2T} I\mathrm{d}t&=\lim_{\tau\to0^+}\int_T^{2T}\int_\Omega\big((f_R(x,p)+G[p])_t+\delta|p|p\big)\frac{1}{\tau}(v(x,t)-v(x,t-\tau))\lambda(t)\mathrm{d}x\mathrm{d}t\\&=\int_T^{2T}\int_\Omega\big((f_R(x,p)+G[p])_t+\delta|p|p\big)v_t\lambda(t)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.43}$$

Put

$$M^*_R(x,p)=\int_0^p\bar\mu(x,s)|s|s\mathrm{d}s.\tag{4.44}$$

Then (4.43) can be rewritten by integration by parts as

$$\lim_{\tau\to0^+}\int_T^{2T} I\mathrm{d}t=\int_T^{2T}\int_\Omega(f_R(x,p)+G[p])_t v_t\lambda(t)\mathrm{d}x\mathrm{d}t-\delta\int_T^{2T}\int_\Omega M^*_R(x,p)\dot\lambda(t)\mathrm{d}x\mathrm{d}t.\tag{4.45}$$

Let us pass to the integral II. We have by definition of v that

$$\mu_R(x,p)\nabla p=\mu_0(x)\left(\nabla v-\nabla_x M_R(x,p)\right).$$

We split the integral into two parts. On the one hand we have

$$\begin{aligned}
&\frac{1}{\tau}\int_T^{2T}\int_\Omega \mu_0(x)\nabla v(x,t)\cdot\big(\nabla v(x,t)-\nabla v(x,t-\tau)\big)\lambda(t)\mathrm{d}x\mathrm{d}t\\
&=\frac{1}{2\tau}\int_T^{2T}\int_\Omega \mu_0(x)\left(|\nabla v(x,t)|^2-|\nabla v(x,t-\tau)|^2+|\nabla v(x,t)-\nabla v(x,t-\tau)|^2\right)\lambda(t)\mathrm{d}x\mathrm{d}t\\
&\geq \frac{1}{2\tau}\int_T^{2T}\int_\Omega \mu_0(x)|\nabla v(x,t)|^2(\lambda(t)-\lambda(t+\tau))\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

hence,

$$\begin{aligned}
\liminf_{\tau\to 0+}\frac{1}{\tau}\int_T^{2T}\int_\Omega \mu_0(x)\nabla v(x,t)\cdot\big(\nabla v(x,t)-\nabla v(x,t-\tau)\big)\lambda(t)\mathrm{d}x\mathrm{d}t&\\
\geq -\int_T^{2T}\int_\Omega \frac{\mu_0(x)}{2}|\nabla v(x,t)|^2\dot{\lambda}(t)\mathrm{d}x\mathrm{d}t.& \qquad (4.46)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_T^{2T}\int_\Omega \mu_0(x)\nabla_x M_R(x,p)\cdot\big(\nabla v(x,t)-\nabla v(x,t-\tau)\big)\lambda(t)\mathrm{d}x\mathrm{d}t\\
&\quad=-\int_T^{2T}\int_\Omega \mu_0(x)(\nabla_x M_R(x,p(x,t+\tau))\lambda(t+\tau)-\nabla_x M_R(x,p(x,t))\lambda(t))\cdot\nabla v(x,t)\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

hence,

$$\begin{aligned}
\lim_{\tau\to 0+}\frac{1}{\tau}\int_T^{2T}\int_\Omega \mu_0(x)\nabla_x M_R(x,p)\cdot\big(\nabla v(x,t)-\nabla v(x,t-\tau)\big)\lambda(t)\mathrm{d}x\mathrm{d}t&\\
=-\int_T^{2T}\int_\Omega \mu_0(x)\left(\nabla_x\bar{\mu}_R(x,p)p_t\lambda(t)+\nabla_x M_R(x,p)\dot{\lambda}(t)\right)\nabla v\mathrm{d}x\mathrm{d}t.& \qquad (4.47)
\end{aligned}$$

Combining (4.46) with (4.47) we obtain

$$\begin{aligned}
\liminf_{\tau\to 0+}\int_T^{2T} II\mathrm{d}t \geq &-\int_T^{2T}\int_\Omega \frac{\mu_0(x)}{2}|\nabla v(x,t)|^2\,\dot{\lambda}(t)\mathrm{d}x\mathrm{d}t\\
&+\int_T^{2T}\int_\Omega \mu_0(x)\left(\nabla_x\bar{\mu}_R(x,p)p_t\lambda(t)+\nabla_x M_R(x,p)\dot{\lambda}(t)\right)\nabla v\mathrm{d}x\mathrm{d}t.
\end{aligned} \qquad (4.48)$$

We integrate III in time and get

$$\begin{aligned}
&-\frac{1}{\tau}\int_T^{2T}\int_\Omega u_t(x,t)\left(\nabla v(x,t)-\nabla v(x,t-\tau)\right)\lambda(t)\mathrm{d}x\mathrm{d}t\\
&\quad=-\frac{1}{\tau}\int_T^{2T}\int_\Omega \left(\nabla v(x,t)u(x,t)\lambda(t)-\nabla(x,t)u_t(x,t+\tau)\lambda(t+\tau)\right)\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$- \frac{1}{\tau} \int_T^{2T} \int_\Omega \nabla v(x,t) \big(u_t(x,t)\lambda(t) - u_t(x,t+\tau)\lambda(t+\tau) \big) \mathrm{d}x\mathrm{d}t.$$

Hence,

$$\lim_{\tau \to 0} \int_T^{2T} III \mathrm{d}t = \int_T^{2T} \int_\Omega \nabla v(x,t) \big(u_{tt}(x,t)\lambda(t) + u_t(x,t)\, \dot{\lambda}(t) \big) \mathrm{d}x\mathrm{d}t. \tag{4.49}$$

Finally, in order to handle the term IV, we consider the function $\hat{M}_R(x,v) = \int_0^v M_R^{-1}(x,v')\mathrm{d}v'$, where M_R^{-1} is as in (4.8). Then $p = M_R^{-1}(x,v)$. The function $\hat{M}_R$ is convex in v because M_R^{-1} is an increasing function of v. This means that the inequality $\hat{M}_R(x,v) - \hat{M}_R(x,\tilde{v}) \leq M_R^{-1}(v)(v-\tilde{v})$ holds for all $v, \tilde{v} \in \mathbb{R}$, in particular

$$\frac{1}{\tau} \Big(\hat{M}_R(x, v(x,t)) - \hat{M}_R(x, v(x,t-\tau)) \Big) \leq \frac{1}{\tau} M_R^{-1}(x, v(x,t))(v(x,t) - v(x,t-\tau)).$$

We thus have

$$\begin{aligned} &\frac{1}{\tau} \int_T^{2T} \int_{\partial\Omega} \alpha(x) \Big(\hat{M}_R(x, v(x,t)) - \hat{M}_R(x, v(x,t-\tau)) \Big) \lambda(t) \mathrm{d}s(x)\mathrm{d}t \\ &\quad \leq \frac{1}{\tau} \int_T^{2T} \int_{\partial\Omega} \alpha(x) p(x,t) \big(v(x,t) - v(x,t-\tau) \big) \lambda(t) \mathrm{d}s(x)\mathrm{d}t, \end{aligned} \tag{4.50}$$

hence

$$\begin{aligned} \liminf_{\tau \to 0+} \frac{1}{\tau} &\int_T^{2T} \int_{\partial\Omega} \alpha(x) p(x,t) \big(v(x,t) - v(x,t-\tau) \big) \lambda(t) \mathrm{d}s(x)\mathrm{d}t \\ &\geq - \int_T^{2T} \int_{\partial\Omega} \alpha(x) \hat{M}_R(x, v(x,t))\, \dot{\lambda}(t) \mathrm{d}s(x)\mathrm{d}t. \end{aligned}$$

In a similar way,

$$\begin{aligned} \lim_{\tau \to 0+} \frac{1}{\tau} &\int_T^{2T} \int_{\partial\Omega} \alpha(x) \bar{p} (v(x,t) - v(x,t-\tau)) \lambda(t) \mathrm{d}s(x)\mathrm{d}t \\ &= - \int_T^{2T} \int_{\partial\Omega} \alpha(x) (\bar{p}_t \lambda(t) + \bar{p}\, \dot{\lambda}(t)) v(x,t) \mathrm{d}s(x)\mathrm{d}t. \end{aligned}$$

Subtracting these two limits we get

$$\liminf_{\tau \to 0+} \int_T^{2T} IV \mathrm{d}t \geq \int_T^{2T} \int_{\partial\Omega} \alpha(x) \Big(\big(\bar{p}_t \lambda(t) + \bar{p}\, \dot{\lambda}(t) \big) v(x,t) - \hat{M}_R(x, v(x,t))\, \dot{\lambda}(t) \Big) \mathrm{d}s(x)\mathrm{d}t. \tag{4.51}$$

Combining (4.42) with (4.45), (4.48), (4.49), and (4.51) yields

$$\int_T^{2T} \int_\Omega (f_R(x,p) + G[p])_t v_t \lambda(t) \mathrm{d}x\mathrm{d}t - \delta \int_T^{2T} \int_\Omega M_R^*(x,p) \dot{\lambda}(t) \mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&- \int_T^{2T}\int_\Omega \frac{\mu_0(x)}{2}|\nabla v|^2\,\dot\lambda(t)\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_\Omega \mu_0(x)\Big(\nabla_x\bar\mu_R(x,p)p_t\lambda(t)+\nabla_x M_R(x,p)\,\dot\lambda(t)\Big)\cdot\nabla v\mathrm{d}x\mathrm{d}t\\
&+ \int_T^{2T}\int_\Omega \big(u_{tt}(x,t)\lambda(t) + u_t(x,t)\,\dot\lambda(t)\big)\cdot\nabla v\mathrm{d}x\mathrm{d}t\\
&+ \int_T^{2T}\int_{\partial\Omega} \alpha(x)\Big(\big(\bar p_t\lambda(t) + \bar p\,\dot\lambda(t)\big)\,v - \hat M_R(x,v)\,\dot\lambda(t)\Big)\mathrm{d}s(x)\mathrm{d}t \le 0.
\end{aligned}
$$

This is an inequality of the form

$$
-\int_T^{2T} A(t)\dot\lambda(t)\mathrm{d}t + \int_T^{2T} B(t)\lambda(t)\mathrm{d}t \le 0 \tag{4.52}
$$

with

$$
\begin{aligned}
A(t) &= \int_\Omega\left(\frac{\mu_0(x)}{2}|\nabla v|^2 + \delta M_R^*(x,p) - \mu_0(x)\nabla_x M_R(x,p)\cdot\nabla v - u_t\cdot\nabla v\right)(x,t)\mathrm{d}x\\
&\quad + \int_{\partial\Omega}\alpha(x)\left(\hat M_R(x,v) - \bar p v\right)(x,t)\mathrm{d}s(x),\\
B(t) &= \int_\Omega\big((f_R(x,p)+G[p])_t v_t + (u_{tt} + \mu_0(x)p_t\nabla_x\bar\mu_R(x,p))\cdot\nabla v\big)\mathrm{d}x + \int_{\partial\Omega}\alpha(x)\bar p_t v\mathrm{d}s(x)\,.
\end{aligned}
$$

We have $G[p]_t v_t \ge 0$ a. e. and

$$
f_R(x,p)_t v_t = \partial_p f(x,p)\bar\mu_R(x,p)|p_t|^2 \ge \frac{\mu^\flat}{\mu^\sharp}(1+R^2)^{-\gamma}\partial_p f_R(x,p)|p_t|^2 = \frac{\mu^\flat}{\mu^\sharp}\Gamma_R\partial_p f_R(x,p)|p_t|^2
$$

according to (3.14), (4.1) and (4.28). On the other hand, by Hypothesis 3.1.1 (i)–(ii), we have $|\nabla_x\bar\mu_R(x,p)| \le C(\partial_p f_R(x,p))^{1/2}$. In particular,

$$
|\mu_0(x)p_t\nabla_x\bar\mu_R(x,p)\cdot\nabla v| \le C\left(f_R(x,p)_t p_t\right)^{1/2}|\nabla v|. \tag{4.53}
$$

Furthermore, by (4.41) we have $|\nabla v| \le |\nabla_x M_R(x,p)| + \bar\mu_R(x,p)|\nabla p|$, hence, by (4.26),

$$
\int_T^{2T}\int_\Omega |\nabla v(x,t)|^2\mathrm{d}x\mathrm{d}t \le C\left(1 + \int_T^{2T}\int_\Omega \mu_R(x,p)|\nabla p|^2\mathrm{d}x\mathrm{d}t\right) \le C. \tag{4.54}
$$

Using (4.53)–(4.54), Hölder's inequality, and the inequality $az^2 - bz \ge \frac{a}{2}z^2 - \frac{b^2}{2a}$ for $a>0$, $b>0$, and $z\in\mathbb{R}$ we conclude that

$$
\begin{aligned}
&\int_T^{2T}\int_\Omega \big((f_R(x,p)+G[p])_t v_t + \mu_0(x)p_t\nabla_x\bar\mu_R(x,p)\cdot\nabla v\big)\mathrm{d}x\mathrm{d}t\\
&\qquad \ge \frac{\mu^\flat}{\mu^\sharp\Gamma_R}\int_T^{2T}\int_\Omega \partial_p f_R(x,p)|p_t|^2\mathrm{d}x\mathrm{d}t - C\left(\int_T^{2T}\int_\Omega \partial_p f_R(x,p)|p_t|^2\mathrm{d}x\mathrm{d}t\right)^{1/2}
\end{aligned}
$$

$$\geq \frac{\mu^\flat}{2\mu^\sharp\Gamma_R}\int_T^{2T}\int_\Omega \partial_p f_R(x,p)|p_t|^2 \mathrm{d}x\mathrm{d}t - \frac{C\mu^\sharp\Gamma_R}{2\mu^\flat} \tag{4.55}$$

with a constant $C > 0$ independent of R, δ, and ε. The remaining terms in $B(t)$ can be estimated using (4.26), (4.30), and (4.54) as follows.

$$\int_T^{2T}\int_\Omega |u_{tt}|\,|\nabla v|\mathrm{d}x\mathrm{d}t \leq C\Gamma_R^{1/2},$$

$$\int_T^{2T}\int_{\partial\Omega} \alpha(x)|\bar{p}_t||v|\mathrm{d}s(x)\mathrm{d}t \leq C.$$

Hence,

$$\int_T^{2T} B(t)\mathrm{d}t \geq \frac{c}{\Gamma_R}\int_T^{2T}\int_\Omega \partial_p f_R(x,p)|p_t|^2\mathrm{d}x\mathrm{d}t - C\Gamma_R \tag{4.56}$$

with constants $C > c > 0$ independent of R, δ, and ε. Using the relations $0 \leq \hat{M}_R(x,v) = \int_0^p \bar{\mu}_R(x,p')p'\mathrm{d}p' \leq (\mu^\sharp/2\mu^\flat)p^2$ and $0 \leq M_R^*(x,p) \leq (\mu^\sharp/\mu^\flat)|p|^3$, we similarly obtain from (4.26) that

$$\int_T^{2T}|A(t)|\mathrm{d}t \leq C\left(1 + \int_T^{2T}\int_\Omega |\nabla v|^2\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega}\alpha(x)p^2\mathrm{d}s(x)\mathrm{d}t\right) \leq C. \tag{4.57}$$

We now choose arbitrary points $2T < r < 3T$ and $T < s < r$ such that $r - s < T$. Then for each $\varepsilon < \frac{r-s}{2}$ we set in (4.52)

$$\lambda(t) = \frac{1}{\varepsilon}(t-s) \ \text{ for } t \in (s, s+\varepsilon), \quad \lambda(t) = \frac{1}{\varepsilon}(r-t) \ \text{ for } t \in (r-\varepsilon, r),$$

choosing λ constant and continuous otherwise, T-periodically extended to the whole real line. All functions under the integrals in (4.52) are T-periodic. We thus can replace the integration domain $[T, 2T]$ with $[s, s+T]$, which yields that

$$\frac{1}{\varepsilon}\int_{r-\varepsilon}^r A(t)\mathrm{d}t \leq \frac{1}{\varepsilon}\int_s^{s+\varepsilon} A(t)\mathrm{d}t - \int_T^{2T} B(t)\mathrm{d}t.$$

Integrating the above inequality over s from $r - T$ to $r - 2\varepsilon$ we obtain

$$\frac{1}{\varepsilon}\int_{r-\varepsilon}^r A(t)\mathrm{d}t \leq \frac{1}{T-2\varepsilon}\int_T^{2T}|A(t)|\mathrm{d}t - \int_T^{2T} B(t)\mathrm{d}t. \tag{4.58}$$

At each of its Lebesgue points $t = r$, the function $A(t)$ admits therefore a pointwise bound given by the right hand side of (4.58), which is in turn estimated by (4.56)–(4.57), that is,

$$\frac{c}{\Gamma_R}\int_T^{2T}\int_\Omega \partial_p f_R(x,p)|p_t|^2\mathrm{d}x\mathrm{d}t + \sup_{t\in[T,2T]}\operatorname{ess} A(t) \le C\Gamma_R. \tag{4.59}$$

To estimate $A(t)$ from below, we first notice that

$$\hat{M}_R(x,v) - \bar{p}v = \int_0^p (p'-\bar{p})\bar{\mu}_R(x,p')\mathrm{d}p' \ge \frac{\mu^\flat}{2\mu^\sharp\Gamma_R}(p^2 - C),$$

and

$$|\nabla v|^2 = \bar{\mu}_R^2(x,p)|\nabla p|^2 \ge \left(\frac{\mu^\flat}{\mu^\sharp\Gamma_R}\right)^2 |\nabla p|^2.$$

The above computations yield that

$$A(t) \ge c\left(\frac{1}{\Gamma_R^2}\int_\Omega |\nabla p|^2(x,t)\mathrm{d}x + \frac{1}{\Gamma_R}\int_{\partial\Omega}\alpha(x)p^2\mathrm{d}s(x)\right) - C.$$

From (4.59) we thus obtain that

$$c\Gamma_R\int_T^{2T}\int_\Omega \partial_p f_R(x,p)|p_t|^2\mathrm{d}x\mathrm{d}t + \sup_{t\in(T,2T)}\operatorname{ess}\left(\int_\Omega |\nabla p|^2(x,t)\mathrm{d}x + \int_{\partial\Omega}\alpha(x)p^2\mathrm{d}s(x)\right) \le C\Gamma_R^3. \tag{4.60}$$

4.4 Uniform estimates of p

To remove the cut-off parameter R, we proceed by Moser iterations and define Lipschitz continuous functions $a_{R,k}(p)$ and their antiderivatives $A_{R,k}(p)$ with indices $k \ge 1$ and with $R > 1$ by the formula

$$a_{R,k}(p) = p(1+\min\{p^2,R^2\})^k,\ \ A_{R,k}(p) = 1 + \int_0^p a_{R,k}(s)\mathrm{d}s\,. \tag{4.61}$$

We have

$$a'_{R,k}(p) = \begin{cases} (1+p^2)^{k-1}(1+(2k+1)p^2) & \text{for } |p| < R\,, \\ (1+R^2)^k & \text{for } |p| \ge R, \end{cases} \tag{4.62}$$

and

$$A_{R,k}(p) = \begin{cases} \frac{1}{2k+2}(1+p^2)^{k+1} & \text{for } |p| < R, \\ \frac{1}{2k+2}(1+R^2)^{k+1} + \frac{1}{2}(p^2-R^2)(1+R^2)^k \ge \frac{1}{2k+2}(1+p^2)(1+R^2)^k & \text{for } |p| \ge R. \end{cases} \tag{4.63}$$

Put

$$\begin{aligned} F_k(x,p) &= \frac{f^\flat}{2k} + \int_0^p \partial_p f_R(x,s) a_{R,k}(s)\mathrm{d}s, \\ V_k[p] &= \int_0^\infty \int_0^{\mathfrak{f}_r[p]} a_{R,k}(v)\psi(x,r,v)\mathrm{d}v\mathrm{d}r. \end{aligned} \tag{4.64}$$

Then

$$G[p]_t a_{R,k}(p) - V_k[p]_t \geq 0 \quad a.\,e. \tag{4.65}$$

by Proposition 3.3.5, and testing (4.25) by $a_{R,k}(p)$ we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (F_k(\cdot,p) + V_k[p])(x,t)\mathrm{d}x + \int_\Omega \mu_R(x,p)|\nabla p|^2 a'_{R,k}(p)\mathrm{d}x + \int_{\partial\Omega} \alpha(x)(p-\bar{p})a_{R,k}(p)\mathrm{d}s(x) \\ &\quad \leq \int_\Omega (u_t\cdot\nabla p)a'_{R,k}(p)\mathrm{d}x \leq \frac{1}{2}\int_\Omega |u_t|^2 \frac{a'_{R,k}(p)}{\mu_R(x,p)}\mathrm{d}x + \int_\Omega \frac{\mu_R(x,p)}{2}|\nabla p|^2 a'_{R,k}(p)\mathrm{d}x. \end{aligned} \tag{4.66}$$

We have omitted the positive δ-term on the left-hand side which does not bring any relevant information. The boundary term will be estimated from below as follows. We first notice that the function $a_{R,k}$ is increasing, hence

$$(p-\bar{p})a_{R,k}(p) \geq A_{R,k}(p) - A_{R,k}(\bar{p}).$$

Since the function $\bar{p}$ is bounded, we may take

$$R > \sup\operatorname{ess}\bar{p} \tag{4.67}$$

and finally get the estimate

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (F_k(\cdot,p) + V_k[p])\mathrm{d}x + c\int_\Omega |\nabla p|^2 \mu_R(x,p)a'_{R,k}(p)\mathrm{d}x + \int_{\partial\Omega} \alpha(x)A_{R,k}(p)\mathrm{d}s(x) \\ &\quad \leq C^{2k+2} + C\int_\Omega |u_t|^2 \frac{a'_{R,k}(p)}{\mu_R(x,p)}\mathrm{d}x \end{aligned} \tag{4.68}$$

with constants c, C independent of δ, ε, R and k. The last term on the right-hand side of (4.68) will again be estimated by Hölder's inequality and (4.40) as follows:

$$\int_\Omega |u_t|^2 \frac{a'_{R,k}(p)}{\mu_R(x,p)}\mathrm{d}x \leq |u_t(t)|_6^2 \left(\int_\Omega \left(\frac{a'_{R,k}(p)}{\mu_R(x,p)}\right)^{3/2}\mathrm{d}x\right)^{2/3} \leq C\Gamma_R \left(\int_\Omega \left(\frac{a'_{R,k}(p)}{\mu_R(x,p)}\right)^{3/2}\mathrm{d}x\right)^{2/3}.$$

Then (4.68) can be reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (F_k(\cdot,p) + V_k[p])\mathrm{d}x + c\int_\Omega |\nabla p|^2 \mu_R(x,p)a'_{R,k}(p)\mathrm{d}x + \int_{\partial\Omega} \alpha(x)A_{R,k}(p)\mathrm{d}s(x)$$

$$\leq C^{2k+2} + C\Gamma_R \left(\int_\Omega \left(\frac{a'_{R,k}(p)}{\mu_R(x,p)} \right)^{3/2} \mathrm{d}x \right)^{2/3} . \tag{4.69}$$

Put

$$w_k = \begin{cases} (1+p^2)^{\frac{k+1-\eta}{2}} & \text{for } |p| < R, \\ (1+p^2)^{\frac{1}{2}}(1+R^2)^{\frac{k-\eta}{2}} & \text{for } |p| \geq R, \end{cases} \qquad \hat{w}_k = \begin{cases} (1+p^2)^{\frac{k+\gamma}{2}} & \text{for } |p| < R, \\ (1+R^2)^{\frac{k+\gamma}{2}} & \text{for } |p| \geq R, \end{cases} \tag{4.70}$$

Using the hypothesis $\gamma + \eta \leq 1$ from (3.13) we obtain the pointwise bounds almost everywhere

$$\begin{aligned} 0 \leq \hat{w}_k &\leq w_k \, , \\ \frac{a'_{R,k}(p)}{\mu_R(x,p)} &\leq \frac{1}{\mu^\flat}(2k+1)|\hat{w}_k|^2 \, , \\ |w_k|^2 &\leq (2k+2)A_{R,k}(p) \, , \\ |\nabla w_k|^2 &\leq \frac{1}{\mu^\flat}(k+1)|\nabla p|^2 \mu_R(x,p) a'_{R,k}(p) \, . \end{aligned} \tag{4.71}$$

From (4.69) we thus obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (F_k(\cdot,p) + V_k[p])\mathrm{d}x + \frac{c}{k+1} \left(\int_\Omega |\nabla w_k|^2 \mathrm{d}x + \int_{\partial\Omega} \alpha(x)|w_k|^2 \mathrm{d}s(x) \right) \\ &\qquad \leq C^{2k+2} + C(k+1)\Gamma_R \left(\int_\Omega |\hat{w}_k|^3 \mathrm{d}x \right)^{2/3} , \end{aligned} \tag{4.72}$$

where c and C are similarly as before constants independent of δ, ε, k and R. For simplicity, put

$$\mathbf{F}_k(t) = \int_\Omega F_k(x, p(x,t))\mathrm{d}x, \quad \mathbf{V}_k(t) = \int_\Omega V_k[p](x,t)\mathrm{d}x. \tag{4.73}$$

Using the symbols $|\cdot|_q$ and $\|\cdot\|_{1,q}$ for $1 \leq q \leq \infty$ to denote the norms in $L^q(\Omega)$ and in $W^{1,q}(\Omega)$, respectively, we have the interpolation inequality

$$|\hat{w}_k(t)|_3 \leq |\hat{w}_k(t)|_{3/2}^{1/3} |\hat{w}_k(t)|_6^{2/3} \leq |\hat{w}_k(t)|_{3/2}^{1/3} |w_k(t)|_6^{2/3}, \tag{4.74}$$

and by embedding of $W^{1,2}(\Omega)$ into $L^6(\Omega)$,

$$|w_k(t)|_6 \leq C\|w_k(t)\|_{1,2}.$$

The right-hand side of (4.72) can thus be estimated from above by

$$C^{2k+2} + C(k+1)\Gamma_R |\hat{w}_k(t)|_{3/2}^{2/3} \|w_k(t)\|_{1,2}^{4/3}. \tag{4.75}$$

We now use Young's inequality

$$a^{2/3} b^{4/3} \leq \frac{1}{3\omega^3} a^2 + \frac{2\omega^{3/2}}{3} b^2$$

with $a = (C(k+1)\Gamma_R)^{3/2} |\hat{w}_k(t)|_{3/2}$, $b = \|w_k(t)\|_{1,2}$, and $\omega = (\frac{c}{k+1})^{2/3}$. This yields (note that $k \geq 1$)

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}_k(t) + \mathbf{V}_k(t)) + \frac{c}{k}\|w_k(t)\|_{1,2}^2 \leq C^{2k+2} + Ck^5\Gamma_R^3 |\hat{w}_k(t)|_{3/2}^2. \tag{4.76}$$

By (3.28) we have

$$0 \leq \mathbf{V}_k(t) \leq CB^{2k+2}, \tag{4.77}$$

for all $t \geq 0$, and from Hypothesis 3.1.1 we get the lower bound for $\mathbf{F}_k$

$$\mathbf{F}_k(t) \geq \frac{f^\flat}{2k} |w_k(t)|_2^2. \tag{4.78}$$

An upper bound for $F_k(x,p)$ for $|p| \leq R$ is still straightforward, namely

$$F_k(x,p) \leq \frac{f^\flat}{2k} + \int_0^p f^\sharp s(1+s^2)^k \mathrm{d}s \leq \frac{f^\sharp}{2k}(1+p^2)^{k+1}. \tag{4.79}$$

The case $p > R$ or $p < -R$ is more delicate. For $p > R$ we have

$$\begin{aligned} F_k(x,p) &\leq \frac{f^\flat}{2k} + \int_0^R f^\sharp s(1+s^2)^k \mathrm{d}s + \int_R^p f^\sharp s(1+R^2)^k \mathrm{d}s \\ &\leq \frac{f^\sharp}{2k}(1+R^2)^k \left(1 + R^2 + k(p^2 - R^2)\right) \leq \frac{f^\sharp}{2}(1+p^2)(1+R^2)^k \\ &\leq \frac{f^\sharp}{2}|w_k|^{q_k} \end{aligned} \tag{4.80}$$

for a. e. $(x,t) \in \Omega \times (T, 2T)$ with

$$q_k = \frac{2(k+1)}{k+1-\eta}. \tag{4.81}$$

Indeed, (4.80) holds for $p < -R$ as well by symmetry. By virtue of (4.70) and (4.78)–(4.80) we thus have

$$\frac{c}{k}|w_k(t)|_2^2 \leq \mathbf{F}_k(t) \leq C|w_k(t)|_{q_k}^{q_k}. \tag{4.82}$$

Note that $q_k \leq 4$ for all $k \geq 1$. It follows from (4.82)–(4.82) and from the embedding $W^{1,2}(\Omega)$ into $L^6(\Omega)$ that

$$\frac{1}{k}\|w_k(t)\|_{1,2}^2 \geq \frac{c}{k}\mathbf{F}_k^{2/q_k}$$

with some constant $c > 0$. From (4.76) we thus obtain (note that $2/q_k < 1$) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{F}_k(t)+\mathbf{V}_k(t)) + \frac{c}{k}(\mathbf{F}_k(t)+\mathbf{V}_k(t))^{2/q_k} &\leq C^{2k+2} + Ck^5\Gamma_R^3|\hat{w}_k(t)|_{3/2}^2 \\ &\leq Ck^5\Gamma_R^3 \max\{C^k, |\hat{w}_k(t)|_{3/2}\}^2 =: M_{R,k}(t).\end{aligned} \tag{4.83}$$

Put $M_{R,k}^* = \sup\operatorname{ess}_{t\in[T,2T]} M_{R,k}(t)$. Then (4.83) is of the form

$$\dot{Y}(t) + bY^\kappa(t) \leq M \tag{4.84}$$

with a T-periodic function $Y = \mathbf{F}_k + \mathbf{V}_k$ and with constants $b = c/k$, $\kappa = 2/q_k$, and $M = M_{R,k}^*$. Let H be the Heaviside function $H(s) = 1$ for $s > 0$, $H(s) = 0$ for $s \leq 0$. Multiplying (4.84) by $H(Y^\kappa(t) - (M/b))$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(Y(t) - (M/b)^{1/\kappa}\big)^+ + (bY^\kappa(t) - M)^+ \leq 0.$$

Integrating from T to $2T$ and using the fact that Y is periodic, we thus have

$$\int_T^{2T} (bY^\kappa(t) - M)^+ \mathrm{d}t \leq 0,$$

which is only possible if $Y(t) \leq (M/b)^{1/\kappa}$ a. e. Hence, (4.83) yields that

$$\mathbf{F}_k(t) \leq C(kM_{R,k}^*)^{q_k/2}. \tag{4.85}$$

Referring again to (4.82) (note that $q_k \leq 4$) we thus conclude that

$$\sup_{t\in[T,2T]}\operatorname{ess} |\hat{w}_k(t)|_2^2 \leq \sup_{t\in[T,2T]}\operatorname{ess} |w_k(t)|_2^2 \leq Ck^{13}\Gamma_R^{3q_k/2}\max\{C^k, \sup_{t\in[T,2T]}\operatorname{ess} |\hat{w}_k(t)|_{3/2}\}^{q_k}. \tag{4.86}$$

Putting for $q \geq 1$

$$\|\hat{w}_k\|_q^* = \sup_{t\in[T,2T]}\operatorname{ess} |\hat{w}_k(t)|_q,$$

we can reformulate (4.86) as

$$\big(\|\hat{w}_k\|_2^*\big)^2 \leq Ck^{13}\Gamma_R^{3q_k/2}\big(\max\{C^k, \|\hat{w}_k\|_{3/2}^*\}\big)^{q_k} \tag{4.87}$$

with a constant C independent of k and R.

The Moser iteration technique will be applied to the new variable

$$w := 1 + \min\{p^2, R^2\}. \tag{4.88}$$

By (4.70), inequality (4.87) can be rewritten as

$$\left(\|w\|^*_{r_k}\right)^{r_k} \leq Qk^{13}\Gamma_R^{3q_k/2} \max\{L^{r_k}, \left(\|w\|^*_{3r_k/4}\right)^{r_k/2}\}^{q_k} \tag{4.89}$$

with $r_k = k + \gamma$ and with some constants $Q \geq 1$, $L \geq 1$ that we keep fixed from now on. We thus have

$$\|w\|^*_{r_k} \leq (Qk^{13})^{1/r_k}\Gamma_R^{3q_k/2r_k} \max\{L, \|w\|^*_{3r_k/4}\}^{q_k/2}. \tag{4.90}$$

We are ready now to start the Moser iterations. We now choose k in (4.90) to be the sequence $\{k_j\}$ for $j \in \mathbb{N} \cup \{0\}$

$$k_j = 3\left(\frac{4}{3}\right)^j - \gamma, \tag{4.91}$$

and put

$$\rho_j = r_{k_j} = k_j + \gamma = 3\left(\frac{4}{3}\right)^j, \quad P_j = \max\{L, \|w\|^*_{\rho_j}\}, \quad \delta_j = \frac{q_{k_j}}{2} - 1 = \frac{\eta}{k_j + 1 - \eta}.$$

Then, by virtue of (4.81), the inequality (4.90) is of the form

$$P_j \leq (\rho_j^{13}Q)^{1/\rho_j}\Gamma_R^{3(1+\delta_j)/\rho_j} P_{j-1}^{1+\delta_j} \quad \text{for } j = 1, 2, \ldots \tag{4.92}$$

The logarithm applied to (4.92) yields

$$\begin{aligned} \log P_j - (1+\delta_j)\log P_{j-1} &\leq \frac{1}{\rho_j}\left(\log(3^{13}Q) + 3(1+\delta_j)\log\Gamma_R + 13\log\left(\frac{4}{3}\right)^j\right) \\ &= \frac{1}{3}\left(\frac{3}{4}\right)^j\left(\log(3^{13}Q) + 3(1+\delta_j)\log\Gamma_R + 13j\log\frac{4}{3}\right), \end{aligned} \tag{4.93}$$

or, equivalently, for $j \in \mathbb{N}$,

$$\begin{aligned} \frac{\log P_j}{\prod_{i=0}^{j}(1+\delta_i)} - \frac{\log P_{j-1}}{\prod_{i=0}^{j-1}(1+\delta_i)} &\leq \frac{1}{3}\left(\frac{3}{4}\right)^j \frac{\log(3^{13}Q) + 3(1+\delta_j)\log\Gamma_R + 13j\log(4/3)}{\prod_{i=0}^{j}(1+\delta_i)} \\ &\leq \frac{1}{3}\left(\frac{3}{4}\right)^j \frac{\log(3^{13}Q) + 3\log\Gamma_R + 13j\log(4/3)}{1+\delta_0}. \end{aligned} \tag{4.94}$$

The sequence on the right-hand side of (4.94) forms a convergent series, more precisely,

$$\sum_{j=1}^{\infty} \left(\frac{3}{4}\right)^j = 3, \quad \sum_{j=1}^{\infty} j \left(\frac{3}{4}\right)^j = 12.$$

Hence,

$$\sup_{j\geq 1} \frac{\log P_j}{\prod_{i=1}^{j}(1+\delta_i)} \leq \log P_0 + \log(3^{13}Q) + 3\log\Gamma_R + 52\log\left(\frac{4}{3}\right). \tag{4.95}$$

We have by virtue of (4.60) that $P_0 \leq C\Gamma_R^3$. We thus have

$$\sup_{j\geq 1} \frac{\log P_j}{\prod_{i=1}^{j}(1+\delta_i)} \leq 6\log\Gamma_R + C \tag{4.96}$$

with a constant $C > 0$ independent of j and R. Note that

$$\delta_i = \frac{\eta}{k_i + 1 - \eta} \leq \frac{1}{k_i + \gamma} = \frac{1}{3}\left(\frac{3}{4}\right)^i.$$

We thus have

$$\prod_{i=1}^{\infty}(1+\delta_i) = \exp\Big(\sum_{i=1}^{\infty}\log(1+\delta_i)\Big) \leq \exp\Big(\sum_{i=1}^{\infty}\delta_i\Big) \leq \mathrm{e},$$

hence, by definition (4.28) of Γ_R,

$$\sup_{j\geq 1} \log P_j \leq 6\gamma\mathrm{e}\log(1+R^2) + C^* \tag{4.97}$$

with a constant $C^* > 0$ independent of j and R, that is,

$$\sup_{j\geq 1} P_j \leq \mathrm{e}^{C^*}(1+R^2)^{\sigma} =: P^*(R). \tag{4.98}$$

with

$$\sigma = 6\gamma\mathrm{e} < 1$$

by Hypothesis (3.13). We see that the norms $\|w\|_k^*$ are bounded by $P^*(R)$ independent of k. We now easily prove that

$$w(x,t) \leq P^*(R) \ \text{ for almost all } \ (x,t) \in \Omega \times (T, 2T). \tag{4.99}$$

Indeed, assume that there exist $a > 0$ and a set $A \subset \Omega \times (T, 2T)$ such that $w(x,t) \geq$

$P^*(R) + a$ for a. e. $(x,t) \in A$. Then

$$(P^*(R)+a)^q|A| \leq \iint_A w^q \mathrm{d}x\mathrm{d}t \leq \int_T^{2T}\int_\Omega w^q \mathrm{d}x\mathrm{d}t \leq T \operatorname*{sup\,ess}_{t\in[T,2T]} \int_\Omega w^q \mathrm{d}x \leq TP^*(R)^q. \tag{4.100}$$

We conclude from (4.100) that

$$|A| \leq T\left(\frac{P^*(R)}{P^*(R)+a}\right)^q$$

for all $q > 1$. This is only possible if $|A| = 0$. Hence, $w(x,t) \leq P^*(R)$ a. e. in agreement with (4.99). We can summarize the above computations as follows.

Proposition 4.4.1. *Let $\bar{R} > 0$ be chosen such that $1 + \bar{R}^2 > \mathrm{e}^{C^*/(1-\sigma)}$ with C^* and σ as in* (4.98)*. Let (u,p) be the solution to* (4.24)*–*(4.25) *given by Proposition 4.1.1 for $R = \bar{R}$. Then we have $|p(x,t)| \leq \bar{R}$ for a. e. $(x,t) \in \Omega \times (T,2T)$, and (u,p) is the solution to the system*

$$\begin{aligned}
&\int_T^{2T}\int_\Omega \big((\rho u_{tt} + \delta|u_t|u_t)\phi + (\mathbf{B}(x)\nabla_s u_t + \mathbf{A}(x)\nabla_s u + \varepsilon\mathbf{B}(x)\nabla_s u_{tt}) : \nabla_s\phi + \nabla p\ \phi\big)\lambda(t)\,dx\,dt \\
&\quad + \int_T^{2T}\int_{\partial\Omega} \beta(x)(\mathbf{C}(x)u + \mathbf{D}(x)u_t - g)\phi\lambda(t)\,ds(x)\,dt = 0, \qquad (4.101)\\
&\int_T^{2T}\int_\Omega \Big(\big((f(x,p)+G[p])_t + \delta|p|p\big)\psi + (\mu(x,p)\nabla p - u_t)\nabla\psi\Big)\lambda(t)\,dx\,dt \\
&\quad + \int_T^{2T}\int_{\partial\Omega} \alpha(x)(p-\bar{p})\psi\lambda(t)\,ds(x)\,dt = 0 \qquad (4.102)
\end{aligned}$$

for all $\phi \in X_3$, $\psi \in X$, and $\lambda \in L^2_T$ with the regularity as in Proposition 4.1.1.

Proof. Assuming that for some (x,t) we have $|p(x,t)| > \bar{R}$ would imply that $w(x,t) = 1 + \min\{p^2(x,t), \bar{R}^2\} = 1 + \bar{R}^2$. By virtue of (4.99) we have

$$1 + \bar{R}^2 = w(x,t) \leq P^*(\bar{R}) = \mathrm{e}^{C^*}(1+\bar{R}^2)^\sigma$$

which contradicts the choice of $\bar{R}$. Hence, the solution satisfies the condition $|p| \leq \bar{R}$ a. e. and the proof is complete. □

We are now ready to finish the proof of Theorem 3.1.3.

Proof of Theorem 3.1.3.

Solutions to (4.101)–(4.102) satisfy, by virtue of (4.26), (4.29), (4.30), (4.36), (4.60) the following estimates independent of δ and ε

$$\int_T^{2T}\int_\Omega \big(|u_{tt}|^2 + \delta|u_t|^3 + \varepsilon\mathbf{B}(x)\nabla_s u_{tt} : \nabla_s u_{tt} + \mathbf{B}(x)\nabla_s u_t : \nabla_s u_t + \mathbf{A}(x)\nabla_s u : \nabla_s u\big)\,\mathrm{d}x\mathrm{d}t$$
$$+ \int_T^{2T}\int_{\partial\Omega} \beta(x)\left(\mathbf{D}(x)u_t\cdot u_t + \mathbf{C}(x)u\cdot u\right)\mathrm{d}s(x)\mathrm{d}t \le C,$$
$$\sup_{(x,t)\in\Omega\times(T,2T)}\mathrm{ess}\, |p(x,t)| + \int_T^{2T}\int_\Omega \left(\delta|p|^3 + |p_t|^2 + |\nabla p|^2\right)\mathrm{d}x\mathrm{d}t + \int_T^{2T}\int_{\partial\Omega}\alpha(x)p^2\mathrm{d}s(x) \le C.$$

The term $\delta|p|p$ converges strongly to 0 in $L^\infty(\Omega\times(T,2T))$ as $\delta\to 0$ due to the uniform upper bound for p. Similarly, u_t are bounded in $L^4_T(L^4(\Omega))$ by Sobolev embeddings, hence $\delta|u_t|u_t$ converge strongly to 0 in $L^2_T(L^2(\Omega))$. Furthermore, for $\phi\in X_3$ and $\lambda\in L^2_T$ we have

$$\int_T^{2T}\int_\Omega \varepsilon\mathbf{B}(x)\nabla_s u_{tt}\,\phi\lambda(t)\mathrm{d}x\mathrm{d}t$$
$$\le C\varepsilon^{1/2}\left(\int_T^{2T}\int_\Omega \varepsilon\mathbf{B}(x)\nabla_s u_{tt} : \nabla_s u_{tt}\mathrm{d}x\mathrm{d}t\right)^{1/2}\left(\int_\Omega|\phi(x)|^2\mathrm{d}x\right)^{1/2}\left(\int_T^{2T}|\lambda(t)|^2\mathrm{d}t\right)^{1/2}$$
$$\le C\varepsilon^{1/2}\left(\int_\Omega|\phi(x)|^2\mathrm{d}x\right)^{1/2}\left(\int_T^{2T}|\lambda(t)|^2\mathrm{d}t\right)^{1/2}.$$

In the linear terms in (4.101)–(4.102), we pass to the limit as $\delta\to 0$ and $\varepsilon\to 0$ by weak convergence. In the nonlinear terms in (4.102), we use the compactness of the embedding of $W^{1,2}_T(L^2(\Omega))\cap L^2_T(X)\cap L^\infty(\Omega\times(T,2T))$ into $L^q(\Omega;C_T)$ for every $q>1$ to check that $p=p^{\delta,\varepsilon}$ converge pointwise almost everywhere, and use the strong continuity of the operator G as well as the Lebesgue Dominated Convergence Theorem. Hence, for every fixed $\phi\in X_3$, $\psi\in X$, and $\lambda\in L^2_T$ we can pass to the limit as $\delta\to 0$ and $\varepsilon\to 0$ and conclude thus the proof of Theorem 3.1.3.

Part III

Obstacle problems applied to anisotropic materials

The aim of Part III is to focus on the analysis of higher differentiability results for solutions to a class of double phase obstacle problems in the scale of Besov spaces. In particular, this third part of the thesis contains the results published in [67, 68]. It is worth noticing that double phase functionals are a useful tool to study the behaviour of strongly anisotropic materials whose hardening properties are strongly dependent on the point and connected to the exponent ruling the growth of the gradient variable. The coefficient $a(\cdot)$ (see (7.1)) regulates the mixture between two different materials, with p and q hardening, respectively (see, for instance, [121, 122]).

The obstacle problem appeared in the mathematical literature in the work of Stampacchia [111] in the special case $\psi = \chi_E$ and related to the capacity of a subset $E \Subset \Omega$; in an earlier independent work, Fichera [51] solved the first unilateral problem, the so-called *Signorini problem* in elastostatics.

It is usually observed that the regularity of solutions to the obstacle problems is influenced by the one of the obstacle; for example, for linear obstacle problems, obstacle and solutions have the same regularity [11, 17, 80]. This does not apply in the nonlinear setting, hence along the years, there have been intense research activities for the regularity of the obstacle problem in this direction. The regularity theory for obstacle problems driven by quasilinear operators of the p-Laplacian type started with the contributions of Duzaar and Fuchs [37], Duzaar [36], Chloe and Lewis [22] and Fuchs [57].
In the case of standard growth conditions, Eleuteri and Passarelli di Napoli [43] proved that an extra differentiability of integer or fractional order of the gradient of the obstacle transfers to the gradient of the solutions, provided the partial map $x \mapsto D_\xi \tilde{F}(x, \xi)$ possesses a suitable differentiability property, where $\tilde{F}$ is a general integrand independent of the $w-$variable.
Recently, it was proved in [61, 62] that the weak differentiability of integer order of the partial map $x \mapsto D_\xi \tilde{F}(x, \xi)$ is a sufficient condition to prove that an extra differentiability of integer order of the gradient of the obstacle transfers to the gradient of the solutions to obstacle problems with p, q-growth conditions. This property was generalized also for fractional differentiability, connected to Besov spaces in [67].
The intermediate case of higher differentiability in the setting of variable exponents case has been carried out in the paper [54].

The regularity properties of local minimizers to double phase functionals have recently been investigated for unconstrained problems. In particular, we quote the work [26] by Colombo and Mingione where the functional $H(x, Du)$ has been considered (see (7.3)), [8] by Baroni, Colombo and Mingione who studied the integrand defined in (7.2) and [27] by Coscia, who dealt with the functional defined by

$$\mathcal{F}(w,\Omega) := \int_{\Omega} b(x,w)[|Dw|^p + a(x)|Dw|^p \log(e + |Dw|)]\mathrm{d}x.$$

Furthermore, a higher fractional differentiability [120] and a Lipschitz continuity result [32] have been proved for solutions to double phase elliptic obstacle problems. We also recall that when referring to p, q-growth conditions, in order to ensure the regularity of minima, a smallness condition on the gap $q/p > 1$ is necessary (see, for instance, the counterexamples in [53, 63, 96]).

Part III is structured as follows. In Chapter 5 we recall some notation and preliminary results. In Chapter 6, we study of the higher differentiablity properties of solutions to (6.3) in case of p, q-growth conditions. We assume that both the gradient of the obstacle and the partial map $x \mapsto \mathcal{A}(x, \xi)$ belong to a suitable Sobolev class of fractional order. The results contained in Chapter 6 are crucial to then prove the main result of Part III in Chapter 7, which is Theorem 7.0.1, namely, higher differentiability properties for a solution to a class of double phase functionals in the scale of Besov spaces.

Chapter 5

Notation and background

In what follows, $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ will denote the ball centered at x of radius r. We shall omit the dependence on the center and on the radius when no confusion arises. For a function $u \in L^1(B)$, the symbol

$$u_B := \fint_B u(x)\mathrm{d}x = \frac{1}{|B|}\int_B u(x)\mathrm{d}x.$$

will denote the integral mean of the function u over the set B.

It is convenient to introduce an auxiliary function

$$V_d(\xi) = |\xi|^{\frac{d-2}{2}}\xi$$

defined for all $\xi \in \mathbb{R}^n$. One can easily check that

$$|\xi|^d = |V_d(\xi)|^2. \tag{5.1}$$

For the auxiliary function V_d, we recall the following estimate (see the proof of [64, Lemma 8.3]):

Lemma 5.0.1. *Let $1 < d < +\infty$. There exists a constant $c = c(n,d) > 0$ such that*

$$c^{-1}(|\xi|^2 + |\eta|^2)^{\frac{d-2}{2}} \le \frac{|V_d(\xi) - V_d(\eta)|^2}{|\xi-\eta|^2} \le c(|\xi|^2 + |\eta|^2)^{\frac{d-2}{2}}$$

for any $\xi, \eta \in \mathbb{R}^n$, $\xi \neq \eta$.

Now we state a well-known iteration lemma (see [64, Lemma 6.1] for the proof).

Lemma 5.0.2. *Let $\Phi : [\frac{R}{2}, R] \to \mathbb{R}$ be a bounded nonnegative function, where $R > 0$. Assume that for all $\frac{R}{2} \le r < s \le R$ it holds*

$$\Phi(r) \le \theta\Phi(s) + A + \frac{B}{(s-r)^2} + \frac{C}{(s-r)^\gamma}$$

where $\theta \in (0,1)$*,* A*,* B*,* $C \geq 0$ *and* $\gamma > 0$ *are constants. Then there exists a constant* $c = c(\theta,\gamma)$ *such that*

$$\Phi\left(\frac{R}{2}\right) \leq c\left(A + \frac{B}{R^2} + \frac{C}{R^\gamma}\right).$$

5.1 Besov-Lipschitz spaces

Let $v : \mathbb{R}^n \to \mathbb{R}$ be a function. As in [70, Section 2.5.12], given $0 < \alpha < 1$ and $1 \leq p, q < \infty$, we say that v belongs to the Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$\|v\|_{B^\alpha_{p,q}(\mathbb{R}^n)} = \|v\|_{L^p(\mathbb{R}^n)} + [v]_{B^\alpha_{p,q}(\mathbb{R}^n)} < \infty,$$

where

$$[v]_{B^\alpha_{p,q}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}\mathrm{d}x\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}} < \infty.$$

Equivalently, we could simply say that $v \in L^p(\mathbb{R}^n)$ and $\frac{\tau_h v}{|h|^\alpha} \in L^q\big(\frac{\mathrm{d}h}{|h|^n}; L^p(\mathbb{R}^n)\big)$. As usual, if one simply integrates for $h \in B(0,\delta)$ for a fixed $\delta > 0$ then an equivalent norm is obtained, because

$$\left(\int_{\{|h|\geq\delta\}}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}\mathrm{d}x\right)^{\frac{q}{p}}\frac{\mathrm{d}h}{|h|^n}\right)^{\frac{1}{q}} \leq c(n,\alpha,p,q,\delta)\|v\|_{L^p(\mathbb{R}^n)}.$$

Similarly, we say that $v \in B^\alpha_{p,\infty}(\mathbb{R}^n)$ if $v \in L^p(\mathbb{R}^n)$ and

$$[v]_{B^\alpha_{p,\infty}(\mathbb{R}^n)} = \sup_{h\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|v(x+h)-v(x)|^p}{|h|^{\alpha p}}\mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Again, one can simply take supremum over $|h| \leq \delta$ and obtain an equivalent norm. By construction, $B^\alpha_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$. One also has the following version of Sobolev embeddings (a proof can be found at [70, Proposition 7.12]).

Lemma 5.1.1. *Suppose that* $0 < \alpha < 1$*.*
(a) If $1 < p < \frac{n}{\alpha}$ *and* $1 \leq q \leq p^*_\alpha = \frac{np}{n-\alpha p}$*, then there is a continuous embedding* $B^\alpha_{p,q}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$*.*
(b) If $p = \frac{n}{\alpha}$ *and* $1 \leq q \leq \infty$*, then there is a continuous embedding* $B^\alpha_{p,q}(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$*, where* BMO *denotes the space of functions with bounded mean oscillations* [64, Chapter 2].

For further needs, we recall the following inclusions ([70, Proposition 7.10 and Formula (7.35)]).

Lemma 5.1.2. *Suppose that* $0 < \beta < \alpha < 1$.
(a) If $1 < p < \infty$ *and* $1 \leq q \leq r \leq \infty$, *then* $B^{\alpha}_{p,q}(\mathbb{R}^n) \subset B^{\alpha}_{p,r}(\mathbb{R}^n)$.
(b) If $1 < p < \infty$ *and* $1 \leq q, r \leq \infty$, *then* $B^{\alpha}_{p,q}(\mathbb{R}^n) \subset B^{\beta}_{p,r}(\mathbb{R}^n)$.
(c) If $1 \leq q \leq \infty$, *then* $B^{\alpha}_{\frac{n}{\alpha},q}(\mathbb{R}^n) \subset B^{\beta}_{\frac{n}{\beta},q}(\mathbb{R}^n)$.

Combining Lemmas 5.1.1 and 5.1.2, we get the following Sobolev type embedding theorem for Besov spaces $B^{\alpha}_{p,\infty}(\mathbb{R}^n)$.

Lemma 5.1.3. *Suppose that* $0 < \alpha < 1$ *and* $1 < p < \frac{n}{\alpha}$. *There is a continuous embedding* $B^{\alpha}_{p,\infty}(\mathbb{R}^n) \subset L^{p^*_\beta}(\mathbb{R}^n)$, *for every* $0 < \beta < \alpha$. *Moreover, the following local estimate*

$$\|F\|_{L^{\frac{np}{n-\beta p}}(B_\varrho)} \leq c(R-\varrho)^{-\delta}(\|F\|_{L^p(B_R)} + [F]_{B^{\alpha}_{p,q}(B_R)}) \tag{5.2}$$

holds for every ball $B_\varrho \subset B_R$, *with* $c = c(n, p, q, \alpha, \beta)$ *and* $\delta = \delta(n, p, q)$.

Given a domain $\Omega \subset \mathbb{R}^n$, we say that v belongs to the local Besov space $B^{\alpha}_{p,q,loc}$ if $\varphi\, v \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ whenever $\varphi \in \mathcal{C}^{\infty}_c(\Omega)$. It is worth noticing that one can prove suitable version of Lemma 5.1.1 and Lemma 5.1.2, by using local Besov spaces.

The following Lemma and its proof can be found in [6].

Lemma 5.1.4. *A function* $v \in L^p_{loc}(\Omega)$ *belongs to the local Besov space* $B^{\alpha}_{p,q,loc}$ *if, and only if,*

$$\left\| \frac{\tau_h v}{|h|^\alpha} \right\|_{L^q\left(\frac{dh}{|h|^n}; L^p(B)\right)} < \infty$$

for any ball $B \subset 2B \subset \Omega$ *with radius* r_B. *Here the measure* $\frac{dh}{|h|^n}$ *is restricted to the ball* $B(0, r_B)$ *on the h-space.*

It is known that Besov-Lipschitz spaces of fractional order $\alpha \in (0,1)$ can be characterized in pointwise terms. Given a measurable function $v : \mathbb{R}^n \to \mathbb{R}$, a *fractional α-Hajlasz gradient for v* is a sequence $\{g_k\}_k$ of measurable, non-negative functions $g_k : \mathbb{R}^n \to \mathbb{R}$, together with a null set $N \subset \mathbb{R}^n$, such that the inequality

$$|v(x) - v(y)| \leq (g_k(x) + g_k(y))|x-y|^\alpha$$

holds whenever $k \in \mathbb{Z}$ and $x, y \in \mathbb{R}^n \setminus N$ are such that $2^{-k} \leq |x-y| < 2^{-k+1}$. We say that $\{g_k\}_k \in l^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ if

$$\|\{g_k\}_k\|_{l^q(L^p)} = \left(\sum_{k\in\mathbb{Z}} \|g_k\|^q_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{q}} < \infty$$

The following result was proved in [81].

Theorem 5.1.5. *Let $0 < \alpha < 1$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let $v \in L^p(\mathbb{R}^n)$. One has $v \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ if, and only if, there exists a fractional α-Hajlasz gradient $\{g_k\}_k \in l^q(\mathbb{Z}; L^p(\mathbb{R}^n))$ for v. Moreover,*

$$\|v\|_{B^{\alpha}_{p,q}(\mathbb{R}^n)} \simeq \inf \|\{g_k\}_k\|_{l^q(L^p)},$$

where the infimum runs over all possible fractional α-Hajlasz gradients for v.

5.2 Difference quotient

We recall some properties of the finite difference quotient operator that will be needed in the sequel. Let us recall that, for every function $F : \mathbb{R}^n \to \mathbb{R}$ the finite difference operator is defined by

$$\tau_{s,h}F(x) = F(x + he_s) - F(x)$$

where $h \in \mathbb{R}^n$, e_s is the unit vector in the x_s direction and $s \in \{1, ..., n\}$.
We start with the description of some elementary properties that can be found, for example, in [64].

Proposition 5.2.1. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \geq 1$, and let us consider the set*

$$\Omega_{|h|} = \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then
(i) $\tau_h F \in W^{1,p}(\Omega_{|h|})$ and

$$D_i(\tau_h F) = \tau_h(D_i F).$$

(ii) If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then

$$\int_{\Omega} F \tau_h G \, dx = \int_{\Omega} G \tau_{-h} F \, dx.$$

(iii) We have

$$\tau_h(FG)(x) = F(x + h)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result about finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 5.2.2. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$ and $F,\ DF \in L^p(B_R)$, then*

$$\int_{B_\rho} |\tau_h F(x)|^p \, dx \leq c(n, p)|h|^p \int_{B_R} |DF(x)|^p \, dx.$$

Moreover,

$$\int_{B_\rho} |F(x+h)|^p\,dx \le \int_{B_R} |F(x)|^p\,dx.$$

We conclude this subsection recalling the following Lemma (see [85]), which can be seen as a consequence of Lemmas 5.1.1 and 5.1.2.

Lemma 5.2.3. *Let $F \in L^2(B_R)$. Suppose that there exist $\rho \in (0,R)$, $0 < \alpha < 1$ and $M > 0$ such that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h}F(x)|^2 dx \le M^2 |h|^{2\alpha},$$

for every h such that $h < \frac{R-\rho}{2}$. Then $F \in L^{\frac{2n}{n-2\beta}}(B_\rho)$ for every $\beta \in (0,\alpha)$ and

$$\|F\|_{L^{\frac{2n}{n-2\beta}}(B_\rho)} \le c(M + \|F\|_{L^2(B_R)}),$$

with $c = c(n, N, R, \rho, \alpha, \beta)$.

5.3 Preliminary results on standard growth conditions

For sake of clarity, we would like to recall the following regularity result (see [43] for the proof), which will be used in order to prove Theorem 6.0.1.

In the case of a regularity of the type $B^{\alpha}_{p,\infty}$, which is the weakest one in the scale of Besov spaces, both on the coefficients and on the gradient of the obstacle, we have the following

Theorem 5.3.1. *Assume that $\mathcal{A}(x,\xi)$ satisfies (A1)-(A3) for an exponent $2 \le p = q < \frac{n}{\alpha}$ and let $u \in \mathcal{K}_\psi(\Omega)$ be the solution to the obstacle problem (6.3). If there exists a non-negative function $k \in L^{\frac{n}{\alpha}}_{loc}(\Omega)$ such that*

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \le |x-y|^\alpha (k(x)+k(y))(\mu^2 + |\xi|^2)^{\frac{p-1}{2}},$$

for a.e. $x, y \in \Omega$ and for every $\xi \in \mathbb{R}^n$, then the following implication

$$D\psi \in B^{\gamma}_{p,\infty,loc}(\Omega) \Rightarrow (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{\alpha}_{2,\infty,loc}(\Omega),$$

holds, provided $0 < \alpha < \gamma < 1$.

Chapter 6

Higher differentiability in the case of lagrangians $\tilde{F}(x, Du)$

In this chapter we study of the higher fractional differentiability properties of the gradient of solutions $u \in W^{1,p}(\Omega)$ to obstacle problems of the form

$$\min\left\{ \int_\Omega \tilde{F}(x, Dw)dx \ : \ w \in \mathcal{K}_\psi(\Omega) \right\}, \tag{6.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$. The function $\psi : \Omega \to [-\infty, +\infty)$, called *obstacle*, belongs to the Sobolev class $W^{1,p}(\Omega)$ and the class $\mathcal{K}_\psi(\Omega)$ is defined as follows

$$\mathcal{K}_\psi(\Omega) = \{w \in W^{1,p}(\Omega) : w \geq \psi \text{ a.e. in } \Omega\}. \tag{6.2}$$

Note that the set $\mathcal{K}_\psi(\Omega)$ is not empty since $\psi \in \mathcal{K}_\psi(\Omega)$.
We here assume that $\tilde{F} : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a Carathéodory function such that there exists a function $\bar{F} : \Omega \times [0, +\infty) \to [0, +\infty)$ satisfying the following equality

$$\tilde{F}(x, \xi) = \bar{F}(x, |\xi|) \tag{$\tilde{\text{F}}1$}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$.
Moreover, we also assume that there exist positive constants $\tilde{\nu}$, $\tilde{L}$, $\tilde{l}$, exponents $2 \leq p < q < +\infty$ and a parameter $\mu \in [0, 1]$, that will allow us to consider in our analysis both the degenerate and the non-degenerate situation, such that the following assumptions are satisfied:

$$\frac{1}{\tilde{l}}(|\xi|^2 - \mu^2)^{\frac{p}{2}} \leq \tilde{F}(x, \xi) \leq \tilde{l}(\mu^2 + |\xi|^2)^{\frac{q}{2}} \tag{$\tilde{\text{F}}2$}$$

$$\langle D_{\xi\xi}\tilde{F}(x, \xi)\lambda, \lambda\rangle \geq \tilde{\nu}(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \tag{$\tilde{\text{F}}3$}$$

$$|D_{\xi\xi}\tilde{F}(x,\xi)| \leq \tilde{L}(\mu^2 + |\xi|^2)^{\frac{q-2}{2}} \tag{$\tilde{\text{F}}4$}$$

for a.e. $x, y \in \Omega$ and every $\xi \in \mathbb{R}^n$.
Very recently, in [31] it has been proved that ($\tilde{\text{F}}$3) and ($\tilde{\text{F}}$4) imply ($\tilde{\text{F}}$2), i.e. if $p < q$, the functional $\tilde{F}$ has non-standard growth conditions of p, q-type, as initially defined and studied by Marcellini [94, 95, 96].
We remark that assumption ($\tilde{\text{F}}$1) is known in the literature as Uhlenbeck structure and it was showed in [114] that it prevents the irregularity phenomenon in problems with non-standard growth.

We say that function $\tilde{F}$ satisfies assumption ($\tilde{\text{F}}$5) if there exist a non-negative function $k \in L^r_{\text{loc}}(\Omega)$, with $r > \frac{n}{\alpha}$ and $0 < \alpha < 1$, such that

$$|D_{\xi}\tilde{F}(x,\xi) - D_{\xi}\tilde{F}(y,\xi)| \leq |x-y|^{\alpha}(k(x) + k(y))(\mu^2 + |\xi|^2)^{\frac{q-1}{2}} \tag{$\tilde{\text{F}}5$}$$

for a.e. $x, y \in \Omega$ and every $\xi \in \mathbb{R}^n$.
On the other hand, we say that assumption ($\tilde{\text{F}}$6) is satisfied if there exists a sequence of measurable non-negative functions $g_k \in L^r_{\text{loc}}(\Omega)$ such that

$$\sum_{k=1}^{\infty} \|g_k\|^{\sigma}_{L^r(\Omega)} < \infty,$$

and at the same time

$$|D_{\xi}\tilde{F}(x,\xi) - D_{\xi}\tilde{F}(y,\xi)| \leq |x-y|^{\alpha}(g_k(x) + g_k(y))(\mu^2 + |\xi|^2)^{\frac{q-1}{2}} \tag{$\tilde{\text{F}}6$}$$

for a.e. $x, y \in \Omega$ such that $2^{-k}\text{diam}(\Omega) \leq |x-y| < 2^{-k+1}\text{diam}(\Omega)$ and for every $\xi \in \mathbb{R}^n$.

It is worth observing that, in the case of standard growth conditions, i.e. $p = q$, $u \in W^{1,p}(\Omega)$ is a solution to the obstacle problem in $\mathcal{K}_{\psi}(\Omega)$ if, and only if, $u \in \mathcal{K}_{\psi}(\Omega)$ solves the variational inequality

$$\int_{\Omega} \langle \mathcal{A}(x, Du), D(\varphi - u)\rangle dx \geq 0 \tag{6.3}$$

for all $\varphi \in \mathcal{K}_{\psi}(\Omega)$, where we set

$$\mathcal{A}(x,\xi) = D_{\xi}\tilde{F}(x,\xi). \tag{6.4}$$

This equivalence has been proved successfully in the case non-standard growth conditions by Eleuteri and Passarelli di Napoli in [45].

From conditions ($\tilde{\text{F}}$2)–($\tilde{\text{F}}$4), we deduce the existence of positive constants ν, L, l such that the following p-ellipticity and q-growth conditions are satisfied by the map $\mathcal{A}$:

$$|\mathcal{A}(x,\xi)| \leq l(\mu^2 + |\xi|^2)^{\frac{q-1}{2}} \tag{A1}$$

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\eta), \xi - \eta \rangle \geq \nu|\xi - \eta|^2(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{A2}$$

$$|\mathcal{A}(x,\xi) - \mathcal{A}(x,\eta)| \leq L|\xi - \eta|(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{q-2}{2}} \tag{A3}$$

for a.e. $x, y \in \Omega$, for every $\xi, \eta \in \mathbb{R}^n$, where we recall that $0 < \alpha < 1$.
Furthermore, if condition ($\tilde{\text{F}}$5) or ($\tilde{\text{F}}$6) holds, then $\mathcal{A}$ satisfies assumptions (A4) or (A5), respectively, that is

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \leq |x - y|^\alpha(k(x) + k(y))(\mu^2 + |\xi|^2)^{\frac{q-1}{2}} \tag{A4}$$

for a.e. $x, y \in \Omega$ and every $\xi \in \mathbb{R}^n$, or

$$|\mathcal{A}(x,\xi) - \mathcal{A}(y,\xi)| \leq |x - y|^\alpha(g_k(x) + g_k(y))(\mu^2 + |\xi|^2)^{\frac{q-1}{2}} \tag{A5}$$

for a.e. $x, y \in \Omega$ such that $2^{-k}\text{diam}(\Omega) \leq |x - y| < 2^{-k+1}\text{diam}(\Omega)$ and for every $\xi \in \mathbb{R}^n$.

Our analysis comes from the fact that the regularity of the solutions to the obstacle problem (6.3) is strictly connected to the analysis of the regularity of the solutions to partial differential equations of the form

$$\text{div} D_\xi \tilde{F}(x, Du) = \text{div} D_\xi \tilde{F}(x, D\psi),$$

whose higher differentiability properties have been widely investigated (see for example [6, 24, 65, 106, 107]). We also notice that previous regularity results concerning local minimizers of integral functionals of the Calculus of Variations, under the assumption (A4), have been obtained by Kristensen and Mingione [85].

The main result of this chapter is a higher differentiability result, which can be seen as an extension of the [43] (see Theorem 5.3.1 in Section 5.3) to the case of functionals with p, q–growth.

Theorem 6.0.1. *Let $\mathcal{A}(x,\xi)$ satisfy (A1)-(A4) for exponents $2 \leq p < \frac{n}{\alpha} < r$, $p < q$ such* *that*

$$\frac{q}{p} < 1 + \frac{\alpha}{n} - \frac{1}{r}. \tag{6.5}$$

Let $u \in \mathcal{K}_\psi(\Omega)$ *be the solution to the obstacle problem* (6.3)*. Then we have*

$$D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega) \Rightarrow (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du \in B^{\alpha}_{2,\infty,loc}(\Omega), \tag{6.6}$$

provided $0 < \alpha < \gamma < 1$.

This Chapter is structured as follows. In order to prove the main result, Theorem 6.0.1, the strategy is to establish the a priori estimate for an approximating solution and then pass to the limit in the approximating problem. Therefore, we present our approximation results in Section 6.1, namely we are able to prove the existence of a sequence of functions with p-growth conditions that monotonically converges to our initial problems. Then in Section 6.2, are able to prove our result. Namely, we derive the a priori estimates in Section 6.2.1 for an approximating problem satisfying standard growth conditions. Then, in Section 6.2.2 we exploit the results of Sections 6.1 and 6.2.1 and using compactness, strictly convexity and weak lower semi-continuity of functional $\tilde{F}$, we are able to prove Theorem 6.0.1.

6.1 Approximation results

We here collect some results which will be used to prove the passage to the limit in Theorems 6.0.1.
We first recall the following Theorem, whose complete version can be found in [30] and which will be used to prove Lemma 6.1.2.

Theorem 6.1.1. *Let* $\tilde{F} : \Omega \times \mathbb{R}^n \to [0, +\infty), \tilde{F} = \tilde{F}(x, \xi)$*, be a Carathéodory function. Then, assumptions* $(\tilde{F}2)$ *and* $(\tilde{F}3)$ *imply that there exist* $c_0(p, q, \nu, R, l, L)$*,* $c_1(p, \nu) > 0$ *and a Carathéodory function* $g : \Omega \times \mathbb{R}^n \to [-c_0, +\infty)$ *s.t. for a.e.* $x \in \Omega$ *and every* $\xi \in \mathbb{R}^n$*,*

$$\tilde{F}(x, \xi) = c_1(\mu^2 + |\xi|^2)^{\frac{p}{2}} + g(x, \xi).$$

In the next lemma, we adapt a well known approximation result, which can be found in [30], to the case when the map $x \mapsto D_\xi \tilde{F}(x, \xi)$ has a Besov regularity.

Lemma 6.1.2. *Let* $\tilde{F} : \Omega \times \mathbb{R}^n \to [0, +\infty), \tilde{F} = \tilde{F}(x, \xi)$*, be a Carathéodory function, convex with respect to* ξ*, satisfying assumptions* $(\tilde{F}1)$*,* $(\tilde{F}2)$*,* $(\tilde{F}3)$ *and* $(\tilde{F}5)$*. Then there exists a sequence* $(\tilde{F}_j)$ *of Carathéodory functions* $\tilde{F}_j : \Omega \times \mathbb{R}^n \to [0, +\infty)$*, convex with respect to the last variable, monotonically convergent to* $\tilde{F}$*, such that*

(i) *for a.e.* $x \in \Omega$ *and every* $\xi \in \mathbb{R}^n$, $\tilde{F}_j(x,\xi) = \bar{F}_j(x,|\xi|)$,

(ii) *for a.e.* $x \in \Omega$, *for every* $\xi \in \mathbb{R}^n$ *and for every* j, $\tilde{F}_j(x,\xi) \leq \tilde{F}_{j+1}(x,\xi) \leq \tilde{F}(x,\xi)$,

(iii) *for a.e.* $x \in \Omega$ *and every* $\xi \in \mathbb{R}^n$, *we have* $\langle D_{\xi\xi}\tilde{F}_j(x,\xi)\lambda,\lambda\rangle \geq \bar{\nu}(\mu^2+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2$, *with* $\bar{\nu}$ *depending only on* p *and* ν,

(iv) *for a.e.* $x \in \Omega$ *and for every* $\xi \in \mathbb{R}^n$, *there exist* L_1, *independent of* j, *and* $\bar{L}_1$, *depending on* j, *such that*

$$\begin{aligned}&1/L_1(\mu+|\xi|)^p \leq \tilde{F}_j(x,\xi) \leq L_1(\mu+|\xi|)^q,\\&\tilde{F}_j(x,\xi) \leq \bar{L}_1(j)(\mu+|\xi|)^p,\end{aligned}$$

(v) *there exists a constant* $C(j) > 0$ *such that*

$$\begin{aligned}&|D_\xi\tilde{F}_j(x,\xi) - D_\xi\tilde{F}_j(y,\xi)| \leq |x-y|^\alpha(k(x)+k(y))(\mu^2+|\xi|^2)^{\frac{q-1}{2}},\\&|D_\xi\tilde{F}_j(x,\xi) - D_\xi\tilde{F}_j(y,\xi)| \leq C(j)|x-y|^\alpha(k(x)+k(y))(\mu^2+|\xi|^2)^{\frac{p-1}{2}}\end{aligned}$$

for a.e. $x, y \in \Omega$ *and for every* $\xi \in \mathbb{R}^n$.

Proof. According to Theorem 6.1.1, which holds under hypotheses ($\tilde{\text{F}}$2) and ($\tilde{\text{F}}$3), there exist the positive constants $c_0 = c_0(p,q,\nu,R,l,L)$ and $c_1 = c_1(p,\nu)$ and a function $g : \Omega \times \mathbb{R}^n \to [-c_0,+\infty)$ s.t.

$$\tilde{F}(x,\xi) = c_1(\mu^2+|\xi|^2)^{\frac{p}{2}} + g(x,\xi) \tag{6.7}$$

with g convex. Moreover there exists $\tilde{g} : \Omega \times [0,+\infty) \to [-c_0,+\infty)$ s.t. $\tilde{g}(x,|\xi|) = g(x,\xi)$ for any $\xi \in \mathbb{R}^n$. Since $n \geq 2$, for a.e. $x \in \Omega$, $t \mapsto \tilde{g}(x,t)$ is convex and increasing. For any $j \in \mathbb{N}$, we might then define $\tilde{g}_j : \Omega \times [0,+\infty) \to [-c_0,+\infty)$ as

$$\begin{aligned}&\tilde{g}_j(x,t) = \tilde{g}(x,t) \qquad \forall (x,t) \in \Omega \times [0,j],\\&\tilde{g}_j(x,t) = \tilde{g}(x,j) + D_t\tilde{g}(x,j)(t-j) \qquad \forall (x,t) \in \Omega \times (j,\infty)\end{aligned}$$

We notice that, by definition, for a.e. $x \in \Omega, t \mapsto \tilde{g}_j(x,t)$ is convex and increasing in $[0,+\infty)$ and $\tilde{g}_j(x,t) \leq \tilde{g}_{j+1}(x,t) \leq \tilde{g}(x,t)$. Combining assumption ($\tilde{\text{F}}$2), the definition of $\tilde{g}_j(x,t)$ and (6.7), we infer

$$\begin{aligned}&\tilde{g}_j(x,t) \leq l(\mu+t)^q,\\&\tilde{g}_j(x,t) \leq c(q,l,j)(\mu+t)^p.\end{aligned} \tag{6.8}$$

We now want to show that $D_t\tilde{g}_j$ has a ($\tilde{\text{F}}$5)-type growth. It is easy to see that $D_t\tilde{g}_j(x,t) = D_t\tilde{g}(x,j)$ for $t \geq j$. In particular, assumption ($\tilde{\text{F}}$5) yields $|D_t\tilde{g}(x,j) - D_t\tilde{g}(y,j)| \leq |x - y|^\alpha(k(x)+k(y))(\mu+j)^{q-1}$. Hence, for a.e. $x \in \Omega$ and every $t > 0$,

$$|D_t\tilde{g}(x,t) - D_t\tilde{g}(y,t)| \leq |x-y|^\alpha(k(x)+k(y))(\mu+t)^{q-1}. \tag{6.9}$$

Moreover, for $t \leq j$, according to (6.7) and (6.9), we obtain

$$\begin{aligned}|D_t\tilde{g}(x,t) - D_t\tilde{g}(y,t)| \leq &|x-y|^\alpha(k(x)+k(y))(\mu+t)^{p-1}(\mu+t)^{q-p}\\ \leq &|x-y|^\alpha(k(x)+k(y))(\mu+t)^{p-1}(\mu+j)^{q-p}\\ \leq &c(j)|x-y|^\alpha(k(x)+k(y))(\mu+t)^{p-1}.\end{aligned}$$

On the other hand, in the same way, for $t > j$, we get

$$\begin{aligned}|D_t\tilde{g}(x,t) - D_t\tilde{g}(y,t)| \leq &|x-y|^\alpha(k(x)+k(y))(\mu+j)^{p-1}(\mu+j)^{q-p}\\ \leq &|x-y|^\alpha(k(x)+k(y))(\mu+t)^{p-1}(\mu+j)^{q-p}\\ \leq &c(j)|x-y|^\alpha(k(x)+k(y))(\mu+t)^{p-1}.\end{aligned}$$

Eventually, for any j, we define $g_j : \Omega \times \mathbb{R}^n \to [-c_0, +\infty)$ as

$$g_j(x,\xi) = \tilde{g}_j(x,|\xi|).$$

Statements $(i), (ii), (iii), (v)$ directly follow by setting $\tilde{F}_j : \Omega \times \mathbb{R}^n \to [0, +\infty)$

$$\tilde{F}_j(x,\xi) := c_1(\mu^2+|\xi|^2)^{\frac{p}{2}} + g_j(x,\xi).$$

Property (iv) is obtained combining (6.7) with (6.8) and the definition of $\tilde{F}_j$. □

Remark 2. It is worth noting that an analogous version of Lemma 6.1.2 can be proved similarly, supposing ($\tilde{\text{F}}$6) instead of ($\tilde{\text{F}}$5). In particular, statement (v) would change as follows.

(v) There exists a constant $C(j) > 0$ such that

$$\begin{aligned}|D_\xi\tilde{F}_j(x,\xi) - D_\xi\tilde{F}_j(y,\xi)| &\leq |x-y|^\alpha(g_k(x)+g_k(y))(\mu^2+|\xi|^2)^{\frac{q-1}{2}},\\ |D_\xi\tilde{F}_j(x,\xi) - D_\xi\tilde{F}_j(y,\xi)| &\leq C(j)|x-y|^\alpha(g_k(x)+g_k(y))(\mu^2+|\xi|^2)^{\frac{p-1}{2}}\end{aligned}$$

for a.e. $x, y \in \Omega$ such that $2^{-k}\text{diam}(\Omega) \leq |x-y| < 2^{-k+1}\text{diam}(\Omega)$ and for every $\xi \in \mathbb{R}^n$.

6.2 Proof of Theorem 6.0.1

In order to prove Theorem 6.0.1, we use the following strategy. In Section 6.2.1, we derive a suitable a priori estimate for minimizers of obstacle problems with p-growth conditions. Then, in Section 6.2.2, we conclude showing that the a priori estimate is preserved when passing to the limit.

6.2.1 A priori estimate

Let us consider

$$\min\left\{\int_\Omega \tilde{F}_j(x, Dw)dx \ : \ w \in \mathcal{K}_\psi(\Omega)\right\}, \tag{6.10}$$

where $\tilde{F}_j : \Omega \times \mathbb{R}^n \to [0, +\infty)$, $\tilde{F}_j = \tilde{F}_j(x, \xi)$, was set in Lemma 6.1.2 and $\mathcal{K}_\psi(\Omega)$ was defined in (6.2).
Setting

$$\mathcal{A}_j(x, \xi) = D_\xi \tilde{F}_j(x, \xi),$$

one can easily check that $\mathcal{A}_j$ satisfies (A1)–(A4) and the following assumptions:

$$|\mathcal{A}_j(x, \xi)| \le l_1(j)(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{6.11}$$

$$|\mathcal{A}_j(x, \xi) - \mathcal{A}_j(x, \eta)| \le L_1(j)|\xi - \eta|(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \tag{6.12}$$

$$|\mathcal{A}_j(x, \xi) - \mathcal{A}_j(y, \xi)| \le \Theta(j)|x - y|^\alpha (k(x) + k(y))(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{6.13}$$

for a.e. $x, y \in \Omega$, for every $\xi, \eta \in \mathbb{R}^n$. It is well known that $u_j \in \mathcal{K}_\psi(\Omega)$ is a minimizer of problem (6.10) if, and only if, the following variational inequality holds

$$\int_\Omega \langle \mathcal{A}_j(x, Du_j), D(\varphi - u_j)\rangle dx \ge 0, \quad \forall \varphi \in \mathcal{K}_\psi(\Omega). \tag{6.14}$$

The following result holds:

Theorem 6.2.1. *Let $\mathcal{A}_j(x, \xi)$ satisfy (A1)–(A4) and* (6.12)– (6.13) *for exponents* $2 \le p < \frac{n}{\alpha} < r$, $p < q$ *satisfying* (6.5). *Let* $u_j \in \mathcal{K}_\psi(\Omega)$ *be the solution to the obstacle problem* (6.14). *Suppose that* $k \in L^r_{loc}(\Omega)$ *and* $D\psi \in B^\gamma_{2q-p,\infty,loc}(\Omega)$, *for* $0 < \alpha < \gamma < 1$. *Then, the following estimate*

$$\int_{B_{R/4}} |\tau_h V_p(Du_j)|^2 dx \le C|h|^{2\alpha}\left\{\int_{B_R} (1 + |Du_j|^p)dx + \|D\psi\|_{B^\gamma_{2q-p,\infty}(B_R)}\right\}^\kappa, \tag{6.15}$$

holds for all balls $B_{R/4} \subset B_R \Subset \Omega$, for positive constants $C := C(R, n, p, q, r, \beta)$, $\kappa := \kappa(n, p, q, r, \beta)$, both independent of j, and for some $0 < \beta < \alpha$.

Proof. We start by observing that, since $p < 2q - p$, we have

$$D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega) \Rightarrow D\psi \in B^{\gamma}_{p,\infty,loc}(\Omega),$$

thus an application of Theorem 5.3.1 implies

$$(\mu^2 + |Du_j|^2)^{\frac{p-2}{4}} Du_j \in B^{\alpha}_{2,\infty,loc}(\Omega),$$

which yields, by applying Lemma 5.2.3,

$$Du_j \in L^{\frac{np}{n-2\beta}}_{loc}(\Omega),$$

for all $0 < \beta < \alpha$. Thus, the integral

$$\int_{\Omega'} (1 + |Du_j|)^{\frac{np}{n-2\beta}}\, dx$$

is finite, for every $\Omega' \Subset \Omega$ and $\beta \in (0, \alpha)$.
In the sequel we will profusely use the following inequality:

$$2q - p \le \frac{r(2q-p)}{r-2} \le \frac{np}{n-2\beta}, \tag{6.16}$$

for $\beta \in (\frac{\alpha n r}{nr + 2(\alpha r - n)}, \alpha)$. The first part of inequality (6.16) is trivial, while the second part comes from (6.5). Namely,

$$\frac{r(2q-p)}{r-2} \le \frac{np}{n-2\beta} \Leftrightarrow \frac{q}{p} \le \frac{nr - n - \beta r}{r(n-2\beta)}$$

and

$$1 + \frac{\alpha}{n} - \frac{1}{r} < \frac{nr - n - \beta r}{r(n-2\beta)} \Leftrightarrow \beta > \frac{\alpha n r}{nr + 2(\alpha r - n)}.$$

Fix $0 < \frac{R}{4} < \rho < s < t < t' < \frac{R}{2}$ such that $B_R \Subset \Omega$ and a cut-off function $\eta \in \mathcal{C}^1_0(B_t)$ such that $0 \le \eta \le 1$, $\eta = 1$ on B_s, $|D\eta| \le \frac{C}{t-s}$.
Now, for $|h| \le t' - t$, we consider functions

$$v_1(x) = \eta^2(x)[(u_j - \psi)(x+h) - (u_j - \psi)(x)]$$

and

$$v_2(x) = \eta^2(x-h)[(u_j-\psi)(x-h)-(u_j-\psi)(x)].$$

Then

$$\varphi_1(x) = u_j(x) + tv_1(x), \tag{6.17}$$

$$\varphi_2(x) = u_j(x) + tv_2(x) \tag{6.18}$$

are admissible test functions for all $t \in [0,1)$.
Inserting (6.17) and (6.18) in (6.14), we obtain

$$\int_\Omega \langle \mathcal{A}_j(x, Du_j), D(\eta^2\tau_h(u_j-\psi))\rangle dx + \int_\Omega \langle \mathcal{A}_j(x, Du_j), D(\eta^2(x-h)\tau_{-h}(u_j-\psi))\rangle dx \geq 0 \tag{6.19}$$

By means of a simple change of variable, we can write the second integral on the left hand side of the previous inequality as follows

$$\int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)), D(-\eta^2\tau_h(u_j-\psi))\rangle dx \tag{6.20}$$

and so inequality (6.19) becomes

$$\int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x, Du_j(x)), D(\eta^2\tau_h(u_j-\psi))\rangle dx \leq 0 \tag{6.21}$$

We can write previous inequality as follows

$$\begin{aligned}
0 \geq & \int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), \eta^2 D\tau_h u_j\rangle dx \\
& - \int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), \eta^2 D\tau_h \psi\rangle dx \\
& + \int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x+h)) - \mathcal{A}_j(x+h, Du_j(x)), 2\eta D\eta\tau_h(u_j-\psi)\rangle dx \\
& + \int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), \eta^2 D\tau_h u_j\rangle dx \\
& - \int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), \eta^2 D\tau_h \psi\rangle dx \\
& + \int_\Omega \langle \mathcal{A}_j(x+h, Du_j(x)) - \mathcal{A}_j(x, Du_j(x)), 2\eta D\eta\tau_h(u_j-\psi)\rangle dx \\
=: & I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned} \tag{6.22}$$

that yields

$$I_1 \leq |I_2| + |I_3| + |I_4| + |I_5| + |I_6| \tag{6.23}$$

The ellipticity assumption (A2) implies

$$I_1 \geq \nu \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \tag{6.24}$$

From the growth condition (A3), Young's and Hölder's inequalities and assumption on $D\psi$, we get

$$\begin{aligned}
|I_2| \leq & L \int_\Omega \eta^2 |\tau_h Du_j| (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{q-2}{2}} |\tau_h D\psi| dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L) \int_\Omega \eta^2 |\tau_h D\psi|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{2q-p-2}{2}} dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L) \left(\int_{B_t} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{2}{2q-p}} \left(\int_{B_{t'}} (1 + |Du_j|)^{2q-p} dx \right)^{\frac{2q-p-2}{2q-p}} \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L, n, p, q) |h|^{2\gamma} [D\psi]^2_{B^\gamma_{2q-p,\infty}(B_R)} \left(\int_{B_{t'}} (1 + |Du_j|)^{2q-p} dx \right)^{\frac{2q-p-2}{2q-p}} \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L, n, p, q) |h|^{2\gamma} [D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)} + C_\varepsilon(L, n, p, q) |h|^{2\gamma} \int_{B_{t'}} (1 + |Du_j|)^{2q-p} dx.
\end{aligned}$$

Therefore, from (6.16), we infer

$$\begin{aligned}
|I_2| \leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
& + C_\varepsilon(L, n, p, q) |h|^{2\gamma} [D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)} + C_\varepsilon(L, n, p, q) |h|^{2\gamma} \left(\int_{B_{t'}} (1 + |Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}}.
\end{aligned} \tag{6.25}$$

Arguing analogously, we get

$$|I_3| \leq 2L \int_\Omega |D\eta| \eta |\tau_h Du_j| (1 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{q-2}{2}} |\tau_h (u_j - \psi)| dx$$

$$
\begin{aligned}
&\leq \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&\quad + \frac{C_\varepsilon(L)}{(t-s)^2} \int_{B_t} |\tau_h(u_j - \psi)|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{2q-p-2}{2}} dx \\
&\leq \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&\quad + \frac{C_\varepsilon(L)}{(t-s)^2} \left(\int_{B_t} |\tau_h \psi|^{2q-p} dx \right)^{\frac{2}{2q-p}} \left(\int_{B_{t'}} (1 + |Du_j|)^{2q-p} dx \right)^{\frac{2q-p-2}{2q-p}} \\
&\quad + \frac{C_\varepsilon(L)}{(t-s)^2} \left(\int_{B_t} |\tau_h u_j|^{2q-p} dx \right)^{\frac{2}{2q-p}} \left(\int_{B_{t'}} (1 + |Du_j|)^{2q-p} dx \right)^{\frac{2q-p-2}{2q-p}}.
\end{aligned}
$$

Using Young's inequality and Lemma 5.2.2, we obtain

$$
\begin{aligned}
|I_3| &\leq \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&\quad + \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2} |h|^2 \left(\int_{B_R} |D\psi|^{2q-p} dx \right)^{\frac{2}{2q-p}} \left(\int_{B_{t'}} (1 + |Du_j|)^{2q-p} dx \right)^{\frac{2q-p-2}{2q-p}} \\
&\quad + \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2} |h|^2 \int_{B_{t'}} (1 + |Du_j|)^{2q-p} dx \\
&\leq \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&\quad + \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2} |h|^2 \int_{B_R} |D\psi|^{2q-p} dx \\
&\quad + \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2} |h|^2 \int_{B_{t'}} (1 + |Du_j|)^{2q-p} dx. \qquad (6.26)
\end{aligned}
$$

Recalling the first inequality of (6.16), we can write

$$
\begin{aligned}
|I_3| &\leq \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&\quad + \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2} |h|^2 \int_{B_R} |D\psi|^{2q-p} dx \\
&\quad + \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2} |h|^2 \left(\int_{B_{t'}} (1 + |Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{r-2}{r}}. \qquad (6.27)
\end{aligned}
$$

In order to estimate the integral I_4, we use assumption (A4), and Young's and Hölder's inequalities as follows

$$
\begin{aligned}
|I_4| &\leq \int_\Omega \eta^2 |\tau_h Du_j| |h|^\alpha (k(x+h) + k(x))(1 + |Du_j(x)|)^{\frac{q-1}{2}} dx \\
&\leq \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx
\end{aligned}
$$

$$
\begin{aligned}
&+ C_\varepsilon |h|^{2\alpha} \int_{B_t} (k(x+h)+k(x))^2 (1+|Du_j|)^{2q-p} dx \\
\leq & \varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&+ C_\varepsilon |h|^{2\alpha} \Big(\int_{B_R} k^r dx \Big)^{\frac{2}{r}} \Big(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \Big)^{\frac{r-2}{r}}. \qquad (6.28)
\end{aligned}
$$

We now take care of I_5. Similarly as above, exploiting assumption (A4) and Hölder's inequality, we infer

$$
\begin{aligned}
|I_5| \leq & \int_\Omega \eta^2 |\tau_h D\psi| |h|^\alpha \left(k(x+h)+k(x)\right) \left(1+|Du_j|^2\right)^{\frac{q-1}{2}} dx \\
\leq & |h|^\alpha \left(\int_{B_{t'}} k^r dx \right)^{\frac{1}{r}} \left(\int_{B_t} |\tau_h D\psi|^{\frac{r}{r-1}} (1+|Du_j|)^{\frac{r(q-1)}{r-1}} dx \right)^{\frac{r-1}{r}} \\
\leq & |h|^\alpha \left(\int_{B_R} k^r dx \right)^{\frac{1}{r}} \left(\int_{B_t} |\tau_h D\psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \left(\int_{B_t} (1+|Du_j|)^{\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r}} dx \right)^{\frac{(r-1)(2q-p)-r}{r(2q-p)}}.
\end{aligned}
$$

Now, we observe

$$
\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r} \leq \frac{r(2q-p)}{r-2} \Leftrightarrow p-2+r(q-p) \geq 0, \qquad (6.29)
$$

which is true by assumption, that is $p \geq 2$, $r > \frac{n}{\alpha} > 2$ and $q > p$. Hence

$$
|I_5| \leq |h|^{\alpha+\gamma} \left(\int_{B_R} k^r dx \right)^{\frac{1}{r}} [D\psi]_{B^\gamma_{2q-p,\infty}(B_R)} \left(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \right)^{\frac{(r-2)(q-1)}{r(2q-p)}}. \qquad (6.30)
$$

From assumption (A4), hypothesis $|D\eta| < \frac{C}{t-s}$ and Hölder's inequality, we infer the following estimate for I_6.

$$
\begin{aligned}
|I_6| \leq & \frac{C}{t-s} |h|^\alpha \int_{B_t} |\tau_h \psi| (k(x+h)+k(x))(1+|Du_j|^2)^{\frac{q-1}{2}} dx \\
&+ \frac{C}{t-s} |h|^\alpha \int_{B_t} |\tau_h u_j| (k(x+h)+k(x))(1+|Du_j|^2)^{\frac{q-1}{2}} dx \\
\leq & \frac{C}{t-s} |h|^\alpha \left(\int_{B_R} k^r dx \right)^{\frac{1}{r}} \left(\int_{B_t} |\tau_h \psi|^{2q-p} dx \right)^{\frac{1}{2q-p}} \\
& \cdot \left(\int_{B_t} (1+|Du_j|)^{\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r}} dx \right)^{\frac{(r-1)(2q-p)-r}{r(2q-p)}} \\
&+ \frac{C}{t-s} |h|^\alpha \left(\int_{B_{t'}} k^r dx \right)^{\frac{1}{r}} \left(\int_{B_t} |\tau_h u_j|^{\frac{r}{r-1}} (1+|Du_j|)^{\frac{r(q-1)}{r-1}} dx \right)^{\frac{r-1}{r}}.
\end{aligned}
$$

Using once again Hölder's inequality, we have

$$
\begin{aligned}
|I_6| \leq & \frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R} k^r dx\right)^{\frac{1}{r}}\left(\int_{B_{t'}}|D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}} \\
& \cdot\left(\int_{B_t}(1+|Du_j|)^{\frac{r(q-1)(2q-p)}{(r-1)(2q-p)-r}}dx\right)^{\frac{(r-1)(2q-p)-r}{r(2q-p)}} \\
& + \frac{C}{t-s}|h|^{\alpha}\left(\int_{B_R} k^r dx\right)^{\frac{1}{r}}\left(\int_{B_t}|\tau_h u_j|^{\frac{rq}{r-1}}dx\right)^{\frac{r-1}{rq}}\left(\int_{B_t}(1+|Du_j|)^{\frac{rq}{r-1}}dx\right)^{\frac{(r-1)(q-1)}{rq}}.
\end{aligned}
$$

Using Lemma 5.2.2, we infer

$$
\begin{aligned}
|I_6| \leq & \frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R} k^r dx\right)^{\frac{1}{r}}\left(\int_{B_R}|D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}} \\
& \cdot\left(\int_{B_t}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{(q-1)(r-2)}{r(2q-p)}} \\
& + \frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R} k^r dx\right)^{\frac{1}{r}}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{rq}{r-1}}dx\right)^{\frac{r-1}{rq}}.
\end{aligned}
$$

We remark that

$$
\frac{rq}{r-1} \leq \frac{r(2q-p)}{r-2} \Leftrightarrow p + r(q-p) \geq 0, \tag{6.31}
$$

which is true by assumption, that is $p \geq 2$, $r > \frac{n}{\alpha} > 2$ and $q > p$. Hence

$$
\begin{aligned}
|I_6| \leq & \frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R} k^r dx\right)^{\frac{1}{r}}\left(\int_{B_R}|D\psi|^{2q-p}dx\right)^{\frac{1}{2q-p}} \\
& \cdot\left(\int_{B_t}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{(q-1)(r-2)}{r(2q-p)}} \\
& + \frac{C}{t-s}|h|^{\alpha+1}\left(\int_{B_R} k^r dx\right)^{\frac{1}{r}}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{q(r-2)}{r(2q-p)}}.
\end{aligned} \tag{6.32}
$$

Inserting estimates (6.24), (6.25), (6.27), (6.28), (6.30) and (6.32) in (6.23), we infer

$$
\begin{aligned}
\nu & \int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
\leq & 3\varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx \\
& + C_\varepsilon(L,n,p,q)|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)} + C_\varepsilon(L,n,p,q)|h|^{2\gamma}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{r-2}{r}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2}|h|^2 \int_{B_R} |D\psi|^{2q-p} dx \\
&+ \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2}|h|^2 \Big(\int_{B_{t'}} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \Big)^{\frac{r-2}{r}} \\
&+ C_\varepsilon |h|^{2\alpha} \Big(\int_{B_R} k^r dx \Big)^{\frac{2}{r}} \Big(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \Big)^{\frac{r-2}{r}} \\
&+ |h|^{\alpha+\gamma} \Big(\int_{B_R} k^r dx \Big)^{\frac{1}{r}} [D\psi]_{B^\gamma_{2q-p,\infty}(B_R)} \Big(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \Big)^{\frac{(r-2)(q-1)}{r(2q-p)}} \\
&+ \frac{C}{t-s}|h|^{\alpha+1} \Big(\int_{B_R} k^r dx \Big)^{\frac{1}{r}} \Big(\int_{B_R} |D\psi|^{2q-p} dx \Big)^{\frac{1}{2q-p}} \\
&\cdot \Big(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \Big)^{\frac{(q-1)(r-2)}{r(2q-p)}} \\
&+ \frac{C}{t-s}|h|^{\alpha+1} \Big(\int_{B_R} k^r dx \Big)^{\frac{1}{r}} \Big(\int_{B_{t'}} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \Big)^{\frac{q(r-2)}{r(2q-p)}} . \qquad (6.33)
\end{aligned}
$$

We now introduce the following interpolation inequality

$$
\|Dw\|_{\frac{r(2q-p)}{r-2}} \le \|Dw\|_p^\delta \|Dw\|^{1-\delta}_{\frac{np}{n-2\beta}}, \qquad (6.34)
$$

where $0 < \delta < 1$ is defined through the condition

$$
\frac{r-2}{r(2q-p)} = \frac{\delta}{p} + \frac{(1-\delta)(n-2\beta)}{np} \qquad (6.35)
$$

which implies

$$
\delta = \frac{nr(p-q) - np + \beta r(2q-p)}{\beta r(2q-p)}, \quad 1-\delta = \frac{n[r(q-p)+p]}{\beta r(2q-p)}.
$$

Hence we get the following inequalities

$$
\begin{aligned}
\Big(\int_{B_{t'}} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \Big)^{\frac{r-2}{r}} \le & \Big(\int_{B_{t'}} (1+|Du_j|)^p dx \Big)^{\frac{\delta(2q-p)}{p}} \\
& \cdot \Big(\int_{B_{t'}} (1+|Du_j|)^{\frac{np}{n-2\beta}} dx \Big)^{\frac{(n-2\beta)[r(q-p)+p]}{\beta pr}}, \qquad (6.36) \\
\Big(\int_{B_t} (1+|Du_j|)^{\frac{r(2q-p)}{r-2}} dx \Big)^{\frac{(r-2)(q-1)}{r(2q-p)}} \le & \Big(\int_{B_t} (1+|Du_j|)^p dx \Big)^{\frac{\delta(q-1)}{p}} \\
& \cdot \Big(\int_{B_t} (1+|Du_j|)^{\frac{np}{n-2\beta}} dx \Big)^{\frac{(n-2\beta)(q-1)p'}{p}}, \qquad (6.37)
\end{aligned}
$$

$$\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{r(2q-p)}{r-2}}dx\right)^{\frac{q(r-2)}{r(2q-p)}} \leq \left(\int_{B_{t'}}(1+|Du_j|)^p dx\right)^{\frac{\delta q}{p}} \cdot \left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{(n-2\beta)q[r(q-p)+p]}{\beta pr(2q-p)}}, \quad (6.38)$$

where $p' = \frac{r(q-p)+p}{\beta r(2q-p)}$.

Inserting (6.36), (6.37) and (6.38) in (6.33), and exploiting the bounds

$$\frac{n[r(q-p)+p]}{\beta pr} < 1, \quad \frac{n(q-1)[r(q-p)+p]}{\beta rp(2q-p)} < 1, \quad \frac{nq[r(q-p)+p]}{\beta pr(2q-p)} < 1, \quad (6.39)$$

which hold by assumption (6.5) and for $\beta \in (\frac{n[r(q-p)+p]}{pr}, \alpha)$, from Young's inequality, we infer

$$\begin{aligned}
&\nu\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\
&\leq 3\varepsilon\int_\Omega \eta^2|\tau_h Du_j|^2(\mu^2+|Du_j(x+h)|^2+|Du_j(x)|^2)^{\frac{p-2}{2}}dx\\
&\quad + C_\varepsilon(L,n,p,q)|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)} + C_{\varepsilon,\theta}(L,n,p,q)|h|^{2\gamma}\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\delta(2q-p)\tilde{p}}{p}}\\
&\quad + \theta|h|^{2\gamma}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}\\
&\quad + \frac{C_\varepsilon(L,n,p,q)}{(t-s)^2}|h|^2\int_{B_R}|D\psi|^{2q-p}dx\\
&\quad + \theta|h|^2\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}\\
&\quad + \frac{C_{\varepsilon,\theta}(L,n,p,q)}{(t-s)^{2\tilde{p}}}|h|^2\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\tilde{p}\delta(2q-p)}{p}}\\
&\quad + C_{\varepsilon,\theta}(n,p,q)|h|^{2\alpha}\left(\int_{B_R}k^r dx\right)^{\frac{2\tilde{p}}{r}}\\
&\quad \cdot\left(\int_{B_R}(1+|Du_j|)^p dx\right)^{\frac{\tilde{p}\delta(2q-p)}{p}}\\
&\quad + \theta|h|^{2\alpha}\left(\int_{B_{t'}}(1+|Du_j|)^{\frac{np}{n-2\beta}}dx\right)^{\frac{n-2\beta}{n}}\\
&\quad + C_\theta|h|^{\alpha+\gamma}\left(\int_{B_R}k^r dx\right)^{\frac{p''}{r}}\\
&\quad \cdot [D\psi]^{p''}_{B^\gamma_{2q-p,\infty}(B_R)}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{\delta(q-1)(2q-p)p''}{p}} \\
&+ \theta |h|^{\alpha+\gamma} \left(\int_{B_t} (1+|Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \\
&+ \frac{C_\theta}{(t-s)^{p''}} |h|^{\alpha+1} \left(\int_{B_R} k^r dx \right)^{\frac{p''}{r}} \left(\int_{B_R} |D\psi|^{2q-p} dx \right)^{\frac{p''}{2q-p}} \\
&\cdot \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{\delta(q-1)p''}{p}} \\
&+ \theta |h|^{\alpha+1} \left(\int_{B_t} (1+|Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \\
&+ \frac{C_\theta}{(t-s)^{p^*}} |h|^{\alpha+1} \left(\int_{B_R} k^r dx \right)^{\frac{p^*}{r}} \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{p^*\delta q}{p}} \\
&+ \theta |h|^{\alpha+1} \left(\int_{B_{t'}} (1+|Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} . \qquad (6.40)
\end{aligned}
$$

for some constant $\theta \in (0,1)$, where we set $\tilde{p} = \frac{\beta pr}{\beta pr - n[r(q-p)+p]}$, $p'' = \frac{\beta rp(2q-p)}{\beta rp(2q-p)-(q-1)n[r(q-p)+p]}$, $p^* = \frac{p}{p-(1-\delta)q}$.
For a better readability we now define

$$
\begin{aligned}
A =& C_\varepsilon(L,n,p,q)[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_R)} + C_{\varepsilon,\theta}(L,n,p,q) \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{\delta(2q-p)\tilde{p}}{p}} \\
&+ C_{\varepsilon,\theta}(n,p,q) \left(\int_{B_R} k^r dx \right)^{\frac{2\tilde{p}}{r}} \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{\tilde{p}\delta(2q-p)}{p}} \\
&+ C_\theta \left(\int_{B_R} k^r dx \right)^{\frac{p''}{r}} [D\psi]^{p''}_{B^\gamma_{2q-p,\infty}(B_R)} \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{\delta(q-1)(2q-p)p''}{p}} \\
B_1 =& C_\varepsilon(L,n,p,q) \int_{B_R} |D\psi|^{2q-p} dx, \\
B_2 =& C_{\varepsilon,\theta}(L,n,p,q) \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{\tilde{p}\delta(2q-p)}{p}}, \\
B_3 =& C_\theta \left(\int_{B_R} k^r dx \right)^{\frac{p''}{r}} \left(\int_{B_R} |D\psi|^{2q-p} dx \right)^{\frac{p''}{2q-p}} \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{\delta(q-1)p''}{p}}, \\
B_4 =& C_\theta \left(\int_{B_R} k^r dx \right)^{\frac{p^*}{r}} \left(\int_{B_R} (1+|Du_j|)^p dx \right)^{\frac{p^*\delta q}{p}},
\end{aligned}
$$

so that we can rewrite the previous estimate as

$$\begin{aligned}
&\nu \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&\quad \leq 3\varepsilon \int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&\quad + \theta(|h|^{2\alpha} + |h|^{\alpha+\gamma} + |h|^{\alpha+1}) \left(\int_{B_t} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \\
&\quad + \theta(|h|^2 + |h|^{2\gamma} + |h|^{\alpha+1}) \left(\int_{B_{t'}} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \\
&\quad + (|h|^{2\gamma} + |h|^{2\alpha} + |h|^{\alpha+\gamma}) A + |h|^2 \frac{B_1}{(t-s)^2} + |h|^2 \frac{B_2}{(t-s)^{2\tilde{p}}} \\
&\quad + |h|^{\alpha+1} \frac{B_3}{(t-s)^{p''}} + |h|^{\alpha+1} \frac{B_4}{(t-s)^{p^*}}.
\end{aligned}$$

Choosing $\varepsilon = \frac{\nu}{6}$, we can reabsorb the first integral in the right hand side of the previous estimate by the left hand side, thus getting

$$\begin{aligned}
&\int_\Omega \eta^2 |\tau_h Du_j|^2 (\mu^2 + |Du_j(x+h)|^2 + |Du_j(x)|^2)^{\frac{p-2}{2}} dx \\
&\quad \leq 3\theta |h|^{2\alpha} \left(\int_{B_t} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} + 3\theta |h|^{2\alpha} \left(\int_{B_{t'}} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \\
&\quad + |h|^{2\alpha} A + |h|^2 \frac{B_1}{(t-s)^2} + |h|^2 \frac{B_2}{(t-s)^{\tilde{p}}} + |h|^{2\alpha} \frac{B_3}{(t-s)^{p''}} + |h|^{2\alpha} \frac{B_4}{(t-s)^{p^*}},
\end{aligned}$$

where we used the fact that $\alpha < \gamma$. Using Lemma 5.0.1 in the left hand side of the previous inequality, recalling that $\eta = 1$ on B_s, we get

$$\begin{aligned}
\int_{B_s} |\tau_h V_p(Du_j)|^2 dx \leq |h|^{2\alpha} \Bigg\{ & 3\theta \left(\int_{B_t} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} + 3\theta \left(\int_{B_{t'}} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \\
& + A + \frac{B_1}{(t-s)^2} + \frac{B_2}{(t-s)^{\tilde{p}}} + \frac{B_3}{(t-s)^{p''}} + \frac{B_4}{(t-s)^{p^*}} \Bigg\}. \quad (6.41)
\end{aligned}$$

Lemma 5.2.3 and inequality (5.1) imply

$$\begin{aligned}
\left(\int_{B_s} |Du_j|^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \leq & 3\theta \left(\int_{B_t} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} + 3\theta \left(\int_{B_{t'}} (1 + |Du_j|)^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \\
& + A + \frac{B_1}{(t-s)^2} + \frac{B_2}{(t-s)^{2\tilde{p}}} + \frac{B_3}{(t-s)^{p''}} + \frac{B_4}{(t-s)^{p^*}}, \quad (6.42)
\end{aligned}$$

for all $\beta \in (0, \alpha)$.
Setting

$$\Phi(r) = \left(\int_{B_r} |Du_j|^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}},$$

we can write inequality (6.42) as

$$\Phi(s) \leq 3\theta\Phi(t) + 3\theta\Phi(t') + A + \frac{B_1}{(t-s)^2} + \frac{B_2}{(t-s)^{2\tilde{p}}} + \frac{B_3}{(t-s)^{p''}} + \frac{B_4}{(t-s)^{p^*}}. \tag{6.43}$$

By virtue of Lemma 5.0.2, choosing $0 < \theta < 1/3$, we obtain

$$\Phi(\varrho) \leq c\left(3\theta\Phi(t') + A + \frac{B_1}{R^2} + \frac{B_2}{R^{2\tilde{p}}} + \frac{B_3}{R^{p''}} + \frac{B_4}{R^{p^*}} \right), \tag{6.44}$$

for some constant $c := c(n, p, q, r, \beta, \theta)$. Then, applying Lemma 5.0.2 again, we get

$$\Phi\left(\frac{R}{4}\right) \leq \tilde{c}\left(A + \frac{B_1}{R^2} + \frac{B_2}{R^{2\tilde{p}}} + \frac{B_3}{R^{p''}} + \frac{B_4}{R^{p^*}} \right), \tag{6.45}$$

with $\tilde{c} := \tilde{c}(n, p, q, r, \beta, \theta)$.
Now, recalling the definition of Φ, we obtain

$$\left(\int_{B_{R/4}} |Du_j|^{\frac{np}{n-2\beta}} dx \right)^{\frac{n-2\beta}{n}} \leq \tilde{c}\left\{ \int_{B_R} (1 + |Du_j|^p) dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(B_R)} \right\}^{\kappa}, \tag{6.46}$$

thus, using Lemma 5.2.3, from inequalities (6.46) and (6.41), we deduce the a priori estimate

$$\int_{B_{R/4}} |\tau_h V_p(Du_j)|^2 dx \leq C|h|^{2\alpha} \left\{ \int_{B_R} (1 + |Du_j|^p) dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(B_R)} \right\}^{\kappa}, \tag{6.47}$$

for some $\beta < \alpha$, where $C := C(R, n, p, q, r, \beta)$ and $\kappa := \kappa(n, p, q, r, \beta)$. □

6.2.2 Passage to the limit

Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to (6.3), and let $\tilde{F}_j$ be defined as in Lemma 6.1.2. From Theorem 6.1.1, there exists $c_1 > 0$ such that

$$|\xi|^p \leq c_1(1 + \tilde{F}_j(x, \xi)), \quad \forall j \in \mathbb{N}. \tag{6.48}$$

Fixed $B_R \Subset \Omega$, let u_j be the solution of the problem

$$\min\left\{\int_{B_R} \tilde{F}_j(x, Dw)dx : w \geq \psi \text{ a.e. in } B_R,\ w \in u + W_0^{1,p}(B_R)\right\}.$$

From (6.48), the minimality of u_j implies

$$\begin{aligned}\int_{B_R} |Du_j|^p dx \leq & c_1 \int_{B_R} (1 + \tilde{F}_j(x, Du_j))dx \\ \leq & c_1 \int_{B_R} (1 + \tilde{F}_j(x, Du))dx \\ \leq & c_1 \int_{B_R} (1 + \tilde{F}(x, Du))dx,\end{aligned} \tag{6.49}$$

where in the last inequality we used Lemma 6.1.2 (*ii*). Thus, up to subsequences,

$$u_j \rightharpoonup \tilde{u} \text{ in } u + W_0^{1,p}(B_R) \tag{6.50}$$

and

$$u_j \to \tilde{u} \text{ in } L^p(B_R). \tag{6.51}$$

For any j, $\tilde{F}_j$ satisfies the assumptions of Theorem 6.2.1. Combining (6.46) and (6.49) we get

$$\|Du_j\|_{L^{\frac{np}{n-2\beta}}(B_{R/4})} \leq \tilde{c}\left\{\int_{B_R} (1 + \tilde{F}(x, Du))dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(B_R)}\right\}^{\tilde{\kappa}}, \tag{6.52}$$

thus, by (6.50), (6.52) and weak lower semicontinuity, we infer

$$\begin{aligned}\|D\tilde{u}\|_{L^{\frac{np}{n-2\beta}}(B_{R/4})} \leq & \liminf_{j\to\infty} \|Du_j\|_{L^{\frac{np}{n-2\beta}}(B_{R/4})} \\ \leq & \tilde{c}\left\{\int_{B_R} (1 + \tilde{F}(x, Du))dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(B_R)}\right\}^{\tilde{\kappa}}.\end{aligned} \tag{6.53}$$

Let $j_0, j \in \mathbb{N}$ s.t. $j_0 < j$. Then, by Lemma 6.1.2 (ii) and the fact that u_j is a minimum for $\tilde{F}_j$, we might write

$$\begin{aligned}\int_{B_R} \tilde{F}_{j_0}(x, Du_j)dx \leq & \int_{B_R} \tilde{F}_j(x, Du_j)dx \\ \leq & \int_{B_R} \tilde{F}_j(x, Du)dx \leq \int_{B_R} \tilde{F}(x, Du)dx.\end{aligned}$$

Now from weak lower semicontinuity of $\tilde{F}_{j_0}$ and (6.50), it holds, for every $j_0 \in \mathbb{N}$,

$$\int_{B_R} \tilde{F}_{j_0}(x, D\tilde{u})dx \leq \liminf_{j \to +\infty} \int_{B_R} \tilde{F}_{j_0}(x, Du_j)dx \leq \int_{B_R} \tilde{F}(x, Du)dx.$$

Combining these last inequalities, we get

$$\int_{B_R} \tilde{F}(x, D\tilde{u})dx = \lim_{j_0 \to +\infty} \int_{B_R} \tilde{F}_{j_0}(x, D\tilde{u})dx \leq \int_{B_R} \tilde{F}(x, Du)dx, \tag{6.54}$$

where we also applied the monotone convergence theorem, according to Lemma 6.1.2 (ii). Moreover, by the weak convergence (6.50), the limit function $\tilde{u}$ still belongs to $\mathcal{K}_\psi(B_R)$, since this set is convex and closed. Thus, we can conclude that

$$\tilde{u} = u \quad \text{a.e. in } B_R \tag{6.55}$$

by strict convexity of $\tilde{F}$, and, recalling estimate (6.53),

$$\|Du\|_{L^{\frac{np}{n-2\beta}}(B_{R/4})} \leq \tilde{c}\left\{\int_{B_R} (1 + \tilde{F}(x, Du))dx + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(B_R)}\right\}^{\tilde{\kappa}}. \tag{6.56}$$

Finally, we can repeat the proof of Theorem 6.2.1 obtaining $V_p(Du) \in B^{\alpha}_{2,\infty,loc}(\Omega)$.

Chapter 7

Higher differentiability in the case of double-phase lagrangians $F(x, u, Du)$

According to the results presented in Chapter 6, we here study the higher fractional differentiability properties of the gradient of the solutions $u \in W^{1,p}(\Omega)$ to variational obstacle problems of the form

$$\min\left\{ \int_\Omega F(x, w, Dw) \mathrm{d}x \ : \ w \in \mathcal{K}_\psi(\Omega) \right\}, \tag{7.1}$$

where the energy density $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$F(x, w, z) = b(x, w)H(x, z), \tag{7.2}$$

being

$$H(x, z) = |z|^p + a(x)|z|^q, \tag{7.3}$$

where $2 \leq p < q$.
Here Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 2$, the obstacle function ψ is the one defined in Chapter 6, as well as the set $\mathcal{K}_\psi(\Omega)$ (see (6.2)).
We assume that the coefficients $a(x)$ and $b(x, w)$ satisfy the following assumptions:

Assumption 1

(i) $a : \Omega \to [0, +\infty)$ is a bounded and measurable function such that

$$|a(x) - a(y)| \leq \omega_a(|x - y|),$$

for all $x, y \in \Omega$, where $\omega_a : \mathbb{R}^+ \to [0, 1]$ is defined by $\omega_a(\rho) = \min\{\rho^\alpha, 1\}$, for some $\alpha \in (0, 1)$;

(ii) the function $b : \Omega \times \mathbb{R} \to (0, +\infty)$ is a bounded Carathéodory function, i.e. there exist $0 < \nu \leq L$ such that

$$0 < \nu \leq b(x, w) \leq L < \infty.$$

Assumption 2

(i) there exists a function $\omega_b : \mathbb{R}^+ \to [0, 1]$ defined by $\omega_b(\rho) = \min\{\rho^\beta, 1\}$, for some $\beta \in (0, 1)$, such that

$$|b(x, u) - b(y, v)| \leq \omega_b(|x - y| + |u - v|),$$

for all $x, y \in \Omega$ and every $u, v \in \mathbb{R}$.

The energy density given by (7.2) is a model case of functions F satisfying the following set of conditions

$$\nu_1 |z|^p \leq F(x, w, z) \leq L_1(1 + |z|^q) \tag{F1}$$

$$\nu_2 |z|^{p-2} |\lambda|^2 \leq \langle D_{zz} F(x, w, z)\lambda, \lambda\rangle \leq L_2(1 + |z|^{q-2})|\lambda|^2 \tag{F2}$$

$$|F(x_1, w_1, z) - F(x_2, w_2, z)| \leq l_1 \omega_\delta(|x_1 - x_2| + |w_1 - w_2|)(1 + |z|^q) \tag{F3}$$

for all $x, x_1, x_2 \in \Omega$, $w, w_1, w_2 \in \mathbb{R}$ and every $z, \lambda \in \mathbb{R}^n$, where $0 < \nu_1 \leq L_1$, $0 < \nu_2 \leq L_2$, $l_1 \geq 1$ are fixed constants and $\omega_\delta : \mathbb{R}^+ \to [0, 1]$ is a function defined by $\omega_\delta(\rho) = \min\{\rho^\delta, 1\}$, for some $\delta \in (0, 1)$ depending on α and β introduced in Assumption 1 and 2 respectively. We point out that the choice of stating Assumption 1 and 2 separately is due to the fact that they are needed independently.

The main difficulty here is the dependence of our double phase functional both on the $x-$variable and the $w-$variable, where the map $w \mapsto b(x, w)H(x, z)$ is non-differentiable. In order to deal with this issue, we follow the strategy proposed in [85] and later used in [44]. Namely, we introduce the so-called "freezed" functional defined in (7.4) and the solution to the corresponding obstacle problem (see (7.5)) for which we prove a higher differentiability result in the scale of Besov spaces following the argument in [67]. The idea is to compare the solution u to the original obstacle problem (7.1) and the solution v to the "freezed" one (7.5). More precisely, we estimate the fractional difference quotients of u and v, in an integral sense, gaining a Besov regularity for u. In order to do so, we also have to derive some ad hoc higher integrability results, both at the interior and up to boundary, that is for the solution u of the original obstacle problem (7.1) and the solution v to the freezed one (7.5) respectively. The first one is obtained adapting the

argument in [38], while the second one generalizes the result by Cupini, Fusco and Petti in [29]. Eventually, we use a boot-strapping argument to get the maximal higher fractional differentiability.
The main result of this Chapter is the following.

Theorem 7.0.1. *Let $u \in W^{1,p}(\Omega)$ be the solution to the obstacle problem (7.1), with F defined by (7.2), under Assumptions 1 and 2, for exponents $2 \le p < \frac{n}{\alpha}$, $p < q$ verifying*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}.$$

If $D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega)$, for $0 < \alpha < \gamma < 1$, then there exists $\tilde{\sigma} := \tilde{\sigma}(p, q, n, \alpha, \beta) \in (0, 1)$ s.t.

$$V_p(Du), \ \sqrt{a(x)}V_q(Du) \in B^{t}_{2,\infty,loc}(\Omega), \quad \forall t \in (0, \tilde{\sigma}).$$

The chapter is organized as follows. In Section 7.1, we show that the solution to the freezed obstacle problem (7.5) satisfies a variational inequality and moreover we present interior and up to the boundary higher integrability properties, which will be crucial for the comparison argument, as already mentioned. In Section 7.2, we exploit the results obtained in Chapter 6 in order to prove the higher fractional differentiability of the solution to the freezed obstacle problem (7.5). We remark that the procedure used in order to do so requires the assumption $p \ge 2$. The comparison argument is presented in Section 7.3. Finally, in Section 7.4, we show that a suitable fractional differentiability property on the gradient of the obstacle transfers to a higher fractional differentiability for the gradient of the minimizer, so that we are eventually able to prove Theorem 7.0.1.

We point out that, in order to prove the higher integrability of the solution to the original obstacle problem (see Theorem 7.1.2) and the higher differentiability of the solution to the freezed obstacle problem in Section 7.2, Assumption 1 (ii) is the only one needed on the function $b(x, w)$. On the other hand, in order to prove the comparison lemma (see Lemma 7.3.2), we require Assumption 2 on the coefficient $b(x, w)$.

7.1 Higher integrability

The results contained in this section will be crucial for the comparison argument presented in Section 7.3.

Let u be a solution to the obstacle problem (7.1) and fix a ball $B = B_{\frac{R}{2}}(x_0) \Subset \Omega$, for a given radius $R > 0$ and $x_0 \in \Omega$. Let us consider the so-called "freezed" functional

$$\int_B \tilde{F}(x, Dw)\mathrm{d}x = \int_B b(x_0, u_B)H(x, Dw)\mathrm{d}x, \tag{7.4}$$

where H was defined in (7.3), and let $v \in u + W_0^{1,p}(B)$ be the solution to

$$\min\left\{\int_B \tilde{F}(x, Dw)\mathrm{d}x \ : \ w \in \mathcal{K}_\psi(\Omega), \ w = u \text{ on } \partial B\right\}. \tag{7.5}$$

Now, we show that a local minimizer of functional (7.4) satisfies a variational inequality. More precisely, we have

Proposition 7.1.1. *A function $v \in u + W_0^{1,p}(B)$ is a solution to (7.5) if and only if it satisfies the following variational inequality*

$$\int_B \langle D_z H(x, Dv), D(\varphi - v)\rangle \, dx \geq 0, \tag{7.6}$$

for every $\varphi \in u + W_0^{1,p}(B) \cap \mathcal{K}_\psi(\Omega)$ such that $H(x, D\varphi) \in L^1(B)$.

Proof. We set $g = v + \varepsilon(\varphi - v)$ for $\varepsilon \in (0, 1)$, which belongs to the obstacle class, indeed

$$g = v + \varepsilon(\varphi - v) = \varepsilon\varphi + (1 - \varepsilon)v \geq \psi.$$

We first notice that $H(x, D(v + \varepsilon(\varphi - v))) \in L^1$. Moreover,

$$\int_B H(x, Dv)\mathrm{d}x \leq \int_B H(x, Dv + \varepsilon D(\varphi - v))\mathrm{d}x,$$

which leads to

$$\int_B H(x, Dv + \varepsilon D(\varphi - v))\mathrm{d}x - \int_B H(x, Dv)\mathrm{d}x \geq 0.$$

From Lagrange's theorem, for $\theta \in (0, 1)$ it holds

$$\int_B \langle D_z H(x, Dv + \varepsilon\theta D(\varphi - v)), \varepsilon D(\varphi - v)\rangle \mathrm{d}x \geq 0.$$

Since $\varepsilon > 0$,

$$\int_B \langle D_z H(x, Dv + \varepsilon\theta D(\varphi - v)), D(\varphi - v)\rangle \mathrm{d}x \geq 0. \tag{7.7}$$

According to [27, Lemma 2.2], it holds

$$|\langle D_z H(x, z), \lambda\rangle| \leq C\left(H(x, z) + H(x, \lambda)\right).$$

Therefore,

$$\begin{aligned}
&|\langle D_z H(x, Dv + \varepsilon\theta D(\varphi - v)), D(\varphi - v)\rangle| \\
\leq& C\left(H(x, Dv + \varepsilon\theta D(\varphi - v)) + H(x, D(\varphi - v))\right) \\
\leq& C\left(H(x, Dv) + H(x, \varepsilon\theta D(\varphi - v)) + H(x, D\varphi) + H(x, Dv)\right) \\
\leq& C\left(H(x, Dv) + (\varepsilon\theta)^p H(x, D\varphi) + (\varepsilon\theta)^p H(x, Dv) + H(x, D\varphi)\right), \qquad (7.8)
\end{aligned}$$

where in the last passage we also used the direct property $H(x, \varepsilon\theta z) \leq (\varepsilon\theta)^p H(x, z)$. For $\varepsilon \to 0$, the second and the third term on the right hand side of (7.8) go to zero. Hence, the right hand side tends to $C\left(H(x, Dv) + H(x, D\varphi)\right)$ in L^1. Then, we can pass to the limit for $\varepsilon \to 0$ in (7.7) applying the Dominated convergence theorem, which concludes the proof. □

If u is a solution to (7.1), then we are able to establish for u a higher integrability result.

Theorem 7.1.2. *Let u be a solution to the obstacle problem* (7.1) *where the integrand satisfies Assumption 1, for exponents $2 \leq p < q$ verifying*

$$\frac{q}{p} \leq 1 + \frac{\alpha}{n}.$$

If the function ψ is s.t. $H(x, D\psi) \in L^{m_1}_{loc}(\Omega)$, for some $m_1 > 1$, then there exist an exponent $m_1 > m_2 > 1$ and a positive constant C s.t. it holds

$$\left(\int_{B_{\frac{R}{2}}} (H(x, Du))^{m_2}\, dx\right)^{\frac{1}{m_2}} \leq C\left[\int_{B_R} H(x, Du)\, dx + \left(\int_{B_R} (H(x, D\psi))^{m_1}\, dx\right)^{\frac{1}{m_1}}\right].$$

for all balls $B_{\frac{R}{2}} \subset B_R \Subset \Omega$.

Proof. Let $\frac{R}{2} \leq t < s \leq R \leq 1$ and let $\eta \in C_0^\infty(B_R)$ be a cut-off function s.t. $0 \leq \eta \leq 1, \eta \equiv 1$ on $B_t, \eta \equiv 0$ outside $B_s, |D\eta| \leq \frac{2}{s-t}$. We set $\varphi = \eta(x)(u(x) - u_{B_R}) - \eta(x)(\psi(x) - \psi_{B_R})$ and $g = u - \varphi \in \mathcal{K}_\psi(\Omega)$. We observe that $g = u$ on ∂B_s and $g = \psi - \psi_{B_R} + u_{B_R}$ on B_t, therefore $Dg = D\psi$ on B_t. Using Assumption 1 (ii) and the fact that u is a local minimizer, we have

$$\begin{aligned}
&\int_{B_t} H(x, Du(x))\mathrm{d}x \\
\leq& C\int_{B_t} F(x, u(x), Du(x))\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{B_s} F(x,g(x),Dg(x))\mathrm{d}x\\
&\leq C\int_{B_s} |Dg(x)|^p + a(x)|Dg(x)|^q\mathrm{d}x\\
&\leq C\int_{B_s} [|D\eta(x)|(\psi(x)-\psi_{B_R})+\eta(x)|D\psi(x)|+|D\eta(x)|(u(x)-u_{B_R})+(1-\eta(x))|Du(x)|]^p\\
&\quad + a(x)\left[|D\eta(x)|(\psi(x)-\psi_{B_R})+\eta(x)|D\psi(x)|+|D\eta(x)|(u(x)-u_{B_R})+(1-\eta(x))|Du(x)|\right]^q\mathrm{d}x\\
&\leq C\int_{B_s}(1-\eta(x))^p\left(|Du|^p+a(x)|Du|^q\right)\mathrm{d}x\\
&\quad + C\int_{B_s}\left[\left|\frac{u(x)-u_{B_R}}{s-t}\right|^p + a(x)\left|\frac{u(x)-u_{B_R}}{s-t}\right|^q\right]\mathrm{d}x\\
&\quad + C\int_{B_s}\left[\left|\frac{\psi(x)-\psi_{B_R}}{s-t}\right|^p + a(x)\left|\frac{\psi(x)-\psi_{B_R}}{s-t}\right|^q\right]\mathrm{d}x\\
&\quad + C\int_{B_s}\left(|D\psi(x)|^p+a(x)|D\psi(x)|^q\right)\mathrm{d}x\\
&\leq C\int_{B_s\setminus B_t} H(x,Du(x))\mathrm{d}x\\
&\quad + \frac{C}{|s-t|^p}\int_{B_R}|u(x)-u_{B_R}|^p\mathrm{d}x + \frac{C}{|s-t|^q}\int_{B_R}a(x)|u(x)-u_{B_R}|^q\mathrm{d}x\\
&\quad + \frac{C}{|s-t|^p}\int_{B_R}|\psi(x)-\psi_{B_R}|^p\mathrm{d}x + \frac{C}{|s-t|^q}\int_{B_R}a(x)|\psi(x)-\psi_{B_R}|^q\mathrm{d}x\\
&\quad + C\int_{B_R} H(x,D\psi(x))\mathrm{d}x.
\end{aligned}
$$

Adding the quantity $C\int_{B_t} H(x,Du(x))\mathrm{d}x$ to both sides of the previous estimate, by Lemma 5.0.2 we get

$$
\begin{aligned}
\int_{B_{\frac{R}{2}}} H(x,Du(x))\mathrm{d}x \leq & C\left[\frac{1}{R^p}\int_{B_R}|u(x)-u_{B_R}|^p\mathrm{d}x + \frac{1}{R^q}\int_{B_R}a(x)|u(x)-u_{B_R}|^q\mathrm{d}x\right.\\
&+\frac{1}{R^p}\int_{B_R}|\psi(x)-\psi_{B_R}|^p\mathrm{d}x + \frac{1}{R^q}\int_{B_R}a(x)|\psi(x)-\psi_{B_R}|^q\mathrm{d}x\\
&\left.+\int_{B_R}H(x,D\psi(x))\mathrm{d}x\right].
\end{aligned}
$$

Setting $\tilde{H}(x,u(x)) := |u(x)|^p + a(x)|u(x)|^q$ and $\tilde{H}(x,\psi(x)) := |\psi(x)|^p + a(x)|\psi(x)|^q$, we can write the previous inequality as

$$
\int_{B_{\frac{R}{2}}} H(x,Du(x))\mathrm{d}x
$$

$$\leq \int_{B_R} \tilde{H}\left(x, \frac{u(x) - u_{B_R}}{R}\right) \mathrm{d}x + \int_{B_R} \tilde{H}\left(x, \frac{\psi(x) - \psi_{B_R}}{R}\right) \mathrm{d}x + \int_{B_R} H(x, D\psi(x)) \mathrm{d}x. \tag{7.9}$$

According to [105, Theorem 2.13] and Hölder's inequality, it holds

$$\begin{aligned} \int_{B_R} \tilde{H}\left(x, \frac{u(x) - u_{B_R}}{R}\right) \mathrm{d}x &\leq \left(\int_{B_R} \left(\tilde{H}\left(x, \frac{u(x) - u_{B_R}}{R}\right)\right)^{d_1} \mathrm{d}x\right)^{\frac{1}{d_1}} \\ &\leq \left(\int_{B_R} (H(x, Du(x)))^{d_2} \,\mathrm{d}x\right)^{\frac{1}{d_2}}, \end{aligned} \tag{7.10}$$

where $d_2 < 1 < d_1$ depend on n, p, q, α. Analogously,

$$\begin{aligned} \int_{B_R} \tilde{H}\left(x, \frac{\psi(x) - \psi_{B_R}}{R}\right) \mathrm{d}x &\leq \left(\int_{B_R} \left(\tilde{H}\left(x, \frac{\psi(x) - \psi_{B_R}}{R}\right)\right)^{d_1} \mathrm{d}x\right)^{\frac{1}{d_1}} \\ &\leq \left(\int_{B_R} (H(x, D\psi(x)))^{d_2} \,\mathrm{d}x\right)^{\frac{1}{d_2}}. \end{aligned} \tag{7.11}$$

Inserting (7.10) and (7.11) in (7.9) and exploiting Hölder's inequality, we infer

$$\int_{B_{\frac{R}{2}}} H(x, Du(x)) \mathrm{d}x \leq C \left[\left(\int_{B_R} (H(x, Du(x)))^{d_2}\right)^{\frac{1}{d_2}} + \int_{B_R} H(x, D\psi(x)) \mathrm{d}x\right]. \tag{7.12}$$

Since $H(x, D\psi(x)) \in L^{m_1}$, for $m_1 > 1$, from Gehring's lemma proved in [64] it follows that there exists $m_1 > m_2 > 1$ s.t. $H(x, Du(x)) \in L^{m_2}$. Then, holding to $d_2 < 1$, we might write

$$\begin{aligned} \int_{B_{\frac{R}{2}}} (H(x, Du(x)))^{m_2} \mathrm{d}x &\leq C \left[\left(\int_{B_R} H(x, Du(x)) \mathrm{d}x\right)^{m_2} + \int_{B_R} (H(x, D\psi(x)))^{m_2} \mathrm{d}x\right] \\ &\leq C \left[\left(\int_{B_R} H(x, Du(x)) \mathrm{d}x\right)^{m_2} + \left(\int_{B_R} (H(x, D\psi(x))^{m_1} \mathrm{d}x\right)^{\frac{m_2}{m_1}}\right]. \end{aligned}$$

Hence,

$$\left(\int_{B_{\frac{R}{2}}} (H(x, Du(x)))^{m_2} \mathrm{d}x\right)^{\frac{1}{m_2}} \leq C \left[\int_{B_R} H(x, Du(x)) \mathrm{d}x + \left(\int_{B_R} (H(x, D\psi(x))^{m_1} \mathrm{d}x\right)^{\frac{1}{m_1}}\right].$$

□

The higher integrability of the minimizer u stated in Theorem 7.1.2 allows us to prove the following higher integrability up to the boundary result for the solution to the freezed obstacle problem (7.5).

Theorem 7.1.3. *Let $v \in u + W_0^{1,p}(B_{\frac{R}{2}})$ be a solution to the obstacle problem (7.5) where the integrand $\tilde{F}$ satisfies Assumption 1, for exponents $2 \le p < q$ verifying*

$$\frac{q}{p} \le 1 + \frac{\alpha}{n}.$$

If the function ψ is s.t. $H(x, D\psi) \in L^{m_1}_{loc}(\Omega)$, for some $m_1 > 1$, then $H(x, Du) \in L^{m_2}_{loc}(\Omega)$, for some $m_1 > m_2 > 1$, and there exist a constant C and an exponent m_3, with $m_1 > m_2 > m_3 > 1$, s.t. $H(x, Dv) \in L^{m_3}_{loc}(\Omega)$ and

$$\left(\int_{B_{\frac{R}{2}}} (H(x,Dv))^{m_3}\, dx\right)^{\frac{1}{m_3}} \le C\left[\left(\int_{B_R} (H(x,Du))^{m_2}\, dx\right)^{\frac{1}{m_2}} + \left(\int_{B_R} (H(x,D\psi))^{m_2}\, dx\right)^{\frac{1}{m_2}}\right].$$

Proof. We start setting

$$w(x) := \begin{cases} v(x) & \text{if } x \in B_{\frac{R}{2}}, \\ u(x) & \text{if } x \in B_R \setminus B_{\frac{R}{2}} \end{cases} \tag{7.13}$$

We first consider $B_\rho(x_1) \subset B_{\frac{R}{2}}$. In this case the Caccioppoli inequality (7.12) holds, namely

$$\int_{B_{\frac{\rho}{2}}} H(x,Dv)\mathrm{d}x \le C\left[\left(\int_{B_\rho} (H(x,Dv))^{d_2}\mathrm{d}x\right)^{\frac{1}{d_2}} + \int_{B_\rho} H(x,D\psi)\mathrm{d}x\right]. \tag{7.14}$$

Let us now focus on the case $B_\rho(x_1) \subset B_R$, with $x_1 \in \partial B_{\frac{R}{2}}$. Fix $\frac{\rho}{2} \le t < s \le \rho$ and a cut-off function η between $B_s(x_1)$ and $B_t(x_1)$, with $|D\eta| \le \dfrac{2}{t-s}$. Let us set $g(x) := (1-\eta(x))v + \eta(x)u(x)$. It is straightforward that $g \in u + W_0^{1,p}$ and $g(x) \ge \psi(x)$. Since v is a minimizer, according to the definition of H and Assumption 1 (ii), we have

$$\begin{aligned}\int_{B_t \cap B_{\frac{R}{2}}} H(x,Dv)\mathrm{d}x &\le C\int_{B_t \cap B_{\frac{R}{2}}} \tilde{F}(x,Dv)\mathrm{d}x \\ &\le C\int_{B_s \cap B_{\frac{R}{2}}} \tilde{F}(x,Dg)\mathrm{d}x.\end{aligned}$$

Therefore, from the definitions of g and η, we get

$$\begin{aligned}&\int_{B_t\cap B_{\frac{R}{2}}} H(x,Dv)\mathrm{d}x\\ \leq& C\left[\int_{B_s\cap B_{\frac{R}{2}}}\left(\frac{1}{(t-s)^p}|u-v|^p+a(x)\frac{1}{(t-s)^q}|u-v|^q\right)\mathrm{d}x\right.\\ &+\int_{(B_s\setminus B_t)\cap B_{\frac{R}{2}}} H(x,Dv)\mathrm{d}x\\ &\left.+\int_{B_s} H(x,Du)\mathrm{d}x\right].\end{aligned}$$

As before, adding the quantity $C\int_{B_t\cap B_{\frac{R}{2}}} H(x,Dv)\mathrm{d}x$ to both sides of the previous inequality, by Lemma 5.0.2 we get

$$\begin{aligned}&\int_{B_{\frac{\rho}{2}}\cap B_{\frac{R}{2}}} H(x,Dv)\mathrm{d}x\\ \leq& C\left[\int_{B_\rho\cap B_{\frac{R}{2}}}\left(\frac{1}{\rho^p}|u-v|^p+a(x)\frac{1}{\rho^q}|u-v|^q\right)\mathrm{d}x\right.\\ &\left.+\int_{B_\rho} H(x,Du)\mathrm{d}x\right].\end{aligned}\tag{7.15}$$

We set

$$\tilde{H}\left(x,\frac{u-v}{\rho}\right):=\frac{1}{\rho^p}|u-v|^p+a(x)\frac{1}{\rho^q}|u-v|^q.$$

Adapting the argument in [26, Remark 2] and [105, Theorem 2.13] and exploiting Hölder's inequality, we have

$$\begin{aligned}\int_{B_\rho\cap B_{\frac{R}{2}}}\tilde{H}\left(x,\frac{u-v}{\rho}\right)\mathrm{d}x&\leq\left(\int_{B_\rho\cap B_{\frac{R}{2}}}\left(\tilde{H}\left(x,\frac{u-v}{\rho}\right)\right)^{d_1}\mathrm{d}x\right)^{\frac{1}{d_1}}\\ &\leq\left(\int_{B_\rho\cap B_{\frac{R}{2}}}\left(H(x,Du-Dv)\right)^{d_2}\mathrm{d}x\right)^{\frac{1}{d_2}},\end{aligned}\tag{7.16}$$

where $d_2 < 1 < d_1$ depend on n, p, q, α. Inserting (7.16) in (7.15), it yields

$$
\begin{aligned}
\int_{B_{\frac{\rho}{2}} \cap B_{\frac{R}{2}}} H(x, Dv) \mathrm{d}x \leq & C \left[\left(\int_{B_\rho} (H(x, Du(x))^{d_2} \, \mathrm{d}x \right)^{\frac{1}{d_2}} \right. \\
& + \left(\int_{B_\rho \cap B_{\frac{R}{2}}} (H(x, Dv))^{d_2} \, \mathrm{d}x \right)^{\frac{1}{d_2}} \\
& \left. + \int_{B_\rho} H(x, Du(x)) \mathrm{d}x \right] \\
\leq & C \left[\left(\int_{B_\rho \cap B_{\frac{R}{2}}} (H(x, Dv))^{d_2} \, \mathrm{d}x \right)^{\frac{1}{d_2}} + \int_{B_\rho} H(x, Du(x)) \mathrm{d}x \right].
\end{aligned}
$$

Therefore, from the definition of w in (7.13), we infer

$$
\begin{aligned}
\int_{B_{\frac{\rho}{2}}} H(x, Dw(x)) \mathrm{d}x \leq & C \left[\left(\int_{B_\rho} (H(x, Dw(x)))^{d_2} \, \mathrm{d}x \right)^{\frac{1}{d_2}} + \int_{B_\rho} H(x, Du(x)) \mathrm{d}x \right. \\
& \left. + \int_{B_\rho} H(x, D\psi(x)) \mathrm{d}x \right]. \qquad (7.17)
\end{aligned}
$$

Hence, by (7.14) it follows that (7.17) holds not only if $B_\rho(x_1) \subset B_{\frac{R}{2}}$ or $B_\rho(x_1) \cap B_{\frac{R}{2}} \neq \emptyset$, but also when $B_\rho(x_1) \subset B_R$ and $x_1 \in \partial B_{\frac{R}{2}}$.

We now take care of the case $B_\rho(x_1) \cap \partial B_{\frac{R}{2}} \neq \emptyset$ and $B_{4\rho} \subset B_R$. We fix $x_2 \in B_\rho(x_1) \cap \partial B_{\frac{R}{2}}$.

$$
\begin{aligned}
& \int_{B_{\frac{\rho}{2}}(x_1)} H(x, Dw) \mathrm{d}x \\
& \leq 3^N \int_{B_{\frac{3\rho}{2}}(x_2)} H(x, Dw) \mathrm{d}x \\
& \leq C \left[\left(\int_{B_{3\rho(x_2)}} (H(x, Dw))^{d_2} \mathrm{d}x \right)^{\frac{1}{d_2}} + \int_{B_{3\rho}(x_2)} H(x, Du) \mathrm{d}x + \int_{B_{3\rho}(x_2)} H(x, D\psi) \mathrm{d}x \right] \\
& \leq C \left[\left(\int_{B_{4\rho(x_1)}} (H(x, Dw))^{d_2} \mathrm{d}x \right)^{\frac{1}{d_2}} + \int_{B_{4\rho}(x_1)} H(x, Du) \mathrm{d}x + \int_{B_{4\rho}(x_1)} H(x, D\psi) \mathrm{d}x \right].
\end{aligned}
$$

Since this estimate holds for every $B_{\frac{\rho}{2}}$ such that $B_{4\rho} \subset B_R$, by a covering argument it follows that inequality (7.17) holds for every $B_{\frac{\rho}{2}}$ such that $B_\rho \subset B_R$. Now, since $H(x, D\psi) \in L^{m_1}$, $m_1 > 1$, Theorem 7.1.2 yields that there exists m_2, with $1 < m_2 < m_1$, s.t. $H(x,Du) \in L^{m_2}$. Therefore, according to Gehring's lemma, there exists m_3, with $1 < m_3 < m_2 < m_1$, such that,

$$\begin{aligned}&\left(\int_{B_{\frac{\rho}{2}}(x_1)} (H(x,Dw))^{m_3}\,\mathrm{d}x\right)^{\frac{1}{m_3}}\\ \leq& C\left[\int_{B_\rho(x_1)} H(x,Dw(x))\mathrm{d}x\right.\\ &\left.+\left(\int_{B_\rho(x_1)} (H(x,Du(x)))^{m_2}\mathrm{d}x\right)^{\frac{1}{m_2}}+\left(\int_{B_\rho(x_1)} (H(x,D\psi(x)))^{m_2}\mathrm{d}x\right)^{\frac{1}{m_2}}\right].\end{aligned}$$

In particular, for $\rho \equiv R$ and $x_1 = x_0$, recalling the definition of w we have

$$\begin{aligned}&\left(\int_{B_{\frac{R}{2}}} (H(x,Dv))^{m_3}\,\mathrm{d}x\right)^{\frac{1}{m_3}}\\ \leq& C\left[\int_{B_{\frac{R}{2}}} H(x,Dv)\mathrm{d}x+\int_{B_R\setminus B_{\frac{R}{2}}} H(x,Du(x))\mathrm{d}x\right.\\ &\left.+\left(\int_{B_R} (H(x,Du(x)))^{m_2}\mathrm{d}x\right)^{\frac{1}{m_2}}+\left(\int_{B_R} (H(x,D\psi(x)))^{m_2}\mathrm{d}x\right)^{\frac{1}{m_2}}\right]\\ \leq& C\left[\int_{B_{\frac{R}{2}}} H(x,Dv)\mathrm{d}x\right.\\ &\left.+\left(\int_{B_R} (H(x,Du(x)))^{m_2}\mathrm{d}x\right)^{\frac{1}{m_2}}+\left(\int_{B_R} (H(x,D\psi(x)))^{m_2}\mathrm{d}x\right)^{\frac{1}{m_2}}\right].\end{aligned}$$

Since v is a minimizer and recalling that $m_2 > 1$, it holds

$$\begin{aligned}&\left(\int_{B_{\frac{R}{2}}} (H(x,Dv))^{m_3}\,\mathrm{d}x\right)^{\frac{1}{m_3}}\\ &\leq C\left[\left(\int_{B_R} (H(x,Du(x)))^{m_2}\mathrm{d}x\right)^{\frac{1}{m_2}}+\left(\int_{B_R} (H(x,D\psi(x)))^{m_2}\mathrm{d}x\right)^{\frac{1}{m_2}}\right],\end{aligned}$$

i.e. the conclusion. □

Remark 3. We point out that Theorems 7.1.2 and 7.1.3 hold true also under the more general hypothesis $q > p > 1$. However, they are stated for $q > p \geq 2$ for later purpose in Section 7.4.

7.2 Higher differentiability for comparison maps

In Chapter 6 we established in Theorem 6.0.1 the higher differentiability of the solution v to (7.5) under more general assumptions on the coefficients. Indeed, the result presented in this Section recalls Theorem 6.0.1 with $r = \infty$. Here, we only give the proof of the a priori bounds, focusing on the differences with respect to the proof carried out in Section 6.2.1. This is crucial in order to establish precise estimates on the difference quotient that will be pivotal for the comparison argument. On the other hand, the approximation procedure is achieved using the same arguments in Section 6.2.2, therefore it will not be presented.
Before stating the result, it is worth noticing that Assumption 1 implies that there exist positive constants $\tilde{l}, \tilde{\nu}, \tilde{L}$ such that the following conditions are satisfied:

$$|D_\xi \tilde{F}(x,\xi)| \leq \tilde{l}(|\xi|^{p-1} + a(x)|\xi|^{q-1}) \tag{$\bar{\mathrm{A}}1$}$$

$$\langle D_\xi \tilde{F}(x,\xi) - D_\xi \tilde{F}(x,\eta), \xi - \eta\rangle \geq \tilde{\nu}(|\xi-\eta|^2(|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} + a(x)|\xi-\eta|^2(|\xi|^2+|\eta|^2)^{\frac{q-2}{2}}) \tag{$\bar{\mathrm{A}}2$}$$

$$|D_\xi \tilde{F}(x,\xi) - D_\xi \tilde{F}(x,\eta)| \leq \tilde{L}(|\xi-\eta|(|\xi|^2+|\eta|^2)^{\frac{p-2}{2}} + a(x)|\xi-\eta|(|\xi|^2+|\eta|^2)^{\frac{q-2}{2}}) \tag{$\bar{\mathrm{A}}3$}$$

$$|D_\xi \tilde{F}(x,\xi) - D_\xi \tilde{F}(y,\xi)| \leq |x-y|^\alpha |\xi|^{q-1} \tag{$\bar{\mathrm{A}}4$}$$

for every $x, y \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$.

The following lemma holds:

Lemma 7.2.1. *Let $v \in u + W_0^{1,p}(B)$ be the solution to (7.5) under Assumption 1, for exponents $2 \leq p < \frac{n}{\alpha}$, $p < q$ satisfying*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}. \tag{7.18}$$

If

$$D\psi \in B^\gamma_{2q-p,\infty}(B),$$

for $0 < \alpha < \gamma < 1$, then

$$V_p(Dv) \in B^\alpha_{2,\infty,loc}(B)$$

and the following estimate

$$\begin{aligned}&\int_{B_{r/4}}(|\tau_h V_p(Dv)|^2+a(x+h)|\tau_h V_q(Dv)|^2)\,dx\\ \le& C|h|^{2\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r}(1+|Dv|^p+|D\psi|^{2q-p})\,dx\right)^{\kappa}+[D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\},\end{aligned}\tag{7.19}$$

holds for all balls $B_{r/4}\subset B_r\Subset B$, for some $\mu<\alpha$, with $C:=C(n,p,q,\mu,\|a\|_\infty,\|D\psi\|_{B^{\gamma}_{2q-p,\infty}})$, $\tilde{p}:=\tilde{p}(n,p,q,\mu)>1$ and $\kappa:=\kappa(n,p,q,\mu)<\tilde{p}$.

Proof. We a priori assume that $Dv\in L^{\frac{np}{n-2\mu}}_{\mathrm{loc}}(B)$, for all $\frac{\alpha n}{n+2\alpha}<\mu<\alpha$.
In the sequel we will profusely use the following inequality:

$$2q-p\le\frac{np}{n-2\mu},\tag{7.20}$$

for $\mu\in[\frac{\alpha n}{n+2\alpha},\alpha)$. Indeed,

$$2q-p\le\frac{np}{n-2\mu}\Leftrightarrow\frac{q}{p}\le\frac{n-\mu}{n-2\mu}$$

and

$$1+\frac{\alpha}{n}\le\frac{n-\mu}{n-2\mu}\Leftrightarrow\mu\ge\frac{\alpha n}{n+2\alpha}.$$

Fix $0<\frac{r}{4}<\rho<s<t<t'<\frac{r}{2}$ such that $B_r\Subset B$ and a cut-off function $\eta\in\mathcal{C}^1_0(B_t)$ such that $0\le\eta\le 1$, $\eta=1$ on B_s, $|D\eta|\le\frac{C}{t-s}$.
Now, for $|h|<\frac{r}{4}$, we consider functions

$$w_1(x)=\eta^2(x)[(v-\psi)(x+h)-(v-\psi)(x)]$$

and

$$w_2(x)=\eta^2(x-h)[(v-\psi)(x-h)-(v-\psi)(x)].$$

Then

$$\varphi_1(x)=v(x)+tw_1(x),\tag{7.21}$$

$$\varphi_2(x)=v(x)+tw_2(x)\tag{7.22}$$

are admissible test functions for all $t\in[0,1)$.
Arguing analogously as in the proof of Theorem 6.2.1, we obtain the following estimate

$$0\ge\int_\Omega\langle D_\xi H(x+h,Dv(x+h))-D_\xi H(x+h,Dv(x)),\eta^2 D\tau_h v\rangle\mathrm{d}x$$

$$
\begin{aligned}
&- \int_\Omega \langle D_\xi H(x+h, Dv(x+h)) - D_\xi H(x+h, Dv(x)), \eta^2 D\tau_h\psi\rangle \mathrm{d}x \\
&+ \int_\Omega \langle D_\xi H(x+h, Dv(x+h)) - D_\xi H(x+h, Dv(x)), 2\eta D\eta \tau_h(v-\psi)\rangle \mathrm{d}x \\
&+ \int_\Omega \langle D_\xi H(x+h, Dv(x)) - D_\xi H(x, Dv(x)), \eta^2 D\tau_h v\rangle \mathrm{d}x \\
&- \int_\Omega \langle D_\xi H(x+h, Dv(x)) - D_\xi H(x, Dv(x)), \eta^2 D\tau_h \psi\rangle \mathrm{d}x \\
&+ \int_\Omega \langle D_\xi H(x+h, Dv(x)) - D_\xi H(x, Dv(x)), 2\eta D\eta \tau_h(v-\psi)\rangle \mathrm{d}x \\
=:& I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned} \tag{7.23}
$$

that yields

$$
I_1 \le |I_2| + |I_3| + |I_4| + |I_5| + |I_6|. \tag{7.24}
$$

The ellipticity assumption ($\bar{\text{A}}2$) and the properties of $a(x)$ imply

$$
\begin{aligned}
I_1 \ge& \tilde{\nu} \int_\Omega \eta^2 |\tau_h Dv|^2 (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} \mathrm{d}x \\
&+ \tilde{\nu} \int_\Omega \eta^2 a(x+h) |\tau_h Dv|^2 (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{q-2}{2}} \mathrm{d}x \\
\ge& \tilde{\nu} \int_\Omega \eta^2 (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2) \mathrm{d}x.
\end{aligned} \tag{7.25}
$$

From the growth condition ($\bar{\text{A}}3$), the boundedness of $a(x)$ and Young's inequality, we get

$$
\begin{aligned}
|I_2| \le& \tilde{L} \int_\Omega \eta^2 |\tau_h Dv| (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} |\tau_h D\psi| \mathrm{d}x \\
&+ \tilde{L} \int_\Omega \eta^2 a(x+h) |\tau_h Dv| (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{q-2}{2}} |\tau_h D\psi| \mathrm{d}x \\
\le& \tilde{L} \int_\Omega \eta^2 |\tau_h Dv| (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} |\tau_h D\psi| \mathrm{d}x \\
&+ \tilde{L} \|a\|_\infty \int_\Omega \eta^2 |\tau_h Dv| (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{q-2}{2}} |\tau_h D\psi| \mathrm{d}x \\
\le& \varepsilon \int_\Omega \eta^2 |\tau_h Dv|^2 (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} \mathrm{d}x \\
&+ C_\varepsilon(\tilde{L}, \|a\|_\infty) \int_\Omega \eta^2 |\tau_h D\psi|^2 (1 + |Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{2q-p-2}{2}} \mathrm{d}x.
\end{aligned}
$$

The calculations performed in Theorem 6.2.1 and Lemma 5.0.1 lead us to the following estimate for the integral I_2

$$\begin{aligned}|I_2| \leq & \varepsilon \int_\Omega \eta^2 |\tau_h V_p(Dv)|^2 \mathrm{d}x \\ & + C_\varepsilon(\tilde{L}, p, q, \|a\|_\infty)|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_r)} \\ & + C_\varepsilon(\tilde{L}, p, q, \|a\|_\infty)|h|^{2\gamma} \int_{B_{t'}} (1+|Dv|)^{2q-p} \mathrm{d}x. \end{aligned} \tag{7.26}$$

Now, we consider the integral I_3. From assumption ($\bar{\mathrm{A}}3$), hypothesis $|D\eta| \leq \frac{C}{t-s}$ and Young's inequality, we get

$$\begin{aligned}|I_3| \leq & 2\tilde{L} \int_\Omega |D\eta|\eta|\tau_h Dv|(|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} |\tau_h(v-\psi)| \mathrm{d}x \\ & + 2\tilde{L}\|a\|_\infty \int_\Omega |D\eta|\eta|\tau_h Dv|(1+|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{q-2}{2}} |\tau_h(v-\psi)| \mathrm{d}x \\ \leq & \varepsilon \int_\Omega \eta^2 |\tau_h Dv|^2 (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} \mathrm{d}x \\ & + \frac{C_\varepsilon(L, \|a\|_\infty)}{(t-s)^2} \int_{B_t} |\tau_h(v-\psi)|^2 (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{2q-p-2}{2}} \mathrm{d}x, \end{aligned}$$

where we also used the boundedness of function $a(x)$.
Arguing analogously as in the proof of Theorem 6.2.1, we can estimate the integral I_3 as follows

$$\begin{aligned}|I_3| \leq & \varepsilon \int_\Omega \eta^2 |\tau_h V_p(Dv)|^2 \mathrm{d}x \\ & + \frac{C_\varepsilon(\tilde{L}, n, p, q, \|a\|_\infty)}{(t-s)^2} |h|^2 \int_{B_r} |D\psi|^{2q-p} \mathrm{d}x \\ & + \frac{C_\varepsilon(\tilde{L}, n, p, q, \|a\|_\infty)}{(t-s)^2} |h|^2 \int_{B_{t'}} (1+|Dv|)^{2q-p} \mathrm{d}x. \end{aligned} \tag{7.27}$$

In order to estimate the integral I_4, we use assumption ($\bar{\mathrm{A}}4$), Young's inequality and Lemma 5.0.1 as follows

$$\begin{aligned}|I_4| \leq & \int_\Omega \eta^2 |\tau_h Dv||h|^\alpha |Dv|^{q-1} \mathrm{d}x \\ \leq & \varepsilon \int_\Omega \eta^2 |\tau_h Dv|^2 (|Dv(x+h)|^2 + |Dv(x)|^2)^{\frac{p-2}{2}} \mathrm{d}x \\ & + C_\varepsilon |h|^{2\alpha} \int_{B_t} |Dv|^{2q-p} \mathrm{d}x \\ \leq & \varepsilon \int_\Omega \eta^2 |\tau_h V_p(Dv)|^2 \mathrm{d}x \end{aligned}$$

$$+ C_\varepsilon |h|^{2\alpha} \int_{B_t} |Dv|^{2q-p} \mathrm{d}x. \tag{7.28}$$

We now take care of I_5. Similarly as above, exploiting assumption ($\bar{\mathrm{A}}4$) and Hölder's inequality, we infer

$$\begin{aligned}
|I_5| \leq & \int_\Omega \eta^2 |\tau_h D\psi| |h|^\alpha |Dv|^{q-1} \mathrm{d}x \\
\leq & |h|^\alpha \left(\int_{B_t} |\tau_h D\psi|^{2q-p} \mathrm{d}x \right)^{\frac{1}{2q-p}} \left(\int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} \mathrm{d}x \right)^{\frac{2q-p-1}{2q-p}}.
\end{aligned}$$

Now, we observe

$$\frac{(q-1)(2q-p)}{2q-p-1} < 2q-p \Leftrightarrow p < q. \tag{7.29}$$

Hence

$$\begin{aligned}
|I_5| \leq & |h|^{\alpha+\gamma} [D\psi]_{B^\gamma_{2q-p,\infty}(B_r)} \left(\int_{B_t} |Dv|^{2q-p} \mathrm{d}x \right)^{\frac{q-1}{2q-p}} \\
\leq & C(q) |h|^{\alpha+\gamma} [D\psi]^q_{B^\gamma_{2q-p,\infty}(B_r)} \\
& + C(q) |h|^{\alpha+\gamma} \left(\int_{B_t} |Dv|^{2q-p} \mathrm{d}x \right)^{\frac{q}{2q-p}}.
\end{aligned} \tag{7.30}$$

From assumption ($\bar{\mathrm{A}}4$), hypothesis $|D\eta| \leq \frac{C}{t-s}$ and Hölder's inequality, we infer the following estimate for I_6.

$$\begin{aligned}
|I_6| \leq & \frac{C}{t-s} |h|^\alpha \int_{B_t} |\tau_h \psi| |Dv|^{q-1} \mathrm{d}x \\
& + \frac{C}{t-s} |h|^\alpha \int_{B_t} |\tau_h v| |Dv|^{q-1} \mathrm{d}x \\
\leq & \frac{C}{t-s} |h|^\alpha \left(\int_{B_t} |\tau_h \psi|^{2q-p} \mathrm{d}x \right)^{\frac{1}{2q-p}} \left(\int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} \mathrm{d}x \right)^{\frac{2q-p-1}{2q-p}} \\
& + \frac{C}{t-s} |h|^\alpha \left(\int_{B_t} |\tau_h v|^{2q-p} \mathrm{d}x \right)^{\frac{1}{2q-p}} \left(\int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} \mathrm{d}x \right)^{\frac{2q-p-1}{2q-p}}.
\end{aligned}$$

Using Lemma 5.2.2, (7.29) and Hölder's and Young's inequality, we have

$$|I_6| \leq \frac{C(n,p,q)}{t-s} |h|^{\alpha+1} \left(\int_{B_{t'}} |D\psi|^{2q-p} \mathrm{d}x \right)^{\frac{1}{2q-p}} \left(\int_{B_t} |Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}} \mathrm{d}x \right)^{\frac{2q-p-1}{2q-p}}$$

$$
\begin{aligned}
&+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{2q-p}\mathrm{d}x\right)^{\frac{1}{2q-p}}\left(\int_{B_t}|Dv|^{\frac{(q-1)(2q-p)}{2q-p-1}}\mathrm{d}x\right)^{\frac{2q-p-1}{2q-p}}\\
\le&\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_r}|D\psi|^{2q-p}\mathrm{d}x\right)^{\frac{1}{2q-p}}\left(\int_{B_t}|Dv|^{2q-p}\mathrm{d}x\right)^{\frac{q-1}{2q-p}}\\
&+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}}\\
\le&\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_r}|D\psi|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}}\\
&+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}}.
\end{aligned}
\tag{7.31}
$$

Inserting estimates (7.25), (7.26), (7.27), (7.28), (7.30) and (7.31) in (7.24), we infer

$$
\begin{aligned}
&\nu\int_\Omega \eta^2(|\tau_h V_p(Dv)|^2+a(x+h)|\tau_h V_p(Dv)|^2)\mathrm{d}x\\
\le&3\varepsilon\int_\Omega \eta^2|\tau_h V_p(Dv)|^2\mathrm{d}x\\
&+C_\varepsilon(\tilde{L},p,q,\|a\|_\infty)|h|^{2\gamma}[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_r)}\\
&+C_\varepsilon(\tilde{L},p,q,\|a\|_\infty)|h|^{2\gamma}\int_{B_{t'}}(1+|Dv|)^{2q-p}\mathrm{d}x\\
&+\frac{C_\varepsilon(\tilde{L},n,p,q,\|a\|_\infty)}{(t-s)^2}|h|^2\int_{B_r}|D\psi|^{2q-p}\mathrm{d}x\\
&+\frac{C_\varepsilon(\tilde{L},n,p,q,\|a\|_\infty)}{(t-s)^2}|h|^2\int_{B_{t'}}(1+|Dv|)^{2q-p}\mathrm{d}x\\
&+C_\varepsilon|h|^{2\alpha}\int_{B_t}|Dv|^{2q-p}\mathrm{d}x\\
&+C(q)|h|^{\alpha+\gamma}[D\psi]^q_{B^\gamma_{2q-p,\infty}(B_r)}+C(q)|h|^{\alpha+\gamma}\left(\int_{B_t}|Dv|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}}\\
&+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_r}|D\psi|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}}\\
&+\frac{C(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}}.
\end{aligned}
\tag{7.32}
$$

We now introduce the following interpolation inequality

$$
\|Dw\|_{2q-p}\le\|Dw\|_p^\delta\|Dw\|^{1-\delta}_{\frac{np}{n-2\mu}},
\tag{7.33}
$$

where $0<\delta<1$ is defined through the condition

$$\frac{1}{(2q-p)}=\frac{\delta}{p}+\frac{(1-\delta)(n-2\mu)}{np} \tag{7.34}$$

which implies

$$\delta=\frac{n(p-q)+\mu(2q-p)}{\mu(2q-p)},\quad 1-\delta=\frac{n(q-p)}{\mu(2q-p)}.$$

Hence we get the following inequalities

$$\int_{B_{t'}}(1+|Dv|)^{2q-p}\mathrm{d}x \le\left(\int_{B_{t'}}(1+|Dv|)^p\mathrm{d}x\right)^{\frac{\delta(2q-p)}{p}}\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{(n-2\mu)(q-p)}{\mu p}}, \tag{7.35}$$

$$\left(\int_{B_{t'}}|Dv|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}} \le\left(\int_{B_{t'}}|Dv|^p\mathrm{d}x\right)^{\frac{\delta q}{p}}\cdot\left(\int_{B_{t'}}|Dv|^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{(n-2\mu)qp'}{p}}, \tag{7.36}$$

where $p'=\frac{q-p}{\mu(2q-p)}$.

Inserting (7.35) and (7.36) in (7.32), and exploiting the bounds

$$\frac{n(q-p)}{\mu p}<1,\quad \frac{nq(q-p)}{\mu p(2q-p)}<1, \tag{7.37}$$

which hold by assumption (7.18) and for $\mu\in(\frac{n(q-p)}{p},\alpha)$, from Young's inequality, we infer

$$\begin{aligned}
&\nu\int_\Omega \eta^2(|\tau_h V_p(Dv)|^2+a(x+h)|\tau_h V_p(Dv)|^2)\mathrm{d}x\\
\le&3\varepsilon\int_\Omega \eta^2|\tau_h V_p(Dv)|^2\mathrm{d}x\\
&+C_\varepsilon(\tilde{L},p,q,\|a\|_\infty)|h|^{2\gamma}[D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\\
&+C_{\varepsilon,\theta}(\tilde{L},n,p,q,\|a\|_\infty)|h|^{2\gamma}\left(\int_{B_r}(1+|Dv|)^p\mathrm{d}x\right)^{\frac{\delta(2q-p)\tilde{p}}{p}}\\
&+\theta|h|^{2\gamma}\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{n-2\mu}{n}}\\
&+\frac{C_\varepsilon(\tilde{L},n,p,q,\|a\|_\infty)}{(t-s)^2}|h|^2\int_{B_r}|D\psi|^{2q-p}\mathrm{d}x\\
&+\theta|h|^2\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{n-2\mu}{n}}+\frac{C_{\varepsilon,\theta}(\tilde{L},n,p,q,\|a\|_\infty)}{(t-s)^{2\tilde{p}}}|h|^2\left(\int_{B_r}(1+|Dv|)^p\mathrm{d}x\right)^{\frac{\tilde{p}\delta(2q-p)}{p}}\\
&+C_{\varepsilon,\theta}|h|^{2\alpha}\left(\int_{B_r}|Dv|^p\mathrm{d}x\right)^{\frac{\tilde{p}\delta(2q-p)}{p}}+\theta|h|^{2\alpha}\left(\int_{B_t}|Dv|^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{n-2\mu}{n}}
\end{aligned}$$

$$
\begin{aligned}
&+ C_\theta(q)|h|^{\alpha+\gamma}[D\psi]^q_{B^\gamma_{2q-p,\infty}(B_r)} + C_\theta(n,p,q)|h|^{\alpha+\gamma}\left(\int_{B_r}|Dv|^p\mathrm{d}x\right)^{\frac{p^*\delta q}{p}}\\
&+ \theta|h|^{\alpha+\gamma}\left(\int_{B_t}|Dv|^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{n-2\mu}{n}}\\
&+ \frac{C_\theta(n,p,q)}{t-s}|h|^{\alpha+1}\left(\int_{B_r}|D\psi|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}}\\
&+ \frac{C_\theta(n,p,q)}{(t-s)^{p^*}}|h|^{\alpha+1}\left(\int_{B_r}|Dv|^p\mathrm{d}x\right)^{\frac{p^*\delta q}{p}} + \theta|h|^{\alpha+1}\left(\int_{B_{t'}}|Dv|^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{n-2\mu}{n}}.
\end{aligned}
\tag{7.38}
$$

for some constant $\theta \in (0,1)$, where we set $\tilde{p} = \frac{\mu p}{\mu p - n(q-p)}$, $p^* = \frac{\mu p(2q-p)}{\mu p(2q-p) - n(q-p)q}$.

For a better readability we now define

$$
\begin{aligned}
A =&C_\varepsilon(\tilde{L},p,q,\|a\|_\infty)[D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_r)} + C_{\varepsilon,\theta}(\tilde{L},n,p,q,\|a\|_\infty)\left(\int_{B_r}(1+|Dv|)^p\mathrm{d}x\right)^{\frac{\delta(2q-p)\tilde{p}}{p}}\\
&+ C_{\varepsilon,\theta}\left(\int_{B_r}|Dv|^p\mathrm{d}x\right)^{\frac{\tilde{p}\delta(2q-p)}{p}}\\
&+ C_\theta(q)|h|^{\alpha+\gamma}[D\psi]^q_{B^\gamma_{2q-p,\infty}(B_r)} + C_\theta(n,p,q)|h|^{\alpha+\gamma}\left(\int_{B_r}|Dv|^p\mathrm{d}x\right)^{\frac{p^*\delta q}{p}}\\
B_1 =&C_\varepsilon(\tilde{L},n,p,q,\|a\|_\infty)\int_{B_r}|D\psi|^{2q-p}\mathrm{d}x,\\
B_2 =&C_{\varepsilon,\theta}(\tilde{L},n,p,q,\|a\|_\infty)\left(\int_{B_r}(1+|Dv|)^p\mathrm{d}x\right)^{\frac{\tilde{p}\delta(2q-p)}{p}},\\
B_3 =&C_\theta(n,p,q)\left(\int_{B_r}|D\psi|^{2q-p}\mathrm{d}x\right)^{\frac{q}{2q-p}},\\
B_4 =&C_\theta(n,p,q)\left(\int_{B_r}|Dv|^p\mathrm{d}x\right)^{\frac{p^*\delta q}{p}},
\end{aligned}
$$

so that we can rewrite the previous estimate as

$$
\begin{aligned}
&\nu\int_\Omega \eta^2(|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_p(Dv)|^2)\mathrm{d}x\\
\leq&3\varepsilon\int_\Omega \eta^2|\tau_h V_p(Dv)|^2\mathrm{d}x\\
&+ \theta(|h|^{2\alpha} + |h|^{\alpha+\gamma})\left(\int_{B_t}(1+|Dv|)^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{n-2\mu}{n}}\\
&+ \theta(|h|^2 + |h|^{2\gamma} + |h|^{\alpha+1})\left(\int_{B_{t'}}(1+|Dv|)^{\frac{np}{n-2\mu}}\mathrm{d}x\right)^{\frac{n-2\mu}{n}}
\end{aligned}
$$

$$+ (|h|^{2\gamma} + |h|^{2\alpha} + |h|^{\alpha+\gamma})A + |h|^2 \frac{B_1}{(t-s)^2} + |h|^2 \frac{B_2}{(t-s)^{2\tilde{p}}} + |h|^{\alpha+1} \frac{B_3}{(t-s)^{p''}} + |h|^{\alpha+1} \frac{B_4}{(t-s)^{p^*}}.$$

Choosing $\varepsilon = \frac{\nu}{6}$, we can reabsorb the first integral in the right hand side of the previous estimate by the left hand side, thus getting

$$\begin{aligned}
&\int_\Omega \eta^2 (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2) \mathrm{d}x \\
\leq& 3\theta |h|^{2\alpha} \left(\int_{B_t} (1+|Dv|)^{\frac{np}{n-2\mu}} \mathrm{d}x \right)^{\frac{n-2\mu}{n}} + 3\theta |h|^{2\alpha} \left(\int_{B_{t'}} (1+|Dv|)^{\frac{np}{n-2\mu}} \mathrm{d}x \right)^{\frac{n-2\mu}{n}} \\
&+ |h|^{2\alpha} A + |h|^2 \frac{B_1}{(t-s)^2} + |h|^2 \frac{B_2}{(t-s)^{2\tilde{p}}} + |h|^{2\alpha} \frac{B_3}{t-s} + |h|^{2\alpha} \frac{B_4}{(t-s)^{p^*}}, \qquad (7.39)
\end{aligned}$$

where we used the fact that $\alpha < \gamma$.

Since the right hand side of (7.39) depends on the integrability of Dv, in order to exploit inequality (7.19), we need to derive an a priori estimate for the gradient of the minimzer v. First, we bound (7.39) from below as follows

$$\begin{aligned}
&\int_{B_s} |\tau_h V_p(Dv)|^2 \mathrm{d}x \\
\leq& \int_{B_s} (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2) \mathrm{d}x \\
\leq& |h|^{2\alpha} \Bigg\{ 2\theta \left(\int_{B_t} (1+|Dv|)^{\frac{np}{n-2\mu}} \mathrm{d}x \right)^{\frac{n-2\mu}{n}} + 3\theta \left(\int_{B_{t'}} (1+|Dv|)^{\frac{np}{n-2\mu}} \mathrm{d}x \right)^{\frac{n-2\mu}{n}} \\
&+ A + \frac{B_1}{(t-s)^2} + \frac{B_2}{(t-s)^{2\tilde{p}}} + \frac{B_3}{t-s} + \frac{B_4}{(t-s)^{p^*}} \Bigg\}, \qquad (7.40)
\end{aligned}$$

where we also used that $\eta = 1$ on B_s. Then, Lemma 5.1.3 and equality (5.1) imply

$$\begin{aligned}
\left(\int_{B_s} |Dv|^{\frac{np}{n-2\mu}} \mathrm{d}x \right)^{\frac{n-2\mu}{n}} \leq& 2\theta \left(\int_{B_t} (1+|Dv|)^{\frac{np}{n-2\mu}} \mathrm{d}x \right)^{\frac{n-2\mu}{n}} + 3\theta \left(\int_{B_{t'}} (1+|Dv|)^{\frac{np}{n-2\mu}} \mathrm{d}x \right)^{\frac{n-2\mu}{n}} \\
&+ A + \frac{B_1}{(t-s)^2} + \frac{B_2}{(t-s)^{2\tilde{p}}} + \frac{B_3}{t-s} + \frac{B_4}{(t-s)^{p^*}}, \qquad (7.41)
\end{aligned}$$

for all $\mu \in (\frac{n(q-p)}{p}, \alpha)$.

Now, applaying the iteration Lemma 5.0.2 twice, we obtain

$$\left(\int_{B_{r/4}} |Dv|^{\frac{np}{n-2\mu}} \mathrm{d}x\right)^{\frac{n-2\mu}{n}} \leq C|h|^{2\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r} (1+|Dv|^p+|D\psi|^{2q-p})\mathrm{d}x\right)^{\kappa} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\}, \tag{7.42}$$

thus, using Lemma 5.1.3, from inequalities (7.42) and (7.40), we deduce the a priori estimate

$$\begin{aligned}&\int_{B_{r/4}} (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2)\mathrm{d}x\\ \leq& C|h|^{2\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r} (1+|Dv|^p+|D\psi|^{2q-p})\mathrm{d}x\right)^{\kappa} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\},\end{aligned} \tag{7.43}$$

for some $\mu < \alpha$, where $C := C(n, p, q, \mu, \|a\|_\infty)$ and $\kappa := \frac{\delta(2q-p)\tilde{p}}{p} < \tilde{p}$. □

According to the previous result, we state the following remarks, which will be crucial for the proof of Theorem 7.4.1.

Remark 4. From Proposition 5.2.1 (*iii*), it follows that

$$|\tau_h(\sqrt{a(x)}V_q(Dv))|^2 \leq Ca(x+h)|\tau_h V_q(Dv)|^2 + C|V_q(Dv)|^2|\tau_h a(x)|. \tag{7.44}$$

Combining (7.43) and (7.44), we obtain

$$\begin{aligned}&\int_{B_{r/4}} |\tau_h(\sqrt{a(x)}V_q(Dv))|^2\mathrm{d}x\\ \leq& \int_{B_{r/4}} (|\tau_h V_p(Dv)|^2 + |\tau_h(\sqrt{a(x)}V_q(Dv))|^2\mathrm{d}x\\ \leq& C\int_{B_{r/4}} (|\tau_h V_p(Dv)|^2 + a(x+h)|\tau_h V_q(Dv)|^2 + |V_q(Dv)|^2|\tau_h a(x)|)\mathrm{d}x\\ \leq& C|h|^{2\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r} (1+|Dv|^p+|D\psi|^{2q-p})\mathrm{d}x\right)^{\kappa} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\} + C|h|^{\alpha}\|Dv\|^q_{L^q(B_r)}\\ \leq& C|h|^{\alpha}\left\{\frac{1}{r^{2\tilde{p}}}\left(\int_{B_r} (1+|Dv|^p+|D\psi|^{2q-p})\mathrm{d}x\right)^{\kappa} + \|Dv\|^q_{L^q(B_r)} + [D\psi]^{2q-p}_{B^{\gamma}_{2q-p,\infty}(B_r)}\right\},\end{aligned} \tag{7.45}$$

which is finite by Theorem 7.2.1. Therefore,

$$\sqrt{a(x)}V_q(Dv) \in B^{\frac{\alpha}{2}}_{2,\infty,\mathrm{loc}}(B).$$

Lemma (5.1.3) yields

$$a(x)|Dv|^q \in L^{\frac{n}{n-2\beta}}_{\mathrm{loc}}(B), \quad \forall \beta < \frac{\alpha}{2}.$$

Remark 5. Choosing $\mu < \alpha$ s.t. $q = \frac{np}{n-2\mu}$, estimates (7.42) and (7.45) yield

$$\begin{aligned}
&\int_{B_{r/4}} (|\tau_h V_p(Dv)|^2 + |\tau_h(\sqrt{a(x)}V_q(Dv))|^2 \mathrm{d}x \\
\le & C|h|^\alpha \left\{ \frac{1}{r^{2\tilde{p}}} \left(\int_{B_r} (1+|Dv|^p + |D\psi|^{2q-p}) \mathrm{d}x \right)^\kappa + [D\psi]^{2q-p}_{B^\gamma_{2q-p,\infty}(B_r)} \right\}^{\frac{q}{p}} \\
\le & C|h|^\alpha \left\{ \frac{1}{r^{2\tilde{p}_1}} \left(\int_{B_r} (1+|Dv|^p + |D\psi|^{2q-p}) \mathrm{d}x \right)^{\kappa_1} + [D\psi]^{q_1}_{B^\gamma_{2q-p,\infty}(B_r)} + 1 \right\}, \qquad (7.46)
\end{aligned}$$

where $\frac{\tilde{p}q}{p} = \tilde{p}_1 > 1$, $\frac{\kappa q}{p} = \kappa_1 < \tilde{p}_1$ and $\frac{(2q-p)q}{p} = q_1 < \tilde{p}_1$, with $\tilde{p}$ and κ introduced in (7.38) and (7.43) respectively.

7.3 Comparison

In this section we prove a comparison Lemma (see Lemma 7.3.2 below), where we estimate the distance between the solution u to the problem (7.1) and the solution v to the problem (7.5). In order to do so, we first need the following lemma.

Lemma 7.3.1. *Let $\tilde{F} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the function defined in* (7.4) *under Assumption 1. Then there exists a positive constant $c = c(r, n, \nu)$ such that the following inequality holds for every $x \in \Omega$ and every $z_1, z_2 \in \mathbb{R}^n$*

$$\begin{aligned}
&c(|V_p(z_1) - V_p(z_2)|^2 + a(x)|V_q(z_1) - V_q(z_2)|^2) \\
&\qquad \le \tilde{F}(x, z_1) - \tilde{F}(x, z_2) - \langle D_\xi \tilde{F}(x, z_2), z_1 - z_2 \rangle. \qquad (7.47)
\end{aligned}$$

Proof. We start proving that for every $r \ge 2$ there exists a constant $c = c(r, n)$ such that

$$c(r,n)|V_r(z_1) - V_r(z_2)|^2 \le g_r(z_1) - g_r(z_2) - \langle D_\xi g_r(z_2), z_1 - z_2 \rangle, \qquad (7.48)$$

where we denote $g_r(z) := |z|^r$.

Let us consider the function $G_r : [0,1] \to \mathbb{R}$ defined by $G_r(t) := g_r(tz_1 + (1-t)z_2)$. Since $G_r \in \mathcal{C}^2([0,1])$, by using Taylor's formula with integral remainder, we obtain

$$G_r(1) = G_r(0) + G_r'(0) + \int_0^1 (1-s) G_r''(s) ds. \qquad (7.49)$$

Since

$$\begin{aligned}
G_r'(t) =& \langle D_\xi g_r(tz_1 + (1-t)z_2), z_1 - z_2 \rangle, \\
G_r''(t) =& \langle D_{\xi\xi} g_r(tz_1 + (1-t)z_2)(z_1 - z_2), z_1 - z_2 \rangle,
\end{aligned}$$

from (7.49) we get

$$\begin{aligned} g_r(z_1) - g_r(z_2) - \langle D_\xi g_r(z_2), z_1 - z_2\rangle & \\ =&\int_0^1 (1-s)\langle D_{\xi\xi} g_r(sz_1 + (1-s)z_2)(z_1 - z_2), z_1 - z_2\rangle ds \\ \geq& c(r)|z_1 - z_2|^2 \int_0^1 (1-s)|sz_1 + (1-s)z_2|^{r-2} ds. \end{aligned} \tag{7.50}$$

Now, we want to estimate from below $|sz_1 + (1-s)z_2|^{r-2}$. If $|z_1| \leq |z_2|$ and $s \in [3/4, 1]$, then $-1/4 \leq s - 1 \leq 0$ and

$$|sz_1 + (1-s)z_2| \geq s|z_1| + (s-1)|z_2| \geq \frac{3}{4}|z_1| - \frac{1}{4}|z_2| \geq \frac{1}{4}(|z_1| + |z_2|),$$

while, if $|z_2| > |z_2|$ and $s \in [0, 1/4]$, then $3/4 \leq 1 - s \leq 1$ and

$$|sz_1 + (1-s)z_2| \geq -s|z_1| + (1-s)|z_2| \geq -\frac{1}{4}|z_1| + \frac{3}{4}|z_2| \geq \frac{1}{4}(|z_1| + |z_2|).$$

Therefore

$$|sz_1 + (1-s)z_2|^{r-2} \geq 4^{2-r}(|z_1| + |z_2|)^{r-2} \tag{7.51}$$

holds on a suitable subinterval of $[0,1]$. Eventually, inserting (7.51) in (7.50) we obtain

$$\begin{aligned} g_r(z_1) - g_r(z_2) - \langle D_\xi g_r(z_2), z_1 - z_2\rangle \geq& c(r)(|z_1| + |z_2|)^{r-2}|z_1 - z_2|^2 \\ \geq& c(r,n)|V_r(z_1) - V_r(z_2)|^2, \end{aligned}$$

where in the last inequality we used Lemma 5.0.1.

At this point, using the bound from below on b in Assumption 1 and estimate (7.47) we deduce

$$\begin{aligned} \tilde{F}(x, z_1) - \tilde{F}(x, z_2) -& \langle D_\xi \tilde{F}(x, z_2), z_1 - z_2\rangle \\ =& b(x_0, u_B)[g_p(z_1) - g_p(z_2) - \langle D_\xi g_p(z_2), z_1 - z_2\rangle \\ &+ a(x)(g_q(z_1) - g_q(z_2) - \langle D_\xi g_q(z_2), z_1 - z_2\rangle)] \\ \geq& c(r,n,\nu)(|V_p(z_1) - V_p(z_2)|^2 + a(x)|V_q(z_1) - V_q(z_2)|^2). \end{aligned}$$

□

Remark 6. In the proof of Lemma 7.3.2 we will take advantage of the higher integrability results established in Section 7.1, in particular in the case $\frac{q}{p} < 1 + \frac{\alpha}{n}$.
Indeed, the assumption $D\psi \in B^\gamma_{2q-p,\infty,\mathrm{loc}}(\Omega)$ and Lemma 5.1.3 imply that $D\psi \in L^{\frac{n(2q-p)}{n-\mu(2q-p)}}$, for every $0 < \mu < \gamma$. Therefore, $H(x, D\psi)$ belong to some L^m, with $m > 1$.

Lemma 7.3.2. *Let u be a solution to (7.1) and $v \in u + W_0^{1,p}(B)$ be the solution to (7.5), under Assumptions 1 and 2, for exponents $2 \le p < q$ verifying*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}.$$

If

$$D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega),$$

for $0 < \alpha < \gamma < 1$, then

$$\int_B |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 \, dx \\ \le CR^{\sigma} \int_{2B} (1 + (H(x, Du))^m + (H(x, D\psi))^m) \, dx, \quad (7.52)$$

with $\sigma = \min\{\beta, m-1\}$, where β is the exponent appearing in the Assumption 2 and where m is the minimum of the two higher integrability exponents of Theorems 7.1.2 and 7.1.3.

Proof. Assumption 1, the definition of $\tilde{F}$ and the minimality of v imply

$$\int_B H(x, Dv) \mathrm{d}x \le C \int_B \tilde{F}(x, Dv) \mathrm{d}x \le \int_B \tilde{F}(x, Du) \mathrm{d}x \le C \int_B H(x, Du) \mathrm{d}x, \quad (7.53)$$

on the other hand, Theorem 7.1.3 yields

$$\int_B (H(x, Dv))^m \mathrm{d}x \le \int_B [(H(x, Du))^m + (H(x, D\psi))^m] \mathrm{d}x, \quad (7.54)$$

for some $m > 1$. From inequality (7.47) we get

$$\int_B |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 \mathrm{d}x \\ \le C \int_B \tilde{F}(x, Du) - \tilde{F}(x, Dv) - \langle D_\xi \tilde{F}(x, Dv), Du - Dv \rangle \mathrm{d}x,$$

moreover, recalling inequality (7.6), i.e.

$$\int_B \langle D_\xi H(x, Dv), Du - Dv \rangle \mathrm{d}x \ge 0, \quad (7.55)$$

and that $b(x_0, u_B) \ge \nu > 0$, we deduce

$$\int_B |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 \mathrm{d}x \le C \int_B \tilde{F}(x, Du) - \tilde{F}(x, Dv) \mathrm{d}x.$$

Hence, we can write the previous estimate as follows

$$
\begin{aligned}
\int_B |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 \mathrm{d}x \\
\leq & C \int_B \tilde{F}(x, Du) - \tilde{F}(x, Dv) \mathrm{d}x \\
= & C \int_B [b(x_0, u_B)H(x, Du) - b(x_0, u_B)H(x, Dv)] \mathrm{d}x \\
= & C \int_B [b(x_0, u_B)H(x, Du) - b(x, u_B)H(x, Du)] \mathrm{d}x \\
& + C \int_B [b(x, u_B)H(x, Du) - b(x, u)H(x, Du)] \mathrm{d}x \\
& + C \int_B [b(x, u)H(x, Du) - b(x, v)H(x, Dv)] \mathrm{d}x \\
& + C \int_B [b(x, v)H(x, Dv) - b(x, v_B)H(x, Dv)] \mathrm{d}x \\
& + C \int_B [b(x, v_B)H(x, Dv) - b(x, u_B)H(x, Dv)] \mathrm{d}x \\
& + C \int_B [b(x, u_B)H(x, Dv) - b(x_0, u_B)H(x, Dv)] \mathrm{d}x \\
= & C[I_1 + I_2 + I_3 + I_4 + I_5 + I_6]. \qquad (7.56)
\end{aligned}
$$

We proceed estimating the various pieces arising up from (7.56).

By Assumption 2 and estimate (7.53), we get

$$
\begin{aligned}
I_1 + I_6 \leq & \int_B \omega_b(|x - x_0|)H(x, Du) \mathrm{d}x + \int_B \omega_b(|x - x_0|)H(x, Dv) \mathrm{d}x \\
\leq & \int_B |x - x_0|^\beta (H(x, Du) + H(x, Dv)) \mathrm{d}x \\
\leq & CR^\beta \int_B H(x, Du) \mathrm{d}x \\
\leq & CR^\beta \int_B [1 + (H(x, Du))^m] \mathrm{d}x. \qquad (7.57)
\end{aligned}
$$

Now, we take care of the integral I_2. From Assumption 2, Young's and Poincaré's inequalities, we infer

$$
\begin{aligned}
I_2 \leq & \int_B \omega_b(|u - u_B|)H(x, Du) \mathrm{d}x \\
= & \int_B \frac{1}{R^{\frac{\sigma}{1+\sigma}}} \omega_b(|u - u_B|) R^{\frac{\sigma}{1+\sigma}} H(x, Du) \mathrm{d}x \\
\leq & C \int_B \frac{1}{R} \omega_b(|u - u_B|)^{\frac{1+\sigma}{\sigma}} + R^\sigma (H(x, Du))^{1+\sigma} \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq CR^{\sigma}\int_B \frac{1}{R^{1+\sigma}}|u-u_B|^{1+\sigma}+(H(x,Du))^{1+\sigma}\mathrm{d}x\\
&\leq CR^{\sigma}\int_B |Du|^{1+\sigma}+(H(x,Du))^{1+\sigma}\mathrm{d}x\\
&\leq CR^{\sigma}\int_B (1+|Du|^{p(1+\sigma)})+(H(x,Du))^{1+\sigma}\mathrm{d}x\\
&\leq CR^{\sigma}\int_B [1+(H(x,Du))^{m}]\mathrm{d}x,
\end{aligned} \tag{7.58}
$$

where $\sigma := \min\{\beta, m-1\}$.

The minimality of u yields that

$$
I_3 \leq 0. \tag{7.59}
$$

Arguing analogously as for the integral I_2, we obtain

$$
\begin{aligned}
I_4 &\leq \int_B \omega_b(|v-v_B|)H(x,Dv)\mathrm{d}x\\
&\leq CR^{\sigma}\int_B \frac{1}{R^{1+\sigma}}|v-v_B|^{1+\sigma}+(H(x,Dv))^{1+\sigma}\mathrm{d}x\\
&\leq CR^{\sigma}\int_B |Dv|^{1+\sigma}+(H(x,Dv))^{1+\sigma}\mathrm{d}x\\
&\leq CR^{\sigma}\int_B [1+(H(x,Dv))^{m}]\mathrm{d}x\\
&\leq CR^{\sigma}\int_B [1+(H(x,Du))^{m}+(H(x,D\psi))^{m}]\mathrm{d}x,
\end{aligned} \tag{7.60}
$$

where in the last inequality we used (7.54).

Since $u = v$ on ∂B, using Poicaré inequality for the function $u - v$, we infer the following estimate for I_5.

$$
\begin{aligned}
I_5 &\leq \int_B \omega_b(|u_B-v_B|)H(x,Dv)\mathrm{d}x\\
&\leq CR^{\sigma}\int_B \frac{1}{R^{1+\sigma}}\omega_b(|u_B-v_B|)^{\frac{1+\sigma}{\sigma}}+(H(x,Du))^{1+\sigma}\mathrm{d}x\\
&\leq CR^{\sigma}\int_B \frac{1}{R^{1+\sigma}}|u-v|^{1+\sigma}+(H(x,Dv))^{1+\sigma}\mathrm{d}x\\
&\leq CR^{\sigma}\int_B |Du|^{1+\sigma}+|Dv|^{1+\sigma}+(H(x,Dv))^{1+\sigma}\mathrm{d}x\\
&\leq CR^{\sigma}\int_B [1+(H(x,Du))^{m}+(H(x,Dv))^{m}]\mathrm{d}x\\
&\leq CR^{\sigma}\int_B [1+(H(x,Du))^{m}+(H(x,D\psi))^{m}]\mathrm{d}x,
\end{aligned} \tag{7.61}
$$

where in the inequality we used estimate (7.53). Finally, inserting estimates (7.57)–(7.61) in (7.56), we get the desired estimate. □

7.4 Main result

In order to prove Theorem 7.0.1 we follow the strategy first proposed in [85].
Before proving our main result, in Section 7.4.1, we fix some further notation and derive a preliminary regularity theorem for solutions to (7.1).
For a ball $\mathcal{B} \Subset \Omega$ of radius R, we will denote by $\mathcal{Q}_1 = \mathcal{Q}_1(\mathcal{B})$ and $\mathcal{Q}_2 = \mathcal{Q}_2(\mathcal{B})$ the largest and the smallest cubes, concentric to $\mathcal{B}$ and with sides parallel to the coordinate axes, contained in $\mathcal{B}$ and containing $\mathcal{B}$ respectively. It is easy to verify that $|\mathcal{B}| \approx |\mathcal{Q}_1| \approx |\mathcal{Q}_2| \approx R^n$. We also denote the enlarged ball by $\hat{\mathcal{B}} = 4\mathcal{B}$. We set

$$\mathcal{Q}_1 = \mathcal{Q}_1(\mathcal{B}) \quad \hat{\mathcal{Q}}_2 = \mathcal{Q}_2(\hat{\mathcal{B}})$$

so that we have the following chain of inclusions

$$\mathcal{Q}_1 \subset \mathcal{B} \Subset 2\mathcal{B} \Subset \mathcal{Q}_1(\hat{\mathcal{B}}) \subset \hat{\mathcal{B}} \subset \hat{\mathcal{Q}}_2.$$

In what follows, we shall always take $\mathcal{B}$ such that $\mathcal{Q}_2(\hat{\mathcal{B}}) \Subset \Omega$.

Our next result shows that a fractional differentiability property on the gradient of the obstacle transfers to a higher fractional differentiability for the gradient of the minimizer.

Theorem 7.4.1. *Let u be a solution to* (7.1) *under Assumptions 1 and 2, for exponents* $2 \le p < \frac{n}{\alpha}$, $p < q$ *verifying*

$$\frac{q}{p} < 1 + \frac{\alpha}{n}.$$

Then the following implication

$$D\psi \in B^{\gamma}_{2q-p,\infty,loc}(\Omega) \Rightarrow V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\sigma_\alpha}_{2,\infty,loc}(\Omega)$$

holds provided $0 < \alpha < \gamma < 1$, *with* $\sigma_\alpha = \sigma_\alpha(p, q, n, \alpha, \beta, m)$, *where β is the exponent appearing in the Assumption 2 and where m is the minimum of the two higher integrability exponents of Theorems 7.1.2 and 7.1.3.*

Proof. Let us fix arbitrary open subsets $\Omega' \Subset \Omega'' \Subset \Omega$ and choose $x_0 \in \Omega'$. We recall the definition of $\tilde{p}_1$ from (7.46). Let $\delta \in \left(0, \frac{\alpha}{2\tilde{p}_1}\right)$ be chosen later and consider the ball $\mathcal{B} = \mathcal{B}(x_0, |h|^\delta)$ with $|h|$ sufficiently small, depending on the dimension n, the parameter δ and the distance between Ω' and the boundary of Ω'' such that $\hat{\mathcal{Q}}_2 \Subset \Omega''$. Furthermore, let $v \in u + W^{1,p}_0(B)$ be the solution to (7.5) with $B = \hat{\mathcal{B}}$.

We estimate the difference quotient for $V_p(Du)$ and $\sqrt{a(x)}V_q(Du)$ as follows

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 \mathrm{d}x \\
=&\int_{\mathcal{B}} |V_p(Du(x+h)) - V_p(Du(x))|^2 \mathrm{d}x \\
&+ \int_{\mathcal{B}} |\sqrt{a(x+h)}V_q(Du(x+h)) - \sqrt{a(x)}V_q(Du(x))|^2 \mathrm{d}x \\
\leq& C\int_{\mathcal{B}} |V_p(Du(x+h)) - V_p(Dv(x+h))|^2 \mathrm{d}x \\
&+ C\int_{\mathcal{B}} |V_p(Dv(x+h)) - V_p(Dv(x))|^2 \mathrm{d}x \\
&+ C\int_{\mathcal{B}} |V_p(Dv(x)) - V_p(Du(x))|^2 \mathrm{d}x \\
&+ C\int_{\mathcal{B}} |\sqrt{a(x+h)}V_q(Du(x+h)) - \sqrt{a(x+h)}V_q(Dv(x+h))|^2 \mathrm{d}x \\
&+ C\int_{\mathcal{B}} |\sqrt{a(x+h)}V_q(Dv(x+h)) - \sqrt{a(x)}V_q(Dv(x))|^2 \mathrm{d}x \\
&+ C\int_{\mathcal{B}} |\sqrt{a(x)}V_q(Dv(x)) - \sqrt{a(x)}V_q(Du(x))|^2 \mathrm{d}x. \qquad (7.62)
\end{aligned}$$

Notice that if $x \in \mathcal{B}$, then $x+h \in \hat{\mathcal{B}}$, for $|h| \leq 1$. Thus, we get

$$\begin{aligned}
&\int_{\mathcal{B}} |V_p(Du(x+h)) - V_p(Dv(x+h))|^2 \mathrm{d}x \\
&+ \int_{\mathcal{B}} |\sqrt{a(x+h)}V_q(Du(x+h)) - \sqrt{a(x+h)}V_q(Dv(x+h))|^2 \mathrm{d}x \\
\leq& \int_{\hat{\mathcal{B}}} |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 \mathrm{d}x. \qquad (7.63)
\end{aligned}$$

Inserting inequality (7.63) in (7.62), we obtain

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 \mathrm{d}x \\
\leq& C\int_{\hat{\mathcal{B}}} |V_p(Du) - V_p(Dv)|^2 + a(x)|V_q(Du) - V_q(Dv)|^2 \mathrm{d}x \\
&+ C\int_{\mathcal{B}} |\tau_h V_p(Dv)|^2 + |\tau_h(\sqrt{a(x)}V_q(Dv))|^2 \mathrm{d}x \\
=:& J_1 + J_2. \qquad (7.64)
\end{aligned}$$

From estimate (7.52) applied over the ball $\hat{\mathcal{B}}$, we infer

$$J_1 \leq C|h|^{\sigma\delta} \int_{\hat{\mathcal{Q}}_2} (1 + (H(x,Du))^m + (H(x,D\psi))^m)\mathrm{d}x, \qquad (7.65)$$

where we used that the radius of $\hat{\mathcal{B}}$ is proportional to $|h|^\delta$. Now estimate (7.46) (see Remark 5) applied over the ball $\mathcal{B}$ yields

$$J_2 \le C|h|^{\alpha-2\delta\tilde{p}_1}\left(\int_{\hat{\mathcal{B}}}(1+|Dv|^p+|D\psi|^{2q-p})\mathrm{d}x\right)^{\kappa_1} + C|h|^\alpha[D\psi]^{q_1}_{B^\gamma_{2q-p,\infty}(\hat{\mathcal{B}})} + C|h|^\alpha, \quad (7.66)$$

recalling that the radius of $\mathcal{B}$ is $|h|^\delta$. Inserting (7.65) and (7.66) in (7.64), we get

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2\mathrm{d}x\\
\le& C|h|^{\sigma\delta}\int_{\hat{\mathcal{Q}}_2}(1+(H(x,Du))^m+(H(x,D\psi))^m)\mathrm{d}x\\
&+C|h|^{\alpha-2\delta\tilde{p}_1}\left(\int_{\hat{\mathcal{B}}}(1+|Dv|^p+|D\psi|^{2q-p})\mathrm{d}x\right)^{\kappa_1} + C|h|^\alpha[D\psi]^{q_1}_{B^\gamma_{2q-p,\infty}(\hat{\mathcal{B}})} + C|h|^\alpha\\
\le& C|h|^{\sigma\delta}\int_{\hat{\mathcal{Q}}_2}(1+(H(x,Du))^m+(H(x,D\psi))^m)\mathrm{d}x\\
&+C|h|^{\alpha-2\delta\tilde{p}_1}\left(\int_{\hat{\mathcal{Q}}_2}(1+H(x,Du))\mathrm{d}x\right)^{\kappa_1} + C|h|^\alpha[D\psi]^{q_1}_{B^\gamma_{2q-p,\infty}(\hat{\mathcal{Q}}_2)}\\
&+C|h|^{\alpha-2\delta\tilde{p}_1}\left(\int_{\hat{\mathcal{Q}}_2}|D\psi|^{2q-p}\mathrm{d}x\right)^{\kappa_1} + C|h|^\alpha,
\end{aligned} \quad (7.67)$$

where in the last inequality we used (7.53).

Now we choose δ in order to minimize the right hand side of the previous estimate. It is easy to check that the best possible estimate is given by the choice

$$\delta = \frac{\alpha}{\sigma+2\tilde{p}_1} \in \left(0, \frac{\alpha}{2\tilde{p}_1}\right).$$

With such a choice of δ estimate (7.67) becomes

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2\mathrm{d}x\\
\le& C|h|^{\frac{\alpha\sigma}{\sigma+2\tilde{p}_1}}\left\{\int_{\hat{\mathcal{Q}}_2}(1+(H(x,Du))^m+(H(x,D\psi))^m)\mathrm{d}x + \|D\psi\|_{B^\gamma_{2q-p,\infty}(\hat{\mathcal{Q}}_2)}+1\right\}^{\kappa_2},
\end{aligned} \quad (7.68)$$

where $\kappa_2 := \kappa_2(n,p,q,\mu)$, for some $\mu < \alpha$.

At this point, arguing as in [85, Lemma 4.5], a covering argument allows us to replace the cubes $\mathcal{Q}_1$ and $\hat{\mathcal{Q}}_2$ with the fixed open subsets Ω' and Ω'', respectively. Indeed for each $|h| \in \mathbb{R}^n$ sufficiently small we can find balls $\mathcal{B}_1 = \mathcal{B}_1(x_1, |h|^\sigma), ..., \mathcal{B}_K = \mathcal{B}_K(x_K, |h|^\sigma)$, being $K = K(h) \in \mathbb{N}$, such that the corresponding inner cubes $\mathcal{Q}_1(\mathcal{B}_1), ..., \mathcal{Q}_1(\mathcal{B}_K)$ are disjoint and satisfy

$$\left|\Omega' \setminus \bigcup_{k=1}^{K} \mathcal{Q}_1(\mathcal{B}_k)\right| = 0.$$

By our assumption we have that $\mathcal{Q}_2(\hat{\mathcal{B}}_k) \subset \Omega''$, for every $k \leq K$ and each of the dilated outer cubes $\mathcal{Q}_2(\hat{\mathcal{B}}_k)$ intersects at most $(16\sqrt{n})$ of the other cubes $\mathcal{Q}_2(\hat{\mathcal{B}}_j)$, with $j \neq k$. Hence, after summing up (7.68) over the inner cubes $\mathcal{Q}_1 \in \{\mathcal{Q}_1(\mathcal{B}_1), ..., \mathcal{Q}_1(\mathcal{B}_K)\}$, and enlarging the constant by a fixed factor only depending on n and p (in particular independent of h), we arrive at

$$\begin{aligned}&\int_{\Omega'} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 \mathrm{d}x \\ \leq& C|h|^{\frac{\alpha\sigma}{\sigma+2\tilde{p}_1}} \left\{ \int_{\Omega''} (1 + (H(x,Du))^m + (H(x,D\psi))^m)\mathrm{d}x + \|D\psi\|_{B^{\gamma}_{2q-p,\infty}(\Omega'')} + 1 \right\}^{\kappa_2}. \quad (7.69)\end{aligned}$$

Since the right hand side of the previous estimate is finite by our assumptions, it follows by arbitrariness of Ω' that

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\frac{\alpha\sigma}{2(\sigma+2\tilde{p}_1)}}_{2,\infty}(\Omega) \quad \text{locally}.$$

Setting

$$\sigma_\alpha := \frac{\alpha\sigma}{2(\sigma + 2\tilde{p}_1)}, \tag{7.70}$$

it follows the conclusion. □

7.4.1 Proof of Theorem 7.0.1

We are now able to give the proof of the main result of this work.

Let us consider the function

$$A(t) = \frac{\alpha\sigma}{2[2(\tilde{p}_1 - \kappa_1 t) + \sigma]}, \qquad \forall t \in \left(0, \frac{\sigma + 2\tilde{p}_1 - \sqrt{(\sigma+2\tilde{p}_1)^2 - 4\kappa_1\alpha\sigma}}{4\kappa_1}\right) =: (0, \tilde{\sigma}), \tag{7.71}$$

where $\tilde{p}_1, \kappa_1$ are defined in (7.46), σ is defined in Lemma 7.3.2 and α is the exponent appearing in Assumption 1.

It is easy to see that $t \mapsto A(t)$ is increasing and that

$$t < A(t) < \tilde{\sigma}, \tag{7.72}$$

$$A(\tilde{\sigma}) = \tilde{\sigma}. \tag{7.73}$$

It is worth noticing that

$$\sigma_\alpha < \tilde{\sigma} < \frac{\alpha\sigma}{2\tilde{p}_1}, \tag{7.74}$$

where σ_α was introduced in (7.70). Indeed, owing to (7.70), the first part of inequality (7.74) holds if, and only if,

$$(\sigma + 2\tilde{p}_1)\sqrt{(\sigma + 2\tilde{p}_1)^2 - 4\kappa_1\alpha\sigma} < (\sigma + 2\tilde{p}_1)^2 - 2\kappa_1\alpha\sigma.$$

The last inequality is satisfied if, and only if,

$$(\sigma + 2\tilde{p}_1)^4 - 4\alpha\kappa_1\sigma(\sigma + 2\tilde{p}_1)^2 < (\sigma + 2\tilde{p}_1)^4 + 4\kappa_1^2\alpha^2\sigma^2 - 4\kappa_1\alpha\sigma(\sigma + 2\tilde{p}_1)^2,$$

that is equivalent to

$$0 < 4\kappa_1^2\alpha^2\sigma^2.$$

On the other hand, the second part of inequality (7.74) is valid if, and only if,

$$\tilde{p}_1(\sigma + 2\tilde{p}_1) - 2\kappa_1\alpha\sigma < \tilde{p}_1\sqrt{(\sigma + 2\tilde{p}_1)^2 - 4\kappa_1\alpha\sigma},$$

or, equivalently,

$$\tilde{p}_1^2(\sigma + 2\tilde{p}_1)^2 + 4\kappa_1^2\alpha^2\sigma^2 - 4\tilde{p}_1\kappa_1\alpha\sigma(\sigma + 2\tilde{p}_1) < \tilde{p}_1^2(\sigma + 2\tilde{p}_1)^2 - 4\tilde{p}_1^2\kappa_1\alpha\sigma.$$

The previous inequality can be written as

$$\kappa_1\alpha\sigma - \tilde{p}_1\sigma < \tilde{p}_1^2,$$

that holds true since $1 < \kappa_1 < \tilde{p}_1$ and $\alpha, \sigma \in (0,1)$. Let us now fix

$$\theta_0 \in \left(0, \frac{\alpha\sigma}{2(\sigma + 2\tilde{p}_1)}\right)$$

and denote

$$\theta_j = A(\theta_{j-1}), \quad \forall j \in \mathbb{N},\ j \geq 1.$$

Hence, the sequence $(\theta_j)_j$ is increasing and

$$\lim_j \theta_j = \tilde{\sigma}. \tag{7.75}$$

Now we define the sequence $(\iota_j)_j$ inductively as follows:

$$\iota_0 = \frac{\theta_0}{2} + \frac{\alpha\sigma}{4(\sigma + 2\tilde{p}_1)} < \frac{\alpha\sigma}{2(\sigma + 2\tilde{p}_1)},$$

$$\iota_j = \frac{\theta_j + A(\iota_{j-1})}{2}.$$

Using the fact that A is increasing and (7.72), (7.73), we obtain

$$\theta_j < \iota_j < \tilde{\sigma}, \qquad \forall j \in \mathbb{N}, \tag{7.76}$$

and therefore, from (7.75), it follows that

$$\lim_j \iota_j = \tilde{\sigma}. \tag{7.77}$$

Arguing by induction, we shall prove that

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\iota_j}_{2,\infty,\mathrm{loc}}(\Omega) \quad \forall j \in \mathbb{N}.$$

The case $j = 0$ follows from Theorem 7.4.1 and our choice of ι_0. Now, let us prove the implication

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\iota_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega) \Rightarrow V_p(Du),\ \sqrt{a(x)}V_q(Du) \in B^{\iota_j}_{2,\infty,\mathrm{loc}}(\Omega). \tag{7.78}$$

By virtue of Lemma 5.1.3, the assumptions $V_p(Du), \sqrt{a(x)}V_q(Du) \in B^{\iota_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega)$ imply

$$V_p(Du),\ \sqrt{a(x)}V_q(Du) \in L^{\frac{2n}{n-2\lambda}}(\hat{\mathcal{Q}}_2),$$

for every $0 < \lambda < \iota_{j-1}$ and so, recalling equality (5.1), we have that

$$|Du|^p,\ a(x)|Du|^q \in L^{\frac{n}{n-2\lambda}}(\hat{\mathcal{Q}}_2).$$

In particular, it follows

$$H(x, Du) \in L^{\frac{n}{n-2\lambda}}(\hat{\mathcal{Q}}_2),$$

for every $0 < \lambda < \iota_{j-1}$. Moreover, the assumption $D\psi \in B^{\gamma}_{2q-p,\infty,\mathrm{loc}}(\Omega)$ and Lemma 5.1.3 imply that $D\psi \in L^{\frac{n(2q-p)}{n-\pi(2q-p)}}(\hat{\mathcal{Q}}_2)$, for every $0 < \pi < \gamma$. Therefore, using Hölder's inequality in estimate (7.67) we infer

$$\begin{aligned}
&\int_{\mathcal{B}} |\tau_h V_p(Du)|^2 + |\tau_h(\sqrt{a(x)}V_q(Du))|^2 \mathrm{d}x \\
\leq &C|h|^{\sigma\delta} \int_{\hat{\mathcal{Q}}_2} (1 + (H(x, Du))^m + (H(x, D\psi))^m)\mathrm{d}x \\
&+ C|h|^{\alpha - 2\delta\tilde{p}_1 + 2\delta\kappa_1\lambda} \left(\int_{\hat{\mathcal{Q}}_2} (1 + H(x, Du))^{\frac{n}{n-2\lambda}} \mathrm{d}x \right)^{\frac{(n-2\lambda)\kappa_1}{n}} + C|h|^{\alpha}[D\psi]^{q_1}_{B^{\gamma}_{2q-p,\infty}(\hat{\mathcal{Q}}_2)}
\end{aligned}$$

$$
\begin{aligned}
&+C|h|^{\alpha-2\delta\tilde{p}_1+(2q-p)\delta\kappa_1\pi}\left(\int_{\hat{\mathcal{Q}}_2}|D\psi|^{\frac{n(2q-p)}{n-\pi(2q-p)}}\mathrm{d}x\right)^{\frac{(n-\pi(2q-p))\kappa_1}{n}}+C|h|^{\alpha}\\
\leq &C|h|^{\sigma\delta}\int_{\hat{\mathcal{Q}}_2}(1+(H(x,Du))^m+(H(x,D\psi))^m)\mathrm{d}x\\
&+C|h|^{\alpha-2\delta\tilde{p}_1+2\delta\kappa_1\lambda}\left(\int_{\hat{\mathcal{Q}}_2}(1+H(x,Du))^{\frac{n}{n-2\lambda}}\mathrm{d}x\right)^{\frac{(n-2\lambda)\kappa_1}{n}}+C|h|^{\alpha}[D\psi]^{q_1}_{B^{\gamma}_{2q-p,\infty}(\hat{\mathcal{Q}}_2)}\\
&+C|h|^{\alpha-2\delta\tilde{p}_1+2\delta\kappa_1\lambda}\left(\int_{\hat{\mathcal{Q}}_2}|D\psi|^{\frac{n(2q-p)}{n-\pi(2q-p)}}\mathrm{d}x\right)^{\frac{(n-\pi(2q-p))\kappa_1}{n}}+C|h|^{\alpha},
\end{aligned}
\tag{7.79}
$$

for some $\pi \geq \frac{2\lambda}{2q-p}$, where we used the fact that the radius of the cube $\hat{\mathcal{Q}}_2$ is proportional to $|h|^{\delta}$. Therefore, choosing δ in order to maximize the right hand side of (7.79), namely

$$
\delta=\frac{\alpha}{\sigma+2(\tilde{p}_1-k_1\lambda)},
$$

we have

$$
\begin{aligned}
&\int_{\mathcal{B}}|\tau_h V_p(Du)|^2+|\tau_h(\sqrt{a(x)}V_q(Du))|^2\mathrm{d}x\\
\leq &C|h|^{\frac{\alpha\sigma}{\sigma+2(\tilde{p}_1-k_1\lambda)}}\left\{\int_{\hat{\mathcal{Q}}_2}(1+(H(x,Du))^m+(H(x,D\psi))^m)\mathrm{d}x\right.\\
&\left.+\int_{\hat{\mathcal{Q}}_2}(1+H(x,Du))^{\frac{n}{n-2\lambda}}\mathrm{d}x+\|D\psi\|_{B^{\gamma}_{2q-p,\infty}(\hat{\mathcal{Q}}_2)}+1\right\}^{\kappa^*},
\end{aligned}
\tag{7.80}
$$

where $\kappa^* := \kappa^*(n,p,q,\mu,\lambda)$. Thus, again through a covering argument, we deduce that

$$
V_p(Du),\ \sqrt{a(x)}V_q(Du)\in B^{\frac{\alpha\sigma}{2[\sigma+2(\tilde{p}_1-k_1\lambda)]}}_{2,\infty,\mathrm{loc}}(\Omega)=B^{A(\lambda)}_{2,\infty,\mathrm{loc}}(\Omega),\quad \forall\lambda<\iota_{j-1}.
$$

We have just proved the following implication

$$
V_p(Du),\ \sqrt{a(x)}V_q(Du)\in B^{\iota_{j-1}}_{2,\infty,\mathrm{loc}}(\Omega)\Rightarrow V_p(Du),\ \sqrt{a(x)}V_q(Du)\in B^{t}_{2,\infty,\mathrm{loc}}(\Omega),\tag{7.81}
$$

for all $t < A(\iota_{j-1})$. Since A is increasing, it follows from (7.76) that $\theta_j < A(\iota_{j-1})$. Moreover, the definition of ι_j implies $\iota_j < A(\iota_{j-1})$. Therefore, (7.78) follows from (7.81). Besides, from (7.76) and (7.77), we infer

$$
V_p(Du),\ \sqrt{a(x)}V_q(Du)\in B^{t}_{2,\infty,\mathrm{loc}}(\Omega),\quad \forall t\in(0,\tilde{\sigma}).
$$

It is worth noting that the exponent $\tilde{\sigma}$ defined in (7.71) is bigger than σ_{α}. Therefore, Theorem 7.0.1 improves the higher fractional differentiability result established in Theorem 7.4.1.

Appendix

Appendix A

Sobolev embeddings and interpolation theory

A.1 2D results

In the sequel we will also frequently use the following 2D interpolation inequalities:

$$\|v\|_{L^4(\Omega)} \leq c\,\|v\|_V^{1/2}\,\|v\|^{1/2}\,, \tag{A.1a}$$

$$\|v\|_{L^\infty(\Omega)} \leq c\,\|v\|_{H^2(\Omega)}^{1/2}\,\|v\|^{1/2}\,, \tag{A.1b}$$

$$\|v\|_{L^r(\Omega)} \leq c\,\|v\|^{\frac{2}{r}}\,\|v\|_V^{1-\frac{2}{r}}\,, \qquad r \in [2,\infty), \tag{A.1c}$$

$$\|v\|_{L^r(\Omega)} \leq c\,\|v\|_{L^s(\Omega)}^{1-\alpha}\,\|v\|_{L^\infty(\Omega}^{\alpha}\,, \qquad \alpha = 1 - \frac{s}{r}, \qquad r \geq 1 \tag{A.1d}$$

$$\|v\|_{H^s(\Omega)} \leq c\,\|v\|_{H^{s_1}(\Omega)}^{1-\theta}\,\|v\|_{H^{s_2}(\Omega)}^{\theta}\,, \qquad \theta = \frac{s-s_1}{s_2-s_1}, \tag{A.1e}$$

$$\|v - v_\Omega\| \leq c\,\|\nabla v\|\,, \tag{A.1f}$$

holding for any sufficiently smooth function v and for suitable embedding constants, all denoted by the same symbol $c > 0$ for brevity. We will also use the following non linear Poincaré inequality (see [66])

$$\left\|v^{p/2}\right\|_V^2 \leq c_p \left(\|v\|_{L^1(\Omega)}^p + \left\|\nabla v^{p/2}\right\|^2\right), \tag{A.2}$$

holding for all non-negative $v \in L^1(\Omega)$ such that $\nabla v^{p/2} \in L^2(\Omega)$, and for all $p \in [2,\infty)$. We also recall that

$$\|v\| \leq c\,\|\nabla v\|^{1/2}\,\|v\|_{V'}^{1/2}\,, \quad \forall v \in V_0, \tag{A.3}$$

as one can prove simply combining the standard interpolation inequality $\|v\| \leq c\,\|v\|_V^{1/2}\,\|v\|_{V'}^{1/2}$ with the Poincaré-Wirtinger inequality (A.1f).

A.2 General results

We first recall here a general interpolation result for L^p- spaces.

Proposition A.2.1. *If $f \in L^p \cap L^s$, then $f \in L^r$, with r s.t.* $\dfrac{1}{r} = \dfrac{\gamma}{p} + \dfrac{1-\gamma}{s}$.

The following interpolation in Sobolev spaces can be found in [90].

Theorem A.2.2. *Let Ω be an open and bounded subset of $\mathbb{R}$. Let X, Y be Hilbert spaces. If $s_1, s_2 \geq 0$,*

(i) then we have

$$[H^{s_1}(\Omega; X), H^{s_2}(\Omega, Y)]_\theta = H^{(1-\theta)s_1+\theta s_2}(\Omega; [X, Y]_\theta); \tag{A.4}$$

(ii) if $s_2 \neq \mu + \frac{1}{2}$ (integer $\mu \geq 0$), then we have

$$[H^{s_1}(\Omega), H^{-s_2}(\Omega)]_\theta = H^{(1-\theta)s_1-\theta s_2}(\Omega) \tag{A.5}$$

if $(1-\theta)s_1 - \theta s_2 \neq -\frac{1}{2} - \nu$ (integer $\nu \geq 0$).

The next theorem can be found in [49].

Theorem A.2.3. *Suppose $u \in L^2(0, T; H_0^1(U))$ with $u' \in L^2(0, T; H^{-1}(U))$.*

(i) Then

$$u \in C([0, T]; L^2(U)). \tag{A.6}$$

(after possibly being redefined on a set of measure zero).

(ii) The mapping

$$t \mapsto \|u(t)\|^2_{L^2(U)}$$

is absolutele continuous, with

$$\frac{d}{dt}\|u(t)\|^2_{L^2(U)} = 2\langle u'(t), u(t)\rangle$$

for a.e. $0 \leq t \leq T$.

(iii) Furthermore, we have the estimate

$$\max_{0\leq t\leq T} \|u(t)\|^2_{L^2(U)} \leq C\left(\|u\|^2_{L^2(0,T;H_0^1(U))} + \|u\|^2_{L^2(0,T;H^{-1}(U))}\right),$$

the constant C depending only on T.

Lemma A.2.4 (Aubin-Lions)**.** *Let X_0, X and X_1 be Banach spaces with $X_0 \subset X \subset X_1$. Suppose that $X_0 \hookrightarrow\hookrightarrow X$ and $X \hookrightarrow X_1$. For $1 \leq p, q \leq +\infty$, let*

$$W = \{u \in L^p([0,T]; X_0) \ s.t. \ u' \in L^q([0,T]; X_1)\}.$$

(i) If $p < +\infty$, then $W \hookrightarrow\hookrightarrow L^p([0,T]; X)$.

ii) If $p = +\infty$ and $q > 1$, then $W \hookrightarrow\hookrightarrow C([0,T]; X)$.

The following compactness embeddings can be found in [12].

Theorem A.2.5. *Let $0 \leq r \leq s$ and let $\Omega \subset \mathbb{R}^n$ be bounded. For $1 \leq p \leq q \leq \infty$, if $r - \frac{n}{q} < s - \frac{n}{p}$, then*

$$W^{s,p}(\Omega) \hookrightarrow\hookrightarrow W^{r,q}(\Omega).$$

Theorem A.2.6. *Let $0 \leq r \leq s$, $1 \leq p \leq q \leq \infty$, $V \hookrightarrow W$ and $r - \frac{1}{q} \leq s - \frac{1}{p}$. Then*

$$W^{s,p}(0,T; V) \hookrightarrow W^{r,q}(0,T; W).$$

In particular, the inclusion is compact if $r - \frac{1}{q} < s - \frac{1}{p}$ and $V \hookrightarrow\hookrightarrow W$.

Appendix B

Topological degree

The content of this Appendix can be found in [119].

Let $\mathcal{A} = \{(f, \Omega, p)$ s.t. $\Omega \subseteq \mathbb{R}^N$ open and bounded, $f : \bar{\Omega} \to \mathbb{R}^N$ continuous, $p \notin \mathbb{R}^N \setminus f(\partial\Omega)\}$.

Definition B.1. The *topological degree* is a function $d : \mathcal{A} \to \mathbb{Z}$ s.t.

d1. NORMALIZATION PROPERTY
$d(\text{id}, \Omega, p) = 1, \quad p \in \Omega.$

d2. DOMAIN ADDITIVITY
Let $(f, \Omega, p) \in \mathcal{A}$ be fixed and $\Omega_1, \Omega_2 \subset \Omega$ open be given s.t. $\Omega_1 \cup \Omega_2 = \emptyset$ and $p \notin f(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$
$\Rightarrow d(f, \Omega, p) = d(f, \Omega_1, p) + d(f, \Omega_2, p).$

d3. HOMOTOPY INVARIANCE
$h : [0, 1] \times \bar{\Omega} \to \mathbb{R}^N$ continuous
$p : [0, 1] \to \mathbb{R}^N$ continuous and s.t. $p(t) \notin h(t, \partial\Omega)$
$\Rightarrow d(h(t, \cdot), \Omega, p(t)) = const$ w.r.t. t.

Proposition B.0.1. *The topological degree has the following properties:*

P0. $d(f, \emptyset, p) = 0.$

P1. EXCISION
$\forall (f, \Omega, p) \in \mathcal{A}, \forall \Omega_1 \subset \Omega$ *open*, $p \notin f(\bar{\Omega} \setminus \Omega_1)$
$\Rightarrow d(f, \Omega, p) = d(f, \Omega_1, p).$

P2. $\forall (f, \Omega, p) \in \mathcal{A}, \forall p \notin f(\bar{\Omega})$
$\Rightarrow d(f, \Omega, p) = 0.$

P3. EXISTENCE
$\forall (f, \Omega, p) \in \mathcal{A}, d(f, \Omega, p) \neq 0$
$\Rightarrow \exists x \in \Omega : f(x) = p$ *with* $p \notin f(\partial\Omega)$.
Namely, the equation has at least one solution.

We also report here the following result, contained in ([119]).

Theorem B.0.2 (Borsuk)**.**
Let $\Omega \subset \mathbb{R}^n$ *be opened bounded and symmetric with respect to* $0 \in \Omega$.
Let $f \in C(\bar{\Omega})$ *be odd and* $0 \notin f(\partial\Omega)$.
Then the topological degree $d(f, \Omega, 0)$ *is odd.*

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