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Nonperturbative aspects of JT gravity and $T\bar{T}$ deformation

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The goal of this dissertation is the study of nonperturbative effects and phenomena in low-dimensional gravity and gauge theories, both in the presence and absence of integrable deformations. The investigation of these nonperturbative aspects is indeed essential to render physical quantities well-definite, allowing for a better understanding of the quantum regime of the relevant theories. Concretely, we focus on JT gravity, one of the rare example of exactly solvable quantum gravitional systems and a remarkable toy model for holography, and 2d Yang-Mills theory, which is very much related to JT gravity in the first order formalism. Due to their high degree of solvability and control, these two models provide a valuable theoretical laboratory to test phenomena and properties likely to occur in their realistic four-dimensional counterparts. Given a theory with an exact quantum solution, it is natural to look for deformations preserving its solvable character: for this reason, we subsequently investigate the flow of these models under the $T\bar{T}$ deformation, which offers the possibility to safely move against the renormalization group flow and possibly describe non-local physics in the deep UV.

The first project concerns the study of the perturbative series associated to bi-local correlators in JT gravity, for positive weight λ of the matter CFT operators. Starting from the known exact expression, derived by CFT and gauge theoretical methods, we reproduce the Schwarzian semi-classical expansion beyond leading order. The computation is done for arbitrary temperature and finite boundary distances, in the case of disk and trumpet topologies. A formula presenting the perturbative result (for $\lambda \in \mathbb{N}/2$) at any given order in terms of generalized Apostol-Bernoulli polynomials is also obtained. The limit of zero temperature is then considered, obtaining a compact expression that allows to discuss the asymptotic behavior of the perturbative series and possibly detect, if any, the relevant nonperturbative completions. Finally we highlight the possibility to express the exact result as particular combinations of Mordell integrals.

Secondly, we turn to investigate the nonperturbative structure of Jackiw–Teitelboim gravity at finite cutoff, as given by its proposed formulation in terms of a $T\bar{T}$-deformed Schwarzian quantum mechanics. Our starting point is a careful computation of the disk partition function to all orders in the perturbative expansion in the cutoff parameter. We show that the perturbative series is asymptotic and that it admits a precise completion exploiting the analytical properties of its Borel transform, as prescribed by resurgence theory. The final result is then naturally interpreted in terms of the nonperturbative branch of the $T\bar{T}$-deformed spectrum. The finite-cutoff trumpet partition function is computed by applying the same strategy. In the second part of the analysis, we propose an extension of this formalism to arbitrary topologies, using the basic gluing rules of the undeformed case. The Weil–Petersson integrations can be safely performed due to the nonperturbative corrections and give results that are compatible with the flow equation associated with the $T\bar{T}$ deformation. We derive exact expressions for general topologies and show that these are captured by a suitable deformation of the Eynard–Orantin topological recursion. Finally, we study the "slope" and "ramp" regimes of the spectral form factor as functions of the cutoff parameter.

The deep link between gravity and $T\bar{T}$ deformation suggests to consider the flow induced by the $T\bar{T}$ operator in another deeply connected two-dimensional model, Yang-Mills. We study the $T\bar{T}$ deformation of this theory on a surface at genus zero by carrying out the analysis at the level of its instanton representation. We first focus on the perturbative sector by considering its power expansion in the deformation parameter μ. By studying the resulting asymptotic series through resurgence theory, we determine the nonperturbative contributions that enter the result for $\mu < 0$. We then extend this analysis to any flux sector by solving the relevant flow equation.

Specifically, we impose boundary conditions corresponding to two distinct regimes: the quantum undeformed theory and the semiclassical limit of the deformed theory. The full partition function is obtained as a sum over all magnetic fluxes. For any $\mu > 0$, only a finite portion of the quantum spectrum survives and the partition function reduces to a sum over a finite set of representations. For $\mu < 0$, nonperturbative contributions regularize the partition function through an intriguing mechanism that generates nontrivial subtractions.
Finally, we continue the analysis by studying the large-N dynamics of $T\bar{T}$-deformed 2d Yang–Mills theory at genus zero. The $1/N$-expansion of the free energy is obtained by exploiting the associated flow equation and the complete phase diagram of the theory is derived for both signs of the rescaled deformation parameter τ . We observe a third-order phase transition driven by instanton condensation, which is the deformed version of the familiar Douglas–Kazakov transition separating the weakly-coupled from the strongly-coupled phase. By studying these phases, we compute the deformation of both the perturbative sector and the Gross–Taylor string expansion. Nonperturbative corrections in τ drive the system into an unexplored disordered phase separated by a novel critical line meeting tangentially the Douglas–Kazakov one at a tricritical point. The associated phase transition is induced by the collision of large-N saddle points, determining its second-order character.

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- L. Griguolo , R. Panerai, J. Papalini, D. Seminara, *Exact $T\bar{T}$ Deformation of Two-Dimensional Maxwell Theory*, Phys.Rev.Lett. 128 (2022) 22, 221601. e-Print: 2203.09683 [hep-th]

- L. Griguolo , R. Panerai, J. Papalini, D. Seminara, *Exact $T\bar{T}$-deformation of two-dimensional Yang-Mills theory on the sphere*, JHEP 10 (2022) 134. e-Print: 2207.05095 [hep-th]

- L. Griguolo , R. Panerai, J. Papalini, D. Seminara, *The phase diagram of $T\bar{T}$-deformed Yang-Mills theory on the sphere*, JHEP 11 (2022) 078. e-Print: 2209.06222 [hep-th]

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Contents

Part I

Introduction

Motivations 1

In this chapter, we provide a general introduction to the content of the dissertation. In 1.1 we give some general background about the area of research that we will be investigating, while in 1.2 we point out the specific goals of this dissertation, summarizing the original results and contributions that this work has produced.

1.1 General background

One of the main issues of our current understanding of the universe is the lack of a complete theory of quantum gravity. If we try to interpret General Relativity as a quantum field theory, this turns out to be non-renormalizable. In other words, General Relativity (GR) can only be regarded as an Effective Field Theory that can describe phenomena occuring at an energy scale below a certain cutoff, where a complete theory of gravity is expected to emerge. Improving our knowledge of the quantum behavior of gravitational theories is one of the most fascinating challenges in contemporary theoretical physics. In particular, deep questions arise in the problem of black hole formation and subsequent evaporation, starting with Hawking's work in the '70s [71, 72]. A promising setup for addressing these long-standing puzzles about quantum gravity is the possibility of reducing the number of physical dimensions of spacetime, allowing gravitational models to make sense at the level of the Euclidean path integral. Due to the absence of local propagating degrees of freedom, gravity theories in lower dimensions are simpler to study but nonetheless key aspects about their higher dimensional avatars can be extrapolated from them, because they still retain quite non-trivial dynamics.

A specific model that has gained a lot of interest in the last few years is Jackiw-Teitelboim (JT) gravity [76] [125]. Interestingly, this particular type of dilaton gravity theory captures the dynamics close to the horizon of near-extremal black holes in higher dimensions and describes the low-energy dynamics of the SYK model [114]. However, it has been only recently, sparked by developments in 2015 by A. Kitaev, that we have reached a far deeper understanding of the quantum mechanical aspects of this model. For instance, a concrete progress on the black hole information paradox [111, 2, 106] has been recently achieved by correctly interpreting how different topological solutions contribute to the gravitational path integral at late time [113, 111]: these results heavily rely on explicit quantum computations and to the special degree of solvability of JT gravity. The theory is often studied with a negative cosmological constant; in fact, by working within the framework of holography, then we have a preferred anchoring point, and are guided by the major breakthroughs in this field throughout the years, the AdS/CFT correspondence [87]. Furthermore, JT gravity is generally considered in the infinite cutoff regime, where the spacetime boundary is pushed at asymptotic distance: a one dimensional quantum mechanics, the Schwarzian theory, arises on the AdS_2 boundary in this limit and allows to obtain the partition function of the dual gravitional theory through localization [119]. The possibility of computing the gravitational path integral and related observables exactly is one the main reasons that render this theory so unique within the landscape of low-dimensional gravity models. Moreover, in [113] the general construction was extended to arbitrary topologies, incorporating to the gravitational path integral a sum over spacetime wormholes that explores the heavy quantum regime even further. The genus expansion of JT gravity, also known as third quantization, was then reinterpreted in terms of a certain double-scaled matrix integral: this feature has identified a novel form of holographic duality where a single quantum-gravity model is dual to an averaged ensemble of boundary Hamiltonians. We point out that these higher genus contributions are proportional to $e^{-\frac{1}{G_N}}$, so they are completely nonperturbative in the gravitational coupling constant: their inclusion is indeed crucial in order to make JT gravity actually describe a sensible finite-entropy black hole Hilbert space. As we mentioned before, the same type of non-trivial geometrical saddles of the gravitational path integral, namely these nonperturbative spacetime wormholes solutions, is responsible for correctly reproducing the Page curve for an evaporating black hole [2].

A fundamental framework to study JT gravity is its reformulation in terms of a gauge theory: in the first order formalism, the vielbein and spin connection of the spacetime manifold can be packaged into a $\mathrm{SL}(2,\mathbb{R})$ gauge connection [12], while the JT action is rephrased in terms of a BF theory with quadratic potential defined on a one-dimensional defect [74]. A natural observable to study in this approach is a boundary anchored Wilson line that, from the boundary perspective, is understood as the insertion of a bilocal correlator in the Schwarzian theory [11]. From a gravitational point of view, the expectation value of a Wilson line can be interpreted as the coupling of a probe particle to a quantum fluctuating spacetime geometry [74]. Besides, a remarkable feature displayed by JT gravity in the first order formalism is its close resemblance with 2d Yang-Mills theory, which takes the equivalent form of a BF-theory action with quadratic potential supported on the whole surface [130]. Interestingly, pure Yang-Mills theory in two spacetime dimensions is one of only a few examples of exactly solvable interacting gauge theories, another being the closely related Chern-Simons theory in three dimensions [128] [1]. Solvability can be traced back to any number of properties: the lack of propagating degrees of freedom; the existence of a subdivision invariant lattice model [96]; invariance under area preserving diffeomorphisms [129]; a type of hidden supersymmetry [130]; the existence of a gauge which allows one to explicitly evaluate the path integral [9]; and probably others. In [130], Witten related the partition function of 2d Yang-Mills on a closed Euclidean manifold, in the topological limit of vanishing coupling constant, to the symplectic volume of the moduli space of flat connections for the gauge group. At finite coupling however, 2d Yang-Mills displays in addition a number of interesting mathematical phenomena whose detailed study may be relevant for more physical models, providing a valuable theoretical laboratory to test properties likely to occur in their realistic four-dimensional counterparts. These phenomena include an asymptotic, but not generally convergent, formal perturbation series, the emerging over the perturbative vacuum of a fully nonpertubative sector of the theory. The latter determines the presence of contributions to both the partition function and the expectation value of other observables from generically unstable instantons.

Given a model with an exact quantum solution, as it happens for JT gravity and 2d YM, it is natural to look for deformations preserving at least part of its solvable character. Usu-

[1] Maybe not so surprisingly, three-dimensional gravity is dual to a double Chern-Simons theory in the Einstein local frame description [131].

ally, one is interested in relevant deformations, which keep the theory well-defined in the UV, disregarding irrelevant ones that should, instead, uncontrollably change it. A notable counterexample is provided by the so-called $T\bar{T}$ deformation [118, 19]. This is a deformation of local relativistic quantum field theories in two dimensions induced by a specific irrelevant local operator, quadratic in the stress-energy tensor. This operator is unambiguously defined in the presence of translational invariance since its point-splitted version has a regular pinching limit, up to total derivatives [135]. The deformation generates a one-parameter family of quantum field theories with strongly-coupled dynamics at high energies, and despite being in general expected to destroy short-distance locality, it exhibits remarkable properties. It preserves many of the symmetries of the original theory, and it is amenable to exact computations. For instance, the finite-volume spectrum of the deformed theory is described by a differential equation of Burgers type [118, 19]. The surprising amount of control that the deformation allows seems to provide a consistent way to move against the renormalization-group flow and explore unconventional fixed points in the ultraviolet. Being triggered by the stress-energy tensor, the deformation appears to be rooted in geometry. In fact, it can equivalently be formulated as a coupling to topological gravity [37, 36, 126] or random background metrics [17]. Much of the literature on the $T\bar{T}$ deformation deals with its application to conformal field theories, where the geometric dependence of the undeformed spectrum is fixed by conformal invariance. In this context, the action of the deformation has been observed leading to radically different regimes according to the sign of the irrelevant coupling μ. For a positive sign, the density of states of a deformed conformal field theory interpolates between the typical Cardy growth and a Hagedorn-like growth [46] signaling nonlocal features of the deformed field theory in the UV, reminiscent of a stringy behavior. For a negative sign, the spectrum seemingly undergoes a partial complexification [1], putting into question the consistency of the theory at finite volume. The presence of nonperturbative effects in the deformation parameter has been advocated [1] to cure this pathological behavior. $T\bar{T}$-deformed conformal field theories with negative μ have also been suggested leading to an extension of the holographic dictionary [89, 46, 80, 20], potentially describing quantum gravity confined in a portion of the AdS_3 bulk of radius $r_c \propto 1/\sqrt{-\mu}$. In particular, imposing suitable boundary conditions on the metric at some finite spatial extent in the bulk should be equivalent to performing a $T\bar{T}$ deformation of the dual boundary CFT [89]. The same idea was then applied to JT gravity by dimensional reduction, showing that JT at finite cutoff is described by a deformation of the Schwarzian action, which is the analogue of the $T\bar{T}$ deformation [63]. JT gravity in this context thus offers the fascinating possibility of probing the behavior of gravity confined in a box, testing finite-size effects of gravitational theories beyond the perturbative regime [73] [2].

1.2 Specific goals and results

The goal of this dissertation, as we shall describe below, is the investigation of nonperturbative effects and phenomena occurring in low-dimensional gravity and gauge theories, both in the presence and absence of integrable deformations. As a result, we find that the study of these nonperturbative aspects is indeed essential to render physical quantities well-definite, allowing for a better understanding of the quantum regime of the relevant theories. To be more concrete,

[2] both in the gravitational coupling constant and in the $T\bar{T}$ parameter.

we will mainly focus on JT gravity, 2d Yang-Mills theory and investigate the flow of these models under the $T\bar{T}$ deformation, exposing the results published in [59, 56, 54, 55, 57].

Since the disk partition function for JT gravity is one-loop exact [119], the perturbative series in the gravitational coupling constant is trivial, because all higher order terms (two-loop, etc.) vanish identically. In order to have a non-trivial perturbative series, we are thus encouraged to consider some observable on the disk topology: the most interesting candidate is a boundary anchored Wilson line, which introduces an interaction between probe matter and gravitational excitations on AdS_2. Since computing gravitational corrections to this observable coming from boundary gravitons in the Schwarzian theory is in general a difficult task, in [59] we started from the exact expression for the Wilson line expectation value [74] [95] on the disk topology and we developed a systematic way to extract from it the full perturbative series expansion, for general value of the inverse temperature β and conformal dimension λ, thus generalizing some of the results in [90]. The perturbative expansion of course misses any non-analytic term in the gravitational coupling constant, which could compete with the contribution coming from higher genus topologies with one holographic boundary. In the limit of zero temperature, we have been able to derive the asymptotic behavior of the general coefficient in the expansion: as an outcome of our analysis, we find the series is asymptotic but nonetheless Borel resummable; however, clearly more studies are needed on this point, to possibly detect and interpret any nonperturbative completion of the series. In this direction, we also computed the same observable on the trumpet configuration [3], which could be used as a building block for directly constructing higher genus topologies contributing to the bilocal correlator [90]. However, it was observed in [92] that, to actually implement periodicity of free matter around the boundary circle, one should include a sum over integers in the Wilson line definition, corresponding to the all possible windings of the line around the neck of the trumpet. Computing the correlators with the improved operator is however a very challenging task, since it implies self-intersections of the line and a consequent a much more complex expression for them; we hope to investigate further on this improved version in the future. As a final comment, in our derivation we also described the analytical structure of the exact Wilson line expression, highlighting the possibility to express it as a particular combinations of Mordell integrals.

We have previously argued that JT gravity at finite cutoff is dual to a $T\bar{T}$-deformed Schwarzian theory on the boundary. However, for large enough energies the perturbative branch of the spectrum of the Schwarzian theory complexifies, implying a potential breakdown of unitarity for the bulk theory at finite cutoff. Moved by the intuition that these ambiguities could be somehow related to the emergence of the nonperturbative branch of the spectrum [4], in [56] we considered the a piori ill-defined expression defining the partition function of JT gravity at finite cutoff as a formal one, encoding a perturbative expansion around the infinite cutoff case. After a careful analysis of the resulting asymptotic series, we then applied the powerful tool of resurgence theory to fix the nonperturbative contributions in the cutoff parameter. This procedure gives a solid prescription to determine the partition function for the classical disk topology; as a bonus, the precise nonperturbative completion provided by resurgence introduces a certain truncation of the spectrum that exactly removes the puzzling complex part mentioned above. The correct inclusion of the relevant nonperturbative term was even more important for the construction of higher genus topologies; in fact, for the trumpet configuration, it completely smooths out the naïve singularity associated with the fact that the cutoff

[3]which corresponds to a disk with an addictional geodesic boundary of length b.

[4]That was initially discarded, not having a well defined undeformed limit.

boundary could overlap with the geodesic boundary. In this way, we could safely use the gluing prescriptions given by [113] to compute the deformed partition functions of arbitrary topologies. Crucially, as a check of the consistency of the construction, the latter satisfy some recursive relations hinting at some matrix-integral origin, representing a one-parameter refinement of the undeformed ones derived by Mirzakhani [99]. We find it noteworthy that the nonperturbative geometrical configurations—the topologies that contribute to the genus expansion—can only be implemented appropriately after the nonperturbative terms in the deformation parameter have been fixed correctly. The latter, at least at a classical level, enjoy a nice physical interpretation as action terms being sensitive to the geometry lying between the finite cutoff boundary and the asymptotic AdS_2 boundary [73], i.e. the portion of space integrated out by the $T\bar{T}$ flow.

Interestingly, $T\bar{T}$-deformed Yang-Mills theory on the sphere displays similar troubling features as the ones encountered in JT gravity. In fact, if one applies the $T\bar{T}$ deformation naively on the spectrum, the partition function of $U(N)$ Yang-Mills on the sphere, written as a sum over irreducible representations of the gauge group, badly diverges. In [54] and [55] we managed to clarify this puzzling aspect by studying the theory in its dual description, i.e. at the level of GNO quantized magnetic fluxes. Specifically, by solving the relevant flow equation for each individual flux sectors and finally summing exactly the "instanton" series, we obtained a well-defined expression for the partition function at arbitrary deformation parameter μ and gauge group $U(N)$, expressed as a deformed version of the Migdal formula. Concretely, we were able to follow the trajectory of each flux sector independently along the flow, by imposing boundary conditions corresponding to two distinct regimes: the full quantum undeformed theory and the semiclassical limit of the deformed theory. In [54] we focused on $N = 1$, i.e. Maxwell theory, where the result holds for any genus g and the computational steps are less complicated to show than in the general $U(N)$ case. For $\mu > 0$, the quantum spectrum of the theory experiences a truncation, the partition function reducing to a sum over a finite set of positive-energy states. As a further outcome, the theory is observed to undergo infinite-order phase transitions for certain values of μ, associated with the vanishing of Polyakov-loop correlators, which act as order parameters for these transitions [5]. For $U(N)$ the same type of truncation is observed for $\mu > 0$, with only the representations satisfying a certain bound depending on μ are left to contribute to the final sum [55]. For $\mu < 0$ instead, the appearance of a nonperturbative contribution in μ drastically modifies the structure of the partition function, regularizing its naive divergences through an instanton-like subtraction. In the general $U(N)$ case, the presence of the Vandermonde determinant renders the structure of the regularizing subtractions more involved, with the number of instanton-like terms growing with N^2 [55].

If the emergence of peculiar nonperturbative terms in the deformation parameter was essential in order to make the partition function well-definite, their role becomes still more important in the large N limit of $T\bar{T}$-deformed Yang-Mills theory on the sphere, determining new unexpected phenomena. Two-dimensional Yang-Mills at large N is a very rich theory already in the undeformed case. In this regime the theory enjoys an exact description as a string theory: this can be shown for instance at the level of the partition function, which can be determined in terms of a sum over maps from a two dimensional worldsheet onto the two dimensional target space [66]. Furthermore, the spherical topology is peculiar in that it exhibits a large-N phase transition in the spacetime area [35], going from a strongly-coupled phase for large area, where the stringy description keeps holding, to a weakly-coupled phase for small area where the large N expansion of the free energy truncates. The discontinuity of the free energy is of third-order

[5] They acquire a non-vanishing vev below the transition.

and this phenomenon is commonly known as Douglas-Kazakov (DK) phase-transition: its physical origin can be understood from the weak-coupling side in terms of instanton condensation [64]. In other words, if below the critical value only the vacuum perturbative sector of the theory survives, beyond the transition the nonperturbative sector of the theory, accounting for unstable instanton configurations, is no longer absent, as the entropy of instantons at large N overwhelms their classical exponential suppression.

In [57] we considered the flow equation for the leading N^2 order of the free energy F, which in this regime reduces to a inviscid Burgers equation solvable through the method of characteristics. This procedure is generalizable and allows to possibly determine in a recursive way all subleading terms in the $1/N^2$ expansion of the free energy. In this setup, we were thus able to carry the undeformed solution along the characteristics and fully characterize the phase diagram of the theory, depending this time on two variables, the oringinal area α of the Riemann surface and the deformation parameter μ. First of all, we could observe the emergence in the diagram of a deformed version of the familiar Douglas-Kazakov phase-transition, separating a weak-coupling phase with non-trivial $1/N^2$ expansion [6] from a strong-coupling phase, described by a consistent deformation of Gross-Taylor string theory. Exploiting the results of [55], we further confirmed that the deformed Douglas-Kazakov phase-transition is driven by instanton condensation also in the $T\bar{T}$-deformed case. Secondly, we noticed that the system of characteristics for $\mu < 0$ has an envelope, i.e. characteristics cumulate around a particular line in the phase diagram; we demonstrated this phenomena is associated with a second order phase-transition where the specific heat diverges with critical exponent 1/2. Moreover, thanks to the finite N results of [55], we could infer that the famous instanton-like corrections in μ are the ones responsible for this new phase transition, crucially driving the system, through a saddle point collision mechanism, into a phase when they are no longer suppressed at large N.

This analysis revealed the existence in the diagram of a tricritical point, where three different phases come in contact with each other: the deformed weak-coupling phase where both instantons in α and μ are absent, the deformed strong-coupling phase where instantons in α are no-longer suppressed but those in μ still are and finally a new phase, which we called "mixed phase", where both instantons in α and μ are active.

Organization of the work. This dissertation is organized as follows. In 2 we introduce in detail the background material that is needed in order to present the original outcomes of this work, focusing in particular on JT gravity, 2d Yang-Mills theory and the $T\bar{T}$ deformation. In 3 we present the perturbative expansion of a boundary-anchored Wilson line in JT gravity, the content of [59]. Chapter 4 is devoted to the study of JT gravity at finite cutoff, the content of [56]. The second part of the work deals with the $T\bar{T}$-deformation of Yang-Mills. In 5 we derive the exact partition function for the theory on the sphere, with $U(N)$ gauge group, presenting the results of [54] and [55]. In 6 we study the large N limit of the deformed theory, the object of [57]. In 7 we draw the final conclusions, outlining some possibile outlooks and related directions of research. A set of final appendices complete the dissertation A.

[6] In the undeformed case, the expansion is trivial in the weak phase.

Review 2

This chapter is devoted to a review of the basic background material useful for this work.

In 2.1 we introduce JT gravity. We fully solve its classical equations of motion 2.1.1, crucially incorporating boundary conditions at the holographic boundary that allow us to map the dynamics to its boundary Schwarzian description. Section 2.1.2 proceeds with the exact quantization of the model, by computing the Euclidean gravitational path integral in the Schwarzian language. In 2.1.3 we quantize the theory in the presence of matter. In particular, we introduce a scalar field in the bulk and compute its boundary to boundary propagator, corresponding to a bilocal operator insertion in the Schwarzian theory: we also consider loop corrections arising from the interaction with boundary gravitons. In 2.1.4 we consider nonperturbative contributions, in the gravity coupling constant, to the gravitational path integral, including non-trivial topological corrections (or wormholes) to the quantum amplitudes and introducing to so called third quantization in JT gravity. In 2.1.5 we reinterpret the genus expansion in JT gravity as a large N expansion of a dual matrix model, in a suitable double scaling limit, introducing the loop equations, or topological recursion equations, enjoyed by general correlators of resolvents in the random Hamiltonian ensemble.

In 2.2 we give a brief summary of the main properties of gauge theories in two dimensions. In 2.2.1 we describe Yang-Mills theory, focusing on the Hamiltonian formalism and deriving the Migdal heat-kernel representation of the exactly solvable partition function for basic topologies. In 2.2.2 we provide a brief digression on the dual representation for Yang-Mills, i.e. the instanton expansion, which expresses the partition function as an infinite sum over unstable flux sectors. In 2.2.3 we specify the Migdal representation to $U(N)$, which is the gauge group studied in this work. In 2.2.4 we specify further to the sphere topology, studying the Douglas–Kazakov phase transition from weak-coupling phase to strong-coupling phase, both at the level of the matrix-model large-N description and at the level of instanton condensation. In 2.2.5 we show that JT gravity in the Einstein local frame can be rephrased as a BF gauge theory, once we further introduce a quadratic boundary potential in the scalar field to reproduce the Schwarzian dynamics. We then apply the rules exposed previously for the Hamiltonian quantization of Yang-Mills to compute the JT gravity partition function on the disk, provided that we choose a specific group theoretical structure, i.e. $\mathrm{SL}^+(2,\mathbb{R})$ representation theory, to obtain the correct density of states. We finally derive in 2.2.6, in the gauge theory language, the exact expression for a boundary anchored Wilson line, further proving that its matrix element between the highest and lowest state of a discrete representation of $\mathrm{SL}^+(2,\mathbb{R})$ labeled by λ is actually equivalent to a bilocal correlator $\langle\mathcal{O}\mathcal{O}\rangle$ in the Schwarzian theory for an operator $\mathcal{O}$ of conformal dimension λ.

In 2.3 we introduce the $T\bar{T}$ deformation. In 2.3.1 we start by pointing out why this type of integrable deformation is remarkable in the spirit of Wilsonian renormalization group flow. In

2.3.2 we define the $T\bar{T}$ operator, showing that its point-splitted version has a regular pinching limit, up to total derivatives. In 2.3.3 and 2.3.4 we apply the $T\bar{T}$ operator to the two theories object of study, i.e. JT gravity and Yang-Mills. Specifically, in 2.3.3 we show that moving a CFT into the bulk with $T\bar{T}$ is actually equivalent to considering 3d gravity in a box, once we evaluate the classical three-dimensional Einstein-Hilbert action on shell. The equivalence, at a classical level, between the finite cutoff version of JT gravity and the $T\bar{T}$ deformation of the Schwarzian follows then directly from dimensional reduction, also allowing to match the $T\bar{T}$-deformed spectrum with the relativistic energy of the black hole in JT gravity, once we impose a radial cutoff on its geometry. In 2.3.4 we instead esplicitely compute the $T\bar{T}$-deformed version of the Yang-Mills Lagrangian, further translating the flow equation for the action into a quantum flow equation for the partition function its self, that we will solve in the 5 and 6.

2.1 JT gravity

Suppose our goal is to find interesting models of gravity in $(1+1)$d. The simplest example we may think of is the Einstein-Hilbert action in two dimensions:

$$S_{\rm EH} = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^2x\sqrt{g}\ R \tag{2.1}$$

Hower in 2d the above is completely topological: in particular, it computes the Euler characteristic of the manifold $\chi = 2-2g$, where g is the genus of the surface. [1]. This means that every metric in the same topological class has the same on-shell value of the action, so there is no sense in extremizing the action and getting interesting equations of motion.

Then one could possibly think of adding matter to (2.1) in a naive way

$$S_{\rm EH} = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^2x\sqrt{g}\ R + S_{\rm matter} \tag{2.2}$$

however the Einstein equation just leads to a vanishing stress-energy tensor $T_{\mu\nu}=0$, implying no energy can consistently be added in this way. This means (2.2) is not useful as a classical toy model to describe black hole formation or evaporation, for instance.

Therefore we need a different coupling with matter, corresponding to the so called dilaton-gravity models whose action take generally this form:

$$S_{\rm dil\ grav} = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^2x\sqrt{g}\ [\Phi R + V(\Phi)] \tag{2.3}$$

where we have introduced a new scalar field, the dilaton Φ, which couples linearly to the Ricci scalar and we have allowed for a generic dilaton potential $V(\Phi)$. If we further specify to a linear dilaton potential $V(\Phi) = -\Lambda\Phi$, we obtain the Jackiw-Teitelboim gravity action

$$S_{\rm JT} = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^2x\sqrt{g}\ \Phi\,(R-\Lambda) \tag{2.4}$$

where Λ is a cosmological constant and Σ is the two-dimensional surface where the theory lives.In particular, we specify to $\Lambda < 0$ [2], because in the following discussion we will be interested in having an asymptotically AdS spacetime to apply holography.

[1]We are considering here a manifold without boundaries, but the same conclusione holds.

[2]We will soon specify $\Lambda = -2$.

2.1.1 Classical solutions

If we define the model (2.4) on a manifold that has a boundary, which is our goal, we have to add a boundary term of this form:

$$S_{\text{bdy}} = \frac{1}{8\pi G}\int_{\partial\Sigma} \mathrm{d}u\ \sqrt{\gamma}\ \Phi|_{\partial\Sigma}\ \kappa \tag{2.5}$$

where u is a one-dimensional time or proper length which parametrizes the boundary $\delta\Sigma$ of the surface and the dilaton is evaluated on the boundary $\Phi|_{\partial\Sigma}$. [3]. This is known as the Gibbons-Hawking boundary term. Therefore the total Jackiw-Teitelboim gravity $S_{\text{bulk}} + S_{\text{bdy}}$ action will read as

$$S_{\text{JT}} = \frac{1}{16\pi G}\int_{\Sigma} \mathrm{d}^2x\sqrt{g}\ \Phi\,(R+\Lambda) + \frac{1}{8\pi G}\int_{\partial\Sigma} \mathrm{d}u\ \sqrt{\gamma}\ \Phi|_{\partial\Sigma}\ \kappa \tag{2.6}$$

This particular choice for the boundary term is needed in order to make the variational principle well defined: in fact we need to add this term in order to get rid of second order derivatives obtained by varying (2.4). In this way we can fix the value of the metric [4] and vary the action with respect to Φ and the metric $g_{\mu\nu}$ to get the following equations of motion

$$\frac{\delta S_{\text{JT}}}{\delta\Phi} = 0 \quad \Longrightarrow R = \Lambda \tag{2.7}$$

$$\frac{\delta S_{\text{JT}}}{\delta g_{\mu\nu}} = 0 \quad \Longrightarrow \nabla_\mu\nabla_\nu\Phi = -\frac{\Lambda}{2}g_{\mu\nu}\Phi \tag{2.8}$$

Let's consider the first equation. In 2d there is only one curvature function, which can be conveniently parametrized as the local Ricci scalar $R(x)$. Therefore having set $R(x) = -2$ means that we know everything about the local geometry. In particular, the geometry is fixed as some patch of AdS_2 manifold, which in Poincaré coordinates can be written as

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 + \mathrm{d}t^2}{z^2} \tag{2.9}$$

where z is the Poincaré radial coordinate and t the Poincaré time coordinate. The holographic boundary is located at $z = 0$. Here we are considering the simplest classical geometry [5], corresponding to the topology of a disk (i.e. a manifold with genus zero and one boundary), as depicted in Figure 2.1.1.

There are several other classical frames to describe the AdS_2 spacetime; we mention the most important ones. If we introduce lightcone coordinates $u = t + iz$ and $v = t - iz$, the metric is rewritten as

$$\mathrm{d}s^2 = -4\frac{\mathrm{d}u\mathrm{d}v}{(u-v)^2} \tag{2.10}$$

This form of the metric is important because it is found in the near-horizon regime of higher dimensional extremal black holes. To see this patch actually represents a black hole geometry,

[3] The one-dimensional metric γ is induced from the two-dimensional metric $g_{\mu\nu}$, while κ is the extrinsic curvature, i.e. the curvature of a space thought as embedded in a higher dimensional one; in this case the one dimensional boundary embedded in Σ.

[4] and not its derivatives too

[5] We will consider higher genus topologies later.

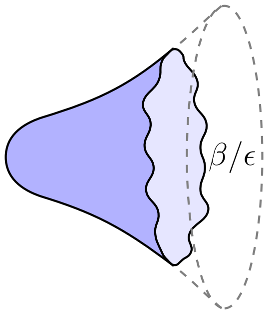


Figure 2.1: The disk topology. The dashed lines represent the full AdS_2 geometry, while the actual manifolds have wiggly boundary of length β/ϵ.

it is convenient to define a new radial coordinate $r = r_h \coth\left(\frac{2\pi}{\beta}(u-v)\right)$ and thus put the metric into the more familiar form:

$$\mathrm{d}s^2 = -\left(r^2 - r_h^2\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 - r_h^2} \tag{2.11}$$

where $r = r_h = \frac{2\pi}{\beta}$ is the black hole horizon. Starting from this solution, we can calculate the total black hole mass in the bulk specetime to be $M \sim r_h^2$. Moreover, by demanding that the Euclidean section of this manifold has no conical deficit, we can deduce the Hawking temperature $T = \beta^{-1} = \frac{r_h}{2\pi}$, justifying the parameter label β as the inverse temperature; this leads to verify the black hole first law $T \propto \sqrt{M}$.

Asymptotic Poincaré boundary conditions We see that the metric (2.9) blows up on the boundary, at $z = 0$. Therefore the induced metric $\gamma_{uu} \equiv \gamma$ on the boundary diverges as $1/z^2$, but if we introduce a cut off at $z = \epsilon$ (i.e. we place the boundary of the disk at $z = \epsilon$) we have

$$\gamma_{uu} = \frac{1}{\epsilon^2} \tag{2.12}$$

In this way the total boundary length will be

$$\oint \mathrm{d}u\, \sqrt{\gamma_{uu}} = \frac{1}{\epsilon} \oint \mathrm{d}u \equiv \frac{\beta}{\epsilon} \tag{2.13}$$

where β plays the role of the renormalized boundary length. As $\epsilon \to 0$ the boundary length diverges and the we recover the full Poincaré disk.
The next task is to find how to parametrize the embedding coordinates $t \equiv t(u)$ and $z \equiv z(u)$ as a function of the boundary time u, as shown in Figure 2.2. That can be done by comparing the line element on the boundary with that of the embedding Poincaré disk: in the limit $\epsilon \to 0$ they should match, i.e.

$$\frac{\mathrm{d}t^2 + \mathrm{d}z^2}{z^2} \approx \frac{\mathrm{d}u^2}{\epsilon^2} \quad \epsilon \to 0 \tag{2.14}$$

We can thus find the relation

$$z \approx \epsilon \sqrt{\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}u}\right)^2} \approx \epsilon \frac{\mathrm{d}t}{\mathrm{d}u} + \mathcal{O}(\epsilon^3) \tag{2.15}$$

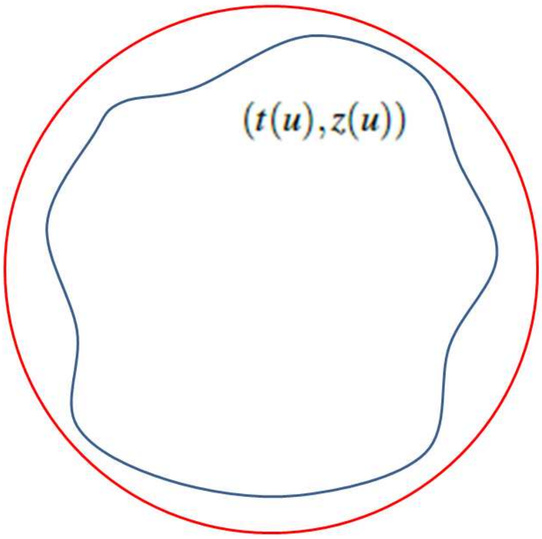


Figure 2.2: A cutout of the Poincaré disk. As $\epsilon \to 0$ the region with boundary labeled by $(t(u), z(u))$ covers the full Poincaré spacetime

because z and so forth $\frac{\mathrm{d}z}{\mathrm{d}u}$ is already of order ϵ and can be ignored. The last result means that the boundary can be parametrized by only one function $t(u)$, which can be as well thought as a field living on the boundary: (2.15) is referred asymptotic Poincaré boundary conditions (this is also known as Fefferman-Graham gauge which is usual pratice in AdS/CFT).

Solution for the dilaton One can prove that the solution of the equation (2.8) (setting $\Lambda = -2$) in Poincaré coordinates is given by

$$\Phi \propto \frac{1}{z} \tag{2.16}$$

Just like the metric, value of the dilaton field diverges at the boundary as $z \to 0$. Therefore introducing as before the cut-off at $z = \epsilon$ we can define the renormalized dilaton field ϕ_r at the boundary as

$$\Phi|_{\partial\Sigma} = \frac{\phi_r}{\epsilon} \tag{2.17}$$

2.1.2 Quantization of the Schwarzian theory

Let's consider the Euclidean gravitational path integral for JT gravity, which is schematically given by

$$Z_{\mathrm{JT}} = \int \mathcal{D}g\mathcal{D}\Phi e^{-S_{\mathrm{JT}}} \tag{2.18}$$

Since the dilaton appears only linearly in the action (2.6), it acts as a Lagrangian multiplier in imposing the constraint $R = \Lambda$. Once we integrate out the dilaton Φ, we are thus lead to evaluate the JT action (2.6) on shell, i.e. on the solutions of the equations of motion. The bulk term vanishes and we are thus left with the boundary action S_{bdy} only. In doing this, we are fixing the bulk geometry to be some interior patch of the Poincaré disk and the gravitational path integral is rephrased as integrating over all possible ways of cutting out some inner portion of the AdS_2 manifold.

Now, we need to compute the extrinsic curvature κ associated with the Poincaré solution (2.9), in addition to the boundary conditions (2.12) and (2.17) for the induced metric and dilaton respectively. The extrinsic curvature can be computed using the formula

$$\kappa = -\frac{h(T, \nabla_T n)}{h(T,T)} \tag{2.19}$$

where T is the tangent vector to the boundary, n is the normal vector(orthogonal to T) and $h(v,w) = g_{\mu\nu} v^\mu w^\nu$ is the scalar product defined with respect to the $2d$ metric in (2.9). This formula can be understood by thinking of the normal vector as parallel transported along the tangent vector through the covariant derivative ∇: the difference between the normal vector n evalueted at two points of the boundary as they get infinitely close is then a measure of the extrinsic curvature, as shown in figure 2.3. We define

$$T^a = \left(t', z'\right) \quad n^a = \frac{z}{\sqrt{(t'^2 + z'^2)}} \left(-z', t'\right) \quad n^a T_a = 0 \tag{2.20}$$

as respectevely the properly normalized tangent and normal vector at the boundary, where $t' = \frac{\mathrm{d}t}{\mathrm{d}u}$ and $z' = \frac{\mathrm{d}z}{\mathrm{d}u}$. After a long but straightforward calculation, the extrinsic curvature yields

$$\kappa = \frac{t'\left(t'^2 + z'^2 + zz''\right) - zz't''}{(t'^2 + z'^2)^{3/2}} \tag{2.21}$$

It can be simplified sobstituting (2.15) and neglecting terms of order $\mathcal{O}(\epsilon^3)$.

$$\kappa = 1 + \{t(u), u\}\epsilon^2 + \mathcal{O}(\epsilon^4) \tag{2.22}$$

where $\{t(u), u\}$ is the Schwarzian derivative defined as

$$\{t(u), u\} = \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2 \tag{2.23}$$

Putting all together, the boundary action evaluated on shell becomes

$$S_{\text{bdy}} = -\frac{1}{8\pi G}\int_{\partial\Sigma} \frac{\mathrm{d}u}{\epsilon}\frac{\phi_r}{\epsilon}\left(1 + \{t(u), u\}\,\epsilon^2 + \ldots\right) \tag{2.24}$$

The $\mathcal{O}(1/\epsilon^2)$ divergent therm is a residue of the UV limit in which we are workig, but as usual practice in AdS/CFT context it can be removed by holographic renormalization, i.e. adding to the boundary theory a counter term in order to eliminate this divergence. Concretely, we consider as our real boundary term the following

$$S_{\text{bdy}} = \frac{1}{8\pi G}\int_{\partial\Sigma} \mathrm{d}u\,\sqrt{\gamma}\,\Phi|_{\partial\Sigma}\,(\kappa + 1) \tag{2.25}$$

In this way we end up with the Schwarzian action as $\epsilon \to 0$

$$S_{\text{Sch}} = -C\int_{\partial\Sigma} \mathrm{d}u\,\{t(u), u\} \tag{2.26}$$

where $C = \frac{\phi_r}{8\pi G}$ and $t(u)$ represents a time reparametrization in terms of the proper time u.

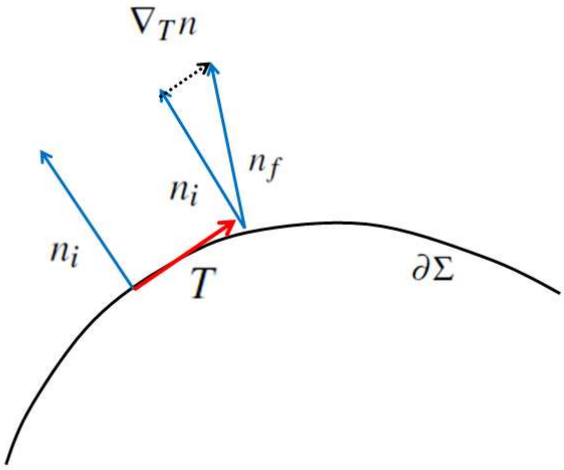


Figure 2.3: The initial normal vector n_i is parallel transported along the tangent vector T to the boundary $\partial\Sigma$. The covariant derivative $\nabla_T n$ consists in comparing it with the final normal vector n_f.

The classical saddle

We now concentrate on the boundary Schwarzian theory, which describes the dynamics of the wiggly boundary curve $t(u)$, the only remaining degree of freedom. The equations of motion are obtained by varying the S_{Sch} with respect to $t(u)$

$$\frac{\delta S_{\text{Sch}}}{\delta t} = \frac{\delta S_{\text{Sch}}}{\delta u}\frac{\mathrm{d}u}{\mathrm{d}t} = 0 \quad \Longrightarrow \frac{\{t(u),u\}'}{t'} = 0 \tag{2.27}$$

This means that the solution to the equation of motion is a non-constant function characterized by a constant Schwarzian [6]. In particular we will restrict to monotone increasing reparametrizations, i.e. $t'(u) > 0$. We know that the Schwarzian derivative vanishes when restricted to the global $\mathrm{SL}(2,\mathbb{R})$ subgroup of the two dimensional conformal group, i.e. under finite transformations of the form

$$t \to \frac{at+b}{ct+d} \qquad ad - bc = 1 \tag{2.28}$$

however these are not true dynamical modes, but instead a redundancy in the description of the cutouts $t(u)$, a gauge symmetry. In fact they represent translations and rotations of a fixed shape around in the hyperbolic disk which do not change the chunk that we are actually cutting out. In order to find different maps with a constant Schwarzian, the true dynamical modes, we

[6] If we compute the Hamiltonian as the Noether charge associated to time translation simmetry, this turns out to be equal to the Lagrangian itself. So the equation of motion is just energy conservation.

perform the following change of variables and evaluate the Schwartzian derivative (2.23) in the new coordinate

$$t(u) = \tan\left(\frac{\tau(u)}{2}\right) \Longrightarrow \{t(u),u\} = \{\tau(u),u\} + \frac{1}{2}\tau'^2 \tag{2.29}$$

If τ is a linear function of u, such as $\tau = au$ with a constant, the action remains invariant in form, up to an addictional constant which does not affect the equation of motion. Since the Poincaré time t must be periodic with period $2\pi/a$, we hence take the constant a to be $\frac{2\pi}{\beta}$, in such a way that the total boundary time length is set to β. For convenience, we can finally set $\beta = 2\pi$. The classical solution is then

$$\tau(u) = u \qquad t(u) = \tan\left(\frac{u}{2}\right) \tag{2.30}$$

We can see from (2.26) that the Schwarzian action S_{Sch} is of order $\mathcal{O}(1/G)$ and so the semi-classical solution (2.30) will be correct only in the limit $G, \hbar \to 0$, because in this regime the saddle point would be a good aproximation to the full path integral. On the other hand, since $C = \frac{\phi_r}{8\pi G}$ has dimension of length, or negative mass dimension, by power counting renormalizability we infer that quantum effects should be important in the IR, that is at long distances or late times. This will be important later when studying the spectral form factor and higher topologies effects.

The Schwarzian path integral is one loop exact

The JT gravity partition function on the disk topology will be given by the full Schwarzian path integral:

$$Z_{\text{JT}} = \int \frac{\mathcal{D}\mu(\tau)}{\text{SL}(2,\mathbb{R})} e^{-S_{\text{Sch}}(\tau)} \tag{2.31}$$

where $\mu(t)$ is some measured to be determined, on the space diff $\left(S^1\right)$ of the 1d change of coordinates on the circle. Moreover τ is supposed to wind once around the circle [7], $\tau(u+2\pi) = \tau(u) + 2\pi$. In terms of τ, the Schwarzian action is rewritten as

$$S_{\text{Sch}}(\tau) = \frac{C}{2}\int_0^{2\pi} \mathrm{d}u \left(\frac{\tau''^2}{\tau'^2} - \tau'^2\right) \tag{2.32}$$

where we used (2.29) and integrated by parts a total derivative. In (2.31), we are explicitly modding out by the SL$(2,\mathbb{R})$ gauge redundancy that we mentioned before.

It turns out that the path integral (2.31) is one-loop exact. This remarkable fact can be explained through localization [119]: in fact the Duistermaat-Heckman (DH) formula shows that the integral over a symplectic manifold of $\exp(H)$ [8], where the Hamiltonian H generates via Poisson brackets a U(1) symmetry of the manifold, is one-loop exact [9].

[7] The boundary curve should not self intersect.

[8] The DH formula can be used whenever the action is said to be Hamiltonian. This holds in our case since the Hamiltonian corresponds to the Schwarzian Lagrangian.

[9] In fact in these cases one can introduce the equivariant extension of the symplectic form $\alpha = \exp(\tilde{\omega}) = \exp(\omega + H)$ which is equivariantly closed and then use the equivariant localization theorem.

First of all, the space over which we are integrating in (2.31), i.e. diff $\left(S^1\right)/\mathrm{SL}(2,\mathbb{R})$ is a symplectic manifold. In fact we can construct on it the following symplectic measure

$$\omega = \int_0^{2\pi} \mathrm{d}u \, \left(\left(\frac{d\tau'}{\tau'} \right) \partial_u \left(\frac{d\tau'}{\tau'} \right) - \mathrm{d}\tau \partial_u \mathrm{d}\tau \right) \tag{2.33}$$

where we omitted the wedge product and we just remind that the abstract exterior derivative $\mathrm{d}\tau$ can be regarded as a fermionic field. The symplectic measure is closed $\mathrm{d}\omega = 0$ and non-degenerate (once we remove its three zero modes corresponding to the action of $\mathrm{SL}(2,\mathbb{R})$ on τ).

Moreover, a key property of ω for localization to work is that it is diff $\left(S^1\right)$ invariant, that is the Lie derivative $\mathcal{L}_{V_\alpha}\omega = 0$ vanishes. Here V_α is the vector field $\delta\tau = \alpha(u)\tau'(u)$ corresponding to an infinistesimal reparametrization on the space of τ's and $\delta u = \alpha(u)$ is an infinitesimal diff $\left(S^1\right)$ transformation. Since ω is closed, the Lie derivative vanishes if $\mathrm{d}\left(i_{V_\alpha}\omega\right) = 0$, which amounts to say

$$i_{V_\alpha}\omega = \mathrm{d}H_\alpha \tag{2.34}$$

in terms of a Hamiltonian function H_α, that via Poisson brackets generates the infinitesimal transformation of τ. In [119] equation (2.34) was proven to hold, starting from the symplectic defined in (2.33) and with Hamiltonian function:

$$H_\alpha = -2\int_0^{2\pi} \mathrm{d}u \; \alpha \left\{ \tan\frac{\tau(u)}{2}, u \right\} \tag{2.35}$$

which, setting $\alpha = 1$, exactly corresponds to the Schwarzian action. This garantees that the Schwarzian path integral is one-loop exact. In A.1 we sketch a proof of the DH formula in more detail. Here we just proceed in the evaluation of the one-loop contribution to the path integral, knowing it will yield the full answer.

Since under diffeomorphisms the differentials $\mathcal{D}\tau$ transform as Grassman variables, one can introduce a periodic Grassmann field ψ to write the Pfaffian of the symplectic form ω as a path integral. We obtain

$$Z_{\mathrm{JT}} = \int \frac{\mathcal{D}\tau\mathcal{D}\psi}{\mathrm{SL}(2,\mathbb{R})} \exp\left\{ -\frac{C}{2}\int_0^{2\pi} \mathrm{d}u \left[\frac{\tau''^2}{\tau'^2} - \tau'^2 - \frac{2}{C}\left(\frac{\psi'\psi''}{\tau'^2} - \psi\psi' \right) \right] \right\} \tag{2.36}$$

By integrating out the fermion ψ, we get the factor $\mathrm{Pf}(\omega)\,\mathcal{D}\tau = \sqrt{\det(\omega)}\mathcal{D}\tau$ yielding the correct measure for the integral over τ.

To perform the one-loop computation, we now consider a small perturbation $\delta(u)$ around the saddle point solution (2.30)

$$\tau(u) = u + \delta(u) \tag{2.37}$$

The ψ field has no classical saddle instead. Dropping total derivatives, the quadratic expansion of the action will be

$$S^{(2)}_{\mathrm{Sch}}(\delta,\psi) = \frac{C}{2}\int_0^{2\pi} \mathrm{d}u \left(\delta''^2 - \delta'^2 \right) + \int_0^{2\pi} \mathrm{d}u \left(\psi''\psi' - \psi'\psi \right) \tag{2.38}$$

We now express $\delta(u)$ and $\psi(u)$ as a discrete Fourier series :

$$\delta(u) = \sum_{n\neq -1,0,1} \delta_n e^{-inu} + \mathrm{h.c.} \qquad \psi(u) = \sum_{n\neq -1,0,1} \psi_n e^{-inu} + \mathrm{h.c.} \tag{2.39}$$

where we have gauge fixed to zero the $n = 0, 1, -1$ modes, which exactly correspond to action of the $SL(2, \mathbb{R})$ generators we are removing from the path integral.

Keeping also track of the leading classical contribution (i.e. the action (2.32) evaluated on the saddle), the quadratic expansion of the action, after exploiting orthogonality of the modes, becomes

$$S_{\text{Sch}}(\delta, \psi) = \pi C + 4\pi C \sum_{n>1} \delta_n \delta_{-n} \left(n^4 - n^2\right) + 4\pi \sum_{n>1} \psi_n \psi_{-n} \left(n^3 - n\right) \tag{2.40}$$

Computing the Gaussian integral in these variables is now very simple. Combining the bosonic and fermionic modes, we get the following one-loop contribution:

$$Z = e^{\pi C} \prod_{n \geq 2} \frac{2\pi}{Cn} = \frac{1}{4\pi^2} C^{\frac{3}{2}} e^{\pi C} \tag{2.41}$$

where we employed ζ function regularization for the infinite product. Restoring the dependence of the inverse temperature β, the full result for the JT gravity partition function on the disk is

$$Z_{\text{disk}}(\beta) = \frac{1}{4\pi^2} \left(\frac{2\pi C}{\beta}\right)^{\frac{3}{2}} e^{\frac{2\pi^2 C}{\beta}} \tag{2.42}$$

We remind that, although we did only a one-loop computation, the result is exact at all orders in perturbation theory, due to the localization argument.

Propagator $\langle \delta(0)\delta(u) \rangle$ With the quadratic action at hand, one can also compute the Schwarzian propagator. To do so we have to invert the Gaussian kernel:

$$\langle \delta(u_1)\delta(u_2) \rangle = \langle \delta(0)\delta(u) \rangle = \frac{2}{C} \sum_{n \neq 0,1,-1} \frac{e^{inu}}{n^2 \left(n^2 - 1\right)} \tag{2.43}$$

where the first equivalence comes from traslation invariance. As proposed in [117], this sum can be evalueted as a countour integral in the complex plane. The details of the computation are fully explained in Appendix A.4, here we present the result

$$\langle \delta(0)\delta(u) \rangle = \frac{2\pi}{C} \left[-\frac{(u-\pi)^2}{2} + (u-\pi)\sin u + 1 + \frac{\pi^2}{6} + \frac{5}{2}\cos u \right] \tag{2.44}$$

The form of the propagator is useful to check esplicitely that the two-loop term, for instance, vanishes. Moreover, it is a key ingredient to compute the gravitational corrections to the connected n-point functions for boundary operators in the Schwarzian theory.

2.1.3 Adding some matter in the bulk

We now perform the calculation with the addition of matter. In particular we consider a massive scalar field χ propagating on the JT black hole background. It couples to gravity in a covariant way

$$S_{\text{matter}} = \frac{1}{2} \int \mathrm{d}^2 x \sqrt{g} \left(g^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi + m^2 \chi^2 \right) \tag{2.45}$$

In Poincaré coordinates the action can be reexpressed as

$$S_{\text{matter}} = \frac{1}{2} \int \mathrm{d}z \mathrm{d}t \frac{1}{z^2} \left[z^2 \left(\partial_z \chi\right)^2 + z^2 \left(\partial_t \chi\right)^2 + m^2 \chi^2 \right] \tag{2.46}$$

The Euler-Lagrange equation for this action is

$$\partial_z(\partial_z\chi)+\partial_t(\partial_t\chi)=\frac{1}{z^2}m^2\chi \tag{2.47}$$

Let's neglect the t therm since we are interested in the behavior of the solution near the boundary $z\to 0$, i.e. the UV region of the potential where the dipendence on t can be ignored. Besides we choose a power law ansatz for the solution $\chi\propto z^{\Delta}$, where the exponent Δ will be soon identified, and we find

$$\Delta(\Delta-1)z^{\Delta-2}=m^2z^{\Delta-2}\Longrightarrow\Delta_{1,2}=\frac{1}{2}\left(1\pm\sqrt{1+4m^2}\right) \tag{2.48}$$

If we denote simply $\Delta_1=\Delta$ (Δ is the biggest root) and $\Delta_2=1-\Delta$ the general solution will be a linear combination

$$\chi(z,t)=z^{1-\Delta}\chi_r^{(1)}(t)+z^{\Delta}\chi_r^{(2)}(t) \tag{2.49}$$

where $\chi_r^{(1)}(t)$ and $\chi_r^{(2)}(t)$ are two coefficients depending only on t. Near the boundary $z\to 0$ the solution with the lowest exponent will be predominant so we have

$$\chi(z,t)\approx z^{1-\Delta}\chi_r(t)\quad z\to 0 \tag{2.50}$$

χ_r as a primary This form suggests that χ_r behaves as a primary with dimension $1-\Delta$. To see this, we rewrite the boundary condition (2.50) in terms of the boundary time u using (2.15)

$$\chi(z,t)\approx\epsilon^{1-\Delta}\left(t'(u)\right)^{1-\Delta}\chi_r(t(u))=\epsilon^{1-\Delta}\chi_r(u)\qquad\chi_r(u)=\left(t'(u)\right)^{1-\Delta}\chi_r(t(u)) \tag{2.51}$$

where the last equation defines $\chi_r(u)$, which we see transforms as a conformal primary of dimension $1-\Delta$ under reparametrizations. Furthermore, according to the AdS/CFT dictionary, we should interpret fields in the bulk as sources for correlation functions of local operators, i.e $\chi_r(t)$ acts as a source for an operator $O(t)$ on the boundary through the standard coupling

$$\int \mathrm{d}t\ O(t)\chi_r(t) \tag{2.52}$$

Up to now, Δ is just a parameter in terms of m but one can easily check it plays the role of the conformal dimension of the dual operator O. [10]

Evaluating the action on shell Now the usual AdS/CFT procedure tells us to evaluate S_{matter} (2.46) on shell, i.e integrating by parts

$$S_{\text{matter}}=\int_{\text{bdy}}\chi\partial_\alpha\chi+\int \mathrm{d}^2x\sqrt{g}\chi\left(-\nabla^2+m^2\right)\chi \tag{2.54}$$

[10] To verify this we have to impose that (2.52) term is invariant to conformal transformations, in particular under riscalings $t\to\lambda t$:

$$\int \mathrm{d}t\ O(t)\chi_r(t)\Longrightarrow\int \mathrm{d}t\lambda\lambda^{\alpha}\lambda^{1-\Delta}\ O(t)\chi_r(t) \tag{2.53}$$

where the first λ comes from the measure, λ^{α} comes from the transformation $O\to\lambda^{\alpha}O$ (where α is to determine) and $\lambda^{1-\Delta}$ from the transformation of χ_r we derived in (2.51). In order (2.53) to be invariant, the scaling dimension α of O has to be $\alpha\equiv\Delta$.

where the second term is zero on the equations of motion. We are left with

$$S_{\text{matter on shell}} \equiv S_{\text{m}} \sim \int_{z=\epsilon} \mathrm{d}t \; \chi \partial_z \chi \tag{2.55}$$

We find that after renormalization (2.55) is rewritten as a function of the source χ_r [11]

$$S_{\text{m}} = D \int \mathrm{d}t \; \mathrm{d}t' \frac{\chi_r(t)\chi_r(t')}{|t-t'|^{2\Delta}} \tag{2.56}$$

up to some proportionality constant D. Let us finally rewrite (2.56) as a function of the time u that parametrizes the boundary curve $t(u)$.

$$S_{\text{m}} = D \int \mathrm{d}u \; \mathrm{d}u' t'(u) t'(u') t'(u)^{\Delta-1} t'(u')^{\Delta-1} \frac{\chi_r(u)\chi_r(u')}{|t(u)-t(u')|^{2\Delta}} \tag{2.57}$$

where the first two derivatives come from the integration measure and the last two from the tranformation (2.51) of χ_r. Recollecting terms we find

$$S_{\text{m}} = D \int \mathrm{d}u \; \mathrm{d}u' \left[\frac{t'(u)t'(u')}{(t(u)-t(u'))^2} \right]^{\Delta} \chi_r(u)\chi_r(u') \tag{2.58}$$

Gravitational corrections to 2 point function $\langle O_\Delta(u_1, u_2) \rangle$

The aim is now to compute perturbately the boundary partition function in presence of a source χ_r [117]

$$Z[\chi_r] = \left\langle e^{-S_{\text{Sch}} - S_{\text{m}}} \right\rangle \tag{2.59}$$

where $\langle\rangle$ indicates the path integral over the Schwarzian theory.

Saddle point approximation First of all, one can try to perform a saddle point approximation of the path integral (2.59) (in the limit $G \to 0$ saddle point becomes exact), i.e evaluate the exponent on the field configuration that minimize it (on the solutions)

$$Z[\chi_r]|_{\text{saddle}} = e^{-S_{\text{Sch}} - S_{\text{m}}|_{t=t_0}} \qquad \left. \frac{\delta(S_{\text{Sch}} + S_{\text{m}})}{\delta t(u)} \right|_{t=t_0} = 0 \tag{2.60}$$

Finding the solution for the boundary metric t coupled to a matter field χ is not a simple task. However, we note that if Δ is not too large(i.e. from (2.48) the bulk scalar is not too heavy), the bulk scalar couples weakly to the backgroud and won't affect the geometry of space time so much with its presence, since the Schwarzian action has a big prefactor $1/G$ with $G \to 0$. This means that we can basically neglect S_{m} in the computation of the saddle $t(u)$ for the Schwarzian theory, or simply arguing that in this regime gravitational and matter degrees of freedom decouple from each other, i.e t does not depend on χ_r.

[11] After removing all divergent terms as $\epsilon \to 0$, the only surviving term is $\int \mathrm{d}t \; \chi_r^{(1)}(t)\chi_r^{(2)}(t)$. Moreover, by carefully studying the solution of (2.47) one finds $\chi_r^{(2)}(t)$ is determined in terms of $\chi_r^{(1)}(t)$ as an integral over a kernel: $\chi_r^{(2)}(t) = \int \mathrm{d}t' \frac{\chi_r^{(1)}(t')}{|t-t'|^{2\Delta}}$

Through this approximation, one can treat the dual field O to the source χ_r as effectively free, as all of its connected correlators vanish except for the two point function which is

$$\frac{\delta^2\left(-\log Z[\chi_r]|_{\text{saddle}}\right)}{\delta\chi_r(u_1)\delta\chi_r(u_2)} = \frac{\delta^2 S_m}{\delta\chi_r(u_1)\delta\chi_r(u_2)} = \langle O_\Delta(u_1,u_2)\rangle = \langle O(u_1)O(u_2)\rangle \propto \left[\frac{t'(u_1)t'(u_2)}{\left(t(u_1)-t(u_2)\right)^2}\right]^\Delta \tag{2.61}$$

obtained through a double functional derivative of the free energy $-\log Z[\chi_r]$, the connected partition function. Since we are assuming the saddle for t is not affected by the presence of χ, we sobstitute the solution (2.30) in (2.61), obtaining

$$\langle O_\Delta(u_1,u_2)\rangle^{\text{class}} = \left[\frac{\frac{\mathrm{d}}{\mathrm{d}u}\tan\left(\frac{u}{2}\right)\Big|_{u=u_1}\frac{\mathrm{d}}{\mathrm{d}u}\tan\left(\frac{u}{2}\right)\Big|_{u=u_2}}{\left(\tan\left(\frac{u_1}{2}\right)-\tan\left(\frac{u_2}{2}\right)\right)^2}\right]^\Delta = \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}} \qquad u_{12}=u_2-u_1 \tag{2.62}$$

where we put the apex "class" to remind that this is computed on the classical saddle. [12]

Beyond the saddle point So far we have worked in the classical limit $G\to 0$ where the saddle point is exact, but actually at G finite we should integrate over all possible field configurations $t(u)$. As shown above, a convenient way to perform this path integral is to consider the one-loop term coming from the expansion around the classical saddle. As we have seen in (2.38), the quadratic Schwarzian action is

$$S^{(2)}_{\text{Sch}}(\delta) = \frac{C}{2}\int \mathrm{d}u\left(\delta''^2-\delta'^2\right) \tag{2.64}$$

where δ represents small quantum fluctuations. This reformulation allows us to work again within the approximation of vanishing backreaction of χ_r on t in the determination of the saddle but enables us to take into account the gravitational loop corrections from $t(u)$. For this purpose, we will need to make us of the expansion

$$O_\Delta(u_1,u_2) = \left[\frac{t'(u_1)t'(u_2)}{\left(t(u_1)-t(u_2)\right)^2}\right]^\Delta = \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\left[1+B(\delta)+C(\delta^2)+\mathcal{O}(\delta^3)\right] \tag{2.65}$$

[12] In fact in the limit $u_{12}\to 0$ we recover the correct behavior

$$\left.\frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\right|_{u_{12}\to 0} \approx \frac{1}{u_{12}^{2\Delta}} \tag{2.63}$$

. As a final comment, if the approximation we adopted didn't hold, i.e the saddle of t depended in some way on χ_r, then we see immediately from (2.58) that S_{m} would not be only quadratic in the source χ_r. Therefore there would be other connected correlators(obtained through higher order functional derivatives $\frac{\delta^n S_{\text{m}}}{\delta\chi_r(u_1)\ldots\delta\chi_r(u_n)}$), not only the two-point function.

obtained by plugging (2.37) into the bilocal correlator (2.61). $B(\delta)$ and $C(\delta^2)$ are respectively the $\mathcal{O}(\delta)$ and $\mathcal{O}(\delta^2)$ terms of the expansion, given by

$$\begin{aligned} B(\delta,u_1,u_2) =&\Delta\left(\delta'(u_1)+\delta'(u_2)-\frac{\delta(u_1)-\delta(u_2)}{\tan\frac{u_{12}}{2}}\right)\\ C(\delta^2,u_1,u_2) =&\frac{\Delta}{\left(2\sin\frac{u_{12}}{2}\right)^2}\Big[(1+\Delta+\Delta\cos u_{12})\,(\delta(u_1)-\delta(u_2))^2\\ &+2\Delta\sin u_{12}\,(\delta(u_2)-\delta(u_1))\left(\delta'(u_1)-\delta'(u_2)\right)\\ &-(\cos u_{12}-1)\left((\Delta-1)(\delta'(u_1)^2+\delta'(u_2)^2)+2\Delta\delta'(u_1)\delta'(u_2)\right)\Big] \end{aligned} \tag{2.66}$$

We now expand in series the exponential e^{-S_m}, plugging in also the expansion (2.65) for O_Δ

$$\begin{aligned} e^{-S_\mathrm{m}} &= 1-D\int \mathrm{d}u_1\mathrm{d}u_2\left[1+B(\delta,u_1,u_2)+C(\delta^2,u_1,u_2)\right]\frac{\chi_r(u_1)\chi_r(u_2)}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\\ &+\frac{D^2}{2}\int \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4\frac{\chi_r(u_1)\chi_r(u_2)\chi_r(u_3)\chi_r(u_4)}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}\left(2\sin\frac{u_{34}}{2}\right)^{2\Delta}}\left[(1+B(\delta,u_1,u_2)+...)(1+B(\delta,u_3,u_4)+...)\right] \end{aligned} \tag{2.67}$$

Recalling that schematically

$$\int \mathcal{D}\delta\ \delta(u_1)\delta(u_2)\ e^{-\frac{C}{2}\int \mathrm{d}u(\delta''^2-\delta'^2)} = \langle\delta(u_1)\delta(u_2)\rangle \tag{2.68}$$

we compute the path integral over δ by simply taking the expectation value for every quadratic appearence in δ. The expectation value of the one point function $\langle\delta\rangle$ vanishes, then $\langle B(\delta)\rangle = 0$. We are interested in the two point function for the operator O that will be given by a double functional derivative of (2.67), setting the sources to zero at the end

$$\langle O_\Delta(u_1,u_2)\rangle = \left.\frac{\delta^2 Z[\chi_r]}{\delta\chi_r(u_1)\delta\chi_r(u_2)}\right|_{\chi=0} = \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\left[1+\left\langle C(\delta^2,u_1,u_2)\right\rangle+...\right] \tag{2.69}$$

where dots stand for higher order corrections. Since the correlation function $G(|u_{12}|) = \langle\delta(u_1)\delta(u_2)\rangle$ depends on the module of the boundary time, we have

$$\left\langle\delta'(u_1)\delta(u_2)\right\rangle = \mathrm{sign}(u_{12})G'(|u_{12}|) \quad \left\langle\delta'(u_1)\delta'(u_2)\right\rangle = -G''(|u_{12}|) \tag{2.70}$$

In this way we can compute $\left\langle C(\delta^2,u_1,u_2)\right\rangle$ sobstituting in (2.66) the expression (2.44) for the propagator $\langle\delta(u_1)\delta(u_2)\rangle$ we computed above, or its derivatives. The final result is

$$\begin{aligned} \langle O_\Delta(u_1,u_2)\rangle = \frac{1}{\left(2\sin\frac{u_{12}}{2}\right)^{2\Delta}}\Bigg(1+\frac{1}{2\pi C}\frac{\Delta}{\left(2\sin\frac{u_{12}}{2}\right)^2}\big[2+4\Delta+u_{12}\,(u_{12}-2\pi)\,(\Delta+1)+\\ (\Delta u_{12}(u_{12}-2\pi)-4\Delta-2)\cos u_{12}+2\,(\pi-u_{12})\,(2\Delta+1)\sin u_{12}\big]+...\Bigg) \end{aligned} \tag{2.71}$$

In the above we recognize the classical leading contribution (2.62) plus a correction that is proportional to G. Small fluctuations of the metric background can be regarded as the interactions of our massive free particle χ with a gravitational particle δ, leading to effective couplings of the form $G^{1/2}\ \delta\delta\chi$ or $G\ \delta\delta\chi\chi$ that correct the classical geodesics of the particle in the bulk.

2.1.4 Higher topologies

We now generalize our construction to the case of n boundaries (or the boundary has n connected components), each described by their own inverse temperatures $\beta_1, \beta_2, \cdots \beta_n$ and define the gravity path integral on a manifold with n boundaries as $Z_{\text{grav}}(\beta_1, \cdots, \beta_n)$.

We then generalize further our set-up by adding to the JT gravity action the topological Einstein-Hilbert term:

$$S_{\text{JT}} = -\frac{S_0}{4\pi}\left[\int_\Sigma \mathrm{d}^2x\sqrt{g}\ R + 2\int_{\partial\Sigma}\mathrm{d}u\ \sqrt{\gamma}\ \ \kappa\right] - \frac{1}{16\pi G}\left[\int_\Sigma \mathrm{d}^2x\sqrt{g}\ \Phi\,(R+2) + 2\int_{\partial\Sigma}\mathrm{d}u\ \sqrt{\gamma}\ \Phi|_{\partial\Sigma}\ (\kappa-1)\right] \tag{2.72}$$

The motivation for this is that the Einstein-Hilbert actioncorresponds to the zero'th order term in the near-extremal near-horizon expansion of higher dimensional black holes, where S_0 plays the role of the near extremal entropy. At first first order we get JT gravity and then we would have an infinite series (completing (2.72)) of higher order corrections as you move back from the near-horizon near-extremal approximation.

Since the first term of the action is proportional to the Euler characteristic χ of the surface, the full path integral decomposes into a sum over different topological sectors weighted by S_0 [13]:

$$Z_{\text{grav}}(\beta_1, \cdots, \beta_n) = \sum_{g=0}^{+\infty} e^{S_0(2-2g-n)} Z_{g,n}(\beta_1, \cdots, \beta_n) \tag{2.73}$$

where $Z_{g,n}$ represents the contribution from geometries with fixed topology. If we fix the number of boundaries n, this series amounts to considering surfaces with increasing number of handles: these are called spacetime wormholes in this context. The Eisntein-Hilbert action is so responsible for weighing amplitudes with topology, determining a supression of these higher wormholes for a given number of boundaries. In the new notation, we identify the disk partition function with $Z_{0,1} \equiv Z_{\text{disk}}(\beta)$.

Weil-Peterson volumes In order to compute $Z_{g,n}$, it is convenient first to study the case of a surface of genus g with n geodesic boundaries of lengths b_i $i = 1, \cdots n$,, and no holographic boundaries. A geodesic boundary follows a geodesic trajectory (null acceleration) and so it is defined to have a null extrisinc curvature, $\kappa = 0$. The path integral on such a configuration thus becomes

$$\int \frac{\mathcal{D}g_{\mu\nu}\mathcal{D}\Phi}{V(\text{Diff})} e^{-S_{\text{JT}}} = \int \frac{\mathcal{D}g_{\mu\nu}}{V(\text{Diff})}\delta\,(R+2) \tag{2.74}$$

where we are modding out by the group of gauge diffeomorphism. In the second step, we have path integrated out the dilaton and we are left with an integral with a null action, that is computing the volume of the moduli space of Riemann surfaces (with $R = -2$) with hyperbolic metric. This goes under the name of Weil–Petersson volume. We denote the moduli space as $\mathcal{M}_{g,n}(\vec{b})$, with real dimension $6g-6+2n$, and with $V_{g,n}$ its volume.

It turns out that the correct measure over $\mathcal{M}_{g,n}$ is the one induced from the Weil-Petersson symplectic form as follows. We can decompose any hyperbolic surface with genus g into a set of pair-of-pants with $3g-3+n$ connecting tubes with lengths b_i and twists τ_i $(i = 1, \cdots, 3g-3+n)$.

[13] This is sometimes referred as third quantization.

The Weil-Petersson form in these length-twist coordinates, is [14]

$$\omega_{g,n} = \sum_{i=1}^{3g-3+n} \mathrm{d}b_i \wedge \mathrm{d}\tau_i \tag{2.75}$$

The volume form extracted from this Weil-Petersson form is $\frac{\omega_{g,n}^{3g-3+n}}{(3g-3+n)!}$.

Formally, the Weil–Petersson volume can then be represented as an integral over the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space of Riemann surfaces of genus g and n marked points p_i,

$$V_{g,n}(b_1,\ldots,b_n) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\omega + \frac{1}{2}\sum_{i=1}^{n} \psi_i\, b_i^2\right), \tag{2.76}$$

where ω is the Weil–Petersson symplectic form and ψ_i the first Chern class of the line bundle whose fiber is the cotangent space at p_i. We refer the reader to [31] for an introduction on the subject. Concretely, the Weil–Petersson is just a polynomial of degree $3g-3+n$ in the geodesic lengths b_i. However, extracting them from (2.76) can be a difficult task. Moreover the the pair-of-pants decomposition is in general not unique. Fortunately,in [99] Mirzakhani found these Weil–Petersson volumes satisfy a topological recursion formula and one can compute all of them recursively.

Trumpet configuration Another key ingredient to compute multi-boundary amplitudes is the partition function of the trumpet configuration, which is a hyperbolic cylinder bounding a holographic boundary, of renormalized length β, and a geodesic boundary, of length b, as shown in Figure 2.4. The metric is of the form

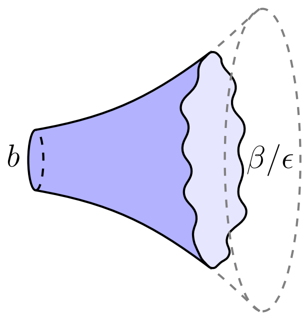


Figure 2.4: The trumpet has an additional boundary, running along a geodesic of length b.

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 + \cosh^2\sigma\ \mathrm{d}\theta^2 \qquad \theta \sim \theta + b \tag{2.77}$$

where the periodic identification of θ breaks the $\mathrm{SL}(2,\mathbb{R})$ symmetry of the hyperbolic plane down to $U(1)$ translations in θ. The wiggly boundary is described by a function $\theta(u)$, and in this case the boundary Schwarzian action becomes

$$S_{\mathrm{Sch}}^{\mathrm{tr}} = -C\int_{\partial\Sigma} \mathrm{d}u\ \{e^{-\theta(u)}, u\} \tag{2.78}$$

[14] It can be deduced from the symplectic form in the first order formulation of JT gravity as a BF gauge theory, as shown in [113]

The good news is that the path integral for this configuration is still one-loop exact as for the disk, so one can obtain the exact answer just by studying the Gaussian integral over small fluctuations about the saddle point $\theta(u) = \frac{2\pi b}{\beta}(u+\delta(u))$.

Decomposing in modes and working out the determinant of the symplectic measure for the trumpet configuration, where $\{e^{-\frac{2\pi b}{\beta}u}, u\} = -\frac{2\pi^2 b^2}{\beta^2}$, we finally get the following answer for the partition function of the trumpet:

$$Z_{\text{tr}}(\beta, b) = \frac{1}{2\pi}\left(\frac{2\pi C}{\beta}\right)^{\frac{1}{2}} e^{-\frac{Cb^2}{2\beta}} \tag{2.79}$$

General topologies With the Weil–Petersson volumes and the trumpet partition function at hand, we can factorize the path integral and exploit the so called topological decomposition. We just need to use the basic Weil–Petersson gluing:

$$\int_0^{+\infty} \mathrm{d}b_i \int_0^{b_i} \mathrm{d}\tau_i = \int_0^{+\infty} \mathrm{d}b_i \; b_i \tag{2.80}$$

where we have integrated over the twists. In this way we are able to compute

$$Z_{g,n}(\beta_1, \cdots, \beta_n) = \int_0^{+\infty} \mathrm{d}b_1 b_1 \cdots \int_0^{+\infty} \mathrm{d}b_n b_n \; V_{g,n}(b_1, \cdots, b_n) \, Z^{\text{tr}}(b_1, \beta_1) \cdots Z^{\text{tr}}(b_n, \beta_n) \tag{2.81}$$

where we are just gluing n external trumpets (corresponding to the n Schwarzian boundaries) with the corresponding Weil–Petersson volume characterized by n boundaries and genus g, along their common tubes of length b_i at each interface.

As an example, we can compute the disk at genus 1, by gluing together a single trumpet and $V_{1,1}$, as shown in Figure 4.2. The result is [94]

$$Z_{1,1}(\beta) = \int_0^{+\infty} \mathrm{d}b \; b \; V_{1,1}(b) \; Z(b, \beta) = \frac{\beta^{\frac{3}{2}}}{24\sqrt{2\pi} C^{\frac{3}{2}}} + \frac{\pi^{\frac{3}{2}}\sqrt{\beta}}{12\sqrt{2C}} \tag{2.82}$$

The cylinder The only exception in the construction above is given by the cyilinder $Z_{0,2}$, our double trumpet (see Figure 4.1), which is obtained by gluing together two trumpets directly, i.e.

$$Z_{0,2}(\beta_1, \beta_2) = \int_0^{+\infty} \mathrm{d}b \; b \; Z^{\text{tr}}(b, \beta_1) \, Z^{\text{tr}}(b, \beta_2) = \frac{\sqrt{\beta_1 \beta_2}}{2\pi(\beta_1 + \beta_2)} \tag{2.83}$$

As a final comment, we note that none of these topologies correspond to classical solutions. In fact none of these manifolds have isometries, while instead the equation of motion for the metric implies that $\epsilon^{\mu\nu}\partial_\nu \Phi$ is a Killing vector.

2.1.5 JT gravity and random matrices

Motivations In this section we will describe the holographic dual of JT gravity, mainly following the recent review [94]. The final outcome of this analysis is that JT gravity is not equivalent to a single boundary hamiltonian but instead to an statistical ensemble. There are two main arguments for this thesis:

- An inverse Laplace transform of the disk partition function (2.42), including the Einstein-Hilbert topological term, reveals the following density of states

$$\rho(E) = \frac{C}{2\pi^2} e^{S_0} \sinh\left(2\pi\sqrt{2CE}\right) \tag{2.84}$$

i.e. the spectral density is not a sum of delta functions, as one would expect for an ordinary quantum system in a finite volume, but instead it is continuous. A continuous spectrum implies the entropy in the microcanonical ensemble is infinite, such that information can be lost inside a black hole. Therefore we expect the disk partition function not to be the whole story, but instead to receive non-perturbative from manifolds with higher genus to effectively render the spectrum discrete. Moreover, a continuos spectrum could emerge from some averaging mechanism.

- the matter bilocal correlator in the Schwarzian theory, that we computed in 2.1.3, has a semiclassical leading behavior given by

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle \simeq \left(\sinh \frac{\pi}{\beta} t\right)^{-2\Delta}$$

that, at late times, decays exponentially [15]. Again, this is not what we would predict for a quantum mechanical system with a bulk dual, where we would expect a late time erratic oscillation around a non-zero mean. Instead of studying matter correlators, a simpler insight with the same outcome is obtained by considering the so called spectral form factor $Z(\beta+it)Z(\beta+it)$, i.e. the modulus squared of an analytically continued partition function. If we consider just the product of two disjoint disks, then the spectral form factor shows exactly that typical t^{-3} decay at late time. However, we have seen in (2.83) that another connected topology, corresponding to a cylinder, is possible for two asymptotic boundaries. Actually, this topology determines the so called ramp-regime in the spectral form factor, which contrasts the forever decay of the disconnected solution. Indeed, including all higher genus connected solutions, is responsibile for the expected late time plateu regime in the spectral form factor. We will treat in more detail the spectral form factor in the finite cutoff JT gravity in 4.6. Here it suffices to say that we can argue the non-factorization in gravity, i.e.

$$Z_{\text{grav}}\left(\beta_1, \beta_2\right) \neq Z(\beta_1)Z(\beta_2)$$

which is a typical property of statistical systems.

We now proceed by constructing the holographic dual of JT gravity. We consider the black hole Hilbert space $\mathcal{H}_{\text{BH}}$ to be of dimension L. A quantum theory is described by an $L\times L$ Hermitian Hamiltonian matrix H acting $\mathcal{H}_{\text{BH}}$. Instead of being dual to a single hamiltonian, we argue the JT gravitational path integral on a manifold with n boundaries of length $\beta_1, \beta_2, \cdots \beta_n$ is equal to an ensemble average over hamiltonians:

$$Z_{\text{grav}}\left(\beta_1, \cdots \beta_n\right) = \int \mathrm{d}H\ e^{-L\,\mathrm{tr}\,V(H)}\ \mathrm{tr}\left(e^{-\beta_1 H}\right)\cdots \mathrm{tr}\left(e^{-\beta_n H}\right) \tag{2.85}$$

[15]Quantum effects actually correct the decay to t^{-3}

where $e^{-L \operatorname{tr} V(H)}$ is the properly normalized probability distribution over theory space, $V(H)$ is the matrix model potential, $Z(\beta_i) = \operatorname{tr}\left(e^{-\beta_i H}\right)$ is the partition function associated to a single boundary and the correlator of these partition functions is hence an observable in the matrix model.

Instead of considering as an observable the partition function, one can either consider the associated spectral density or resolvent, which are related by the following trasformations

$$R(E) = \int_0^{+\infty} \mathrm{d}\beta e^{\beta E} Z(\beta) \qquad Z(\beta) = \int_0^{+\infty} \mathrm{d}E e^{-\beta E} \rho(E) \qquad R(E+i\epsilon) - R(E-i\epsilon) = -2\pi i \rho(E) \tag{2.86}$$

In the following we will consider the resolvent as our main object. We can then define correlation functions on the matrix integral side with n insertions of resolvents

$$R(E_1, \cdots, E_n) = \langle R(E_1) \cdots R(E_n) \rangle \tag{2.87}$$

where we denote with $\langle\rangle$ the ensemble average.Correlation functions of resolvents in the matrix integral are known to have a $1/L$ expansion in the large L limit, taking the form

$$R(E_1, \cdots, E_n)_{\text{conn}} = \sum_{g=0}^{+\infty} \frac{R_{g,n}(E_1, \cdots, E_n)}{L^{2g-2+n}} \tag{2.88}$$

where we refer to the connected piece only. The series is only asymptotic and there are hence non-perturbative corrections to it. This expansion arises from a perturbation theory in 't Hooft double line diagrams, where $R_{g,n}$ is the contribution corresponding to the diagram with n insertions and genus g. [16]

We note that (2.88) clearly resembles the expansion (2.73) of the gravitational path integral, also known as third quantization, once we clarify the connecton between L and e^{S_0}. To do that we need to introduce the concept of a double scaled matrix model.

Double scaled matrix model Although the single hamiltonian H has a discrete spectrum for finite L, after averaging over H and taking the large L limit, the resulting density of states will be continuous. This is exactly what happens in JT gravity. So we should find a potential $V(H)$ such that its density of states to leading order at large L matches with JT gravity on the disk. However, this is not possible, since JT gravity has an infinite support over $E \in [0, +\infty)$. So how can we intepret it as a density? The resolution of this puzzle is that JT gravity is dual not to an ordinary matrix integral, but to a double-scaled matrix integral.

Let us see how it works. We wish to obtain a distribution that has support on the entire positive real axis, so we begin by regularizing the disk density of states, and constructing a matrix potential $V(H)$, such that to leading order in L one has

$$\rho_0(E) = \langle \rho(E) \rangle_{L \to \infty} = e^{S_0} \frac{C}{2\pi^2} \sinh\left(2\pi \sqrt{2CE \frac{a-E}{a}}\right) \tag{2.89}$$

which has finite support on $E \in [0, a]$. We then wish to send $a \to \infty$. This runs into an issue: since we have a finite number of eigenvalues L to be distributed on the whole real line, the

[16] the genus of the surface needed to embed the particular double-line diagram.

spectral density $\rho_0(E)$ goes to zero. To resolve this problem, we need to send the number of eigenvalues to infinity $L \to \infty$ at the same time. We do so by keeping e^{S_0} fixed. After having performed this double scaling limit, the topological expansion (2.88) becomes thus written in terms of e^{S_0} instead of L, assuming exactly the same form as the gravitational expansion (2.73).

Loop equations Once the matrix potential is tuned to give the disk JT gravity density of states in the large e^{S_0} limit, all other subleading terms in the topological expansion will automatically match between the matrix model and the gravity calculation [113]. This miracle is due to the so called loop-equations of the matrix model, which provide a recursive way to determine all $R_{g,n}$. The loop equations are in fact Schwinger–Dyson-like identities which one can write down starting from the following master identity, which comes from differential over a single eigenvalue inside the matrix integral:

$$0 = \int \mathrm{d}^L\lambda \frac{\partial}{\partial\lambda_a}\left[\frac{1}{E-\lambda_a}R(E_1)\cdots R(E_k)\prod_{i<j}(\lambda_i-\lambda_j)^2 e^{-L\sum_j V(\lambda_j)}\right] \tag{2.90}$$

Once we evaluate the derivative, sum over a and expand in powers of $1/L$, for each fixed power in the expansion one can get recursive equations that can be solved order by order, making it possible to systematically compute all of the $R_{g,n}$ with $R_{0,1}$ and $R_{0,2}$ as the only inputs. The former corresponds to the leading density $\rho_0 = \langle\rho(E)\rangle_{L\to\infty}$, which is determined by the potential through

$$V'(E) = \int \mathrm{d}\lambda \frac{\rho_0(\lambda)}{E-\lambda} \tag{2.91}$$

by imposing the stationary saddle point equation for the effective potential $V_{\text{eff}} = V(E) - 2\int \mathrm{d}\lambda\rho_0(\lambda)\log(\lambda - E)$ in the large L limit [17]. The latter, i.e. $R_{0,2}$, represents the density-density correlator $\langle\rho(E)\rho(E')\rangle_{L\to\infty}$ which is universal and indipendent of the potential. The loop equations themselves were studied a long time ago, but they were put in its most useful form by B. Eynard [39].

In order to better show the recursion relations, he proved it is useful to introduce the following objects $W_{g,n}$ related to the resolvent correlators by:

$$W_{g,n}(z_1, z_2, \cdots, z_n) = (-2)^n z_1 \cdots z_n \; R_{g,n}(-z_1^2, -z_2^2, \cdots, -z_n^2)_{\text{conn}} \qquad E = -z^2 \tag{2.92}$$

where in the case of JT gravity

$$W_{0,1}(z) = 2zy(z) \qquad W_{0,2}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} \tag{2.93}$$

The function $y(z) = -i\pi\rho(-z^2) = \frac{C}{2\pi}\sin\left(2\pi\sqrt{2C}z\right)$ is called spectral curve, while $W_{0,2}$ can be easily inferred from the cylinder partition function (2.83). In terms of $W_{g,n}$ the Eynard-Orantin recursion relations [41] are concretely expressed through the formula (4.98) that we will present and exploit later on in the context of JT gravity at finite cutoff.

[17] The Vandermonde comes from diagonalizing H.

The duality We now finally present the duality proposed by Saad-Shenker-Stanford in [113], where they demonstrate JT gravity is actually dual to a double scaled matrix model. The quantities $W_{g,n}$ are related to $Z_{g,n}$ by the following integral transform

$$W_{g,n}(E_1,\cdots,E_n) = 2^n z_1 \cdots z_n \int_0^{+\infty} \mathrm{d}\beta_1 \cdots \mathrm{d}\beta_n e^{-\left(\beta_1 z_1^2 + \beta_2 z_2^2 + \cdots \beta_n z_n^2\right)} Z_{g,n}(\beta_1,\cdots,\beta_n) \tag{2.94}$$

We now express $Z_{g,n}$ as an n-dimensional integral with n trumpets attached to the Weil-Petersson volume $V_{g,n}$, as prescribed in (2.81). Then we exchange the integrals over b_i and β_i and perform the integral over β_i exploiting:

$$2z_i \int_0^{+\infty} \mathrm{d}\beta_i e^{-\beta_i z_i^2} \frac{1}{2\pi} \left(\frac{2\pi C}{\beta_i}\right)^{\frac{1}{2}} e^{-\frac{C b_i^2}{2\beta_i}} = \sqrt{2}\sqrt{C} e^{-\sqrt{2} b_i \sqrt{C} z_i} \tag{2.95}$$

We find

$$W_{g,n}(E_1,\cdots,E_n) = (2C)^{\frac{n}{2}} \left(\prod_{i=1}^n \int_0^{+\infty} \mathrm{d}b_i \; b_i \; e^{-\sqrt{2C} b_i z_i}\right) V_{g,n}(b_1,\cdots b_n) \tag{2.96}$$

Eynard and Orantin showed that Mirzakhani's recursion relations for $V_{g,n}$ imply that $W_{g,n}$ satisfy the loop equations of a double scaled matrix model with spectral curve

$$y(z) = \frac{C}{2\pi} \sin\left(2\pi\sqrt{2C} z\right) \tag{2.97}$$

. We therefore conclude that the sum over topologies in JT gravity reproduces the genus expansion of a particular double-scaled matrix integral, introducing a new form of holographic duality where a single gravity bulk theory is dual to an ensemble of boundary hamiltonians. This implies that an n-point connected correlator at "genus" g for the boundary theory is dual to a connected topology with n boundaries and genus g on the gravity side: holography, through the JT gravity example, gives then a meaningful physical interpretation of the mathematical duality.

2.2 Gauge theories in two dimensions

2.2.1 Yang-Mills theory: general properties

Let us consider a Euclidean gauge theory on a compact orientable Riemann surface Σ of genus $\mathbf{g}$. We denote the gauge group and its Lie algebra with G and $\mathfrak{g}$, respectively. In our conventions, the gauge fields are hermitian, and we define the curvature in terms of the gauge connection A as $F = \mathrm{d}A - \mathrm{i}A \wedge A$. The action of pure Yang–Mills theory

$$S_{\mathrm{ym}} = \frac{1}{2g_{\mathrm{ym}}^2} \int_\Sigma \mathrm{tr}\, F \wedge \star F \tag{2.98}$$

can be rewritten in terms of a single $\mathfrak{g}$-valued scalar $f = \star F$ as

$$S_{\mathrm{ym}} = \frac{1}{2g_{\mathrm{ym}}^2} \int_\Sigma \eta \,\mathrm{tr}\, f^2 \,, \tag{2.99}$$

where η is the volume form on Σ. An alternative action for the theory can be obtained by introducing an auxiliary $\mathfrak{g}$-valued scalar ϕ,

$$S_{\mathrm{ym}} = \mathrm{i}\int_\Sigma \mathrm{tr}(\phi F) + \frac{g_{\mathrm{ym}}^2}{2}\int_\Sigma \eta\, \mathrm{tr}\,\phi^2 \,. \tag{2.100}$$

which is also known as first-order formulation of Yang-Mills theory.

This last expression shows that the theory is invariant under a large group of local symmetries, known as *area-preserving diffeomorphisms* [29]. As a consequence, the partition function is sensitive to the underlying geometry only through the total area $a = \mathrm{vol}\,\Sigma$; by virtue of this, we state the theory is almost topological. In fact, since the action is invariant under an appropriate simultaneous rescaling of a and of the Yang–Mills coupling g_{ym}, the dependence on such couplings comes only through the combination $\alpha = g_{\mathrm{ym}}^2 a$.

The Migdal heat-kernel expansion

The previous statement extends to the Yang-Mills partition function, which can be defined via the usual path integral formulation:

$$Z(\alpha) = \int \mathcal{D}A e^{-S_{ym}(A)} \tag{2.101}$$

It turns out the full quantum theory is solvable. The first solution was found by Migdal and Rusakov ([110][96]) and relies on the lattice formulation in terms of Wilson action. A similar method was developed by Witten ([129] [130]) and consist in cutting the Riemann surface with a circle and factorize the path integral integration in order to write it as a product of vectors on a Hilbert space. In particular, the Hilbert space consists of class functions on the gauge group G.

The resulting partition function, known as Migdal representation, reads:

$$Z = \sum_R (\dim R)^{2-2\mathbf{g}}\, e^{-g_{\mathrm{ym}}^2 a C_2(R)/2} \,, \tag{2.102}$$

and consists in a sum over inequivalent irreducible representations R of the gauge group G, weighted by the Euler characteristic $\chi = 2-2g$ of the Riemann surface and multiplied by a Boltzmann factor. It exhibits explicit dependence only on the effective area parameter α; $c_2(R)$ is the eigenvalue of the quadratic Casimir operator of a given representation. The proof by Witten holds also for a genus g surface Σ which contains a number b of one-dimensional boundaries, the corresponding partition function is

$$Z(\alpha; U_1, \ldots, U_b) = \sum_R (\dim R)^{2-2\mathbf{g}-b}\, e^{-\alpha C_2(R)/2} \chi_R(U_1)\ldots\chi_R(U_b) \,. \tag{2.103}$$

For each boundary b_i we can associate a gauge invariant holonomy U_i and define the character $\chi_R(U_i)$, which is the trace of the holonomy U_i in the representation R. Indeed it is possible to obtain higher genus surfaces by gluing two boundaries. The operation, at the level of the partition function, consists in integrating the two characters corresponding to the holonomies with respect to the Haar measure of the group G. For instance, we can obtain the torus from a finite cylinder by gluing the two boundaries, the resulting partition function will be of the form (2.102) with $g = 1$.

Below, we review the strategy proposed by Witten in [130], following an Hamiltonian quantization of the theory. Specifically, we will derive the Migdal representation for the basic topologies of the disk and sphere, that will be relevant for this dissertation.

Hamiltonian approach Let us consider an initial-value circle $C \subset \Sigma$ in our surface Σ. In a neighborhood of C, we can write the volume form in terms of local coordinates as $\eta = \mathrm{d}x \wedge \mathrm{d}t$, where C corresponds to $t = 0$ and x is a coordinate along C such that $\oint \mathrm{d}x = 1$. Since the action (2.100) is linear in F, the Hamiltonian reads [18]

$$H = \frac{g_{\mathrm{ym}}^2}{2} \oint_C \mathrm{d}x \ \mathrm{tr}\left(\phi^2\right) , \tag{2.104}$$

and generates translations along t. The Hilbert space of states will be defined on circular slices at constant time, while the hamiltonian will give the propagation along the time coordinate. When quantizing the theory on a spatial circle C, due to the Gauss law constraint A.5, the wave functions are simply class functions $\psi(U)$ [19]of the holonomy U around that circle:

$$U = \mathrm{Pexp} \oint_C A \,. \tag{2.105}$$

Consequently a natural basis for the Hilbert space of states, the so called “representation basis”, is provided by the characters in the irreducible unitary representations, i.e.

$$\langle R|U\rangle = \chi_R(U) = \mathrm{tr}_R(U) \tag{2.106}$$

We note from (2.100) that the canonical momentum conjugate to the space component of the gauge field $\pi_{A_x^a}$ is ϕ^a. Hence in the quantum theory, by canonical quantization, if the A_x^a are multiplication operators, then ϕ^a are functional derivatives operators:

$$\phi^a = -i \frac{\delta}{\delta A_x^a} \tag{2.107}$$

As a consequence the momentum acting on the wavefunctions $\chi_R(g)$ is given by

$$\pi_{A_x^a} \chi_R \left(\mathrm{Pexp} \oint A_x^a T^a\right) = -i \chi_R \left(T^a U\right) \tag{2.108}$$

while the hamiltonian (2.104) will be

$$H = \frac{g_{\mathrm{ym}}^2}{2} \oint \mathrm{d}x \frac{\delta}{\delta A_x^a(x)} \frac{\delta}{\delta A_x^a(x)} \tag{2.109}$$

and therefore acts diagonally on the characters. Let’s find the associated eigenvalues E_R

$$\frac{g_{\mathrm{ym}}^2}{2} \oint \mathrm{d}x \ \frac{\delta^2}{\delta A_1^a(x) \delta A_1^a(x)} \ \chi_R(U) = \frac{g_{\mathrm{ym}}^2}{2} \oint \mathrm{d}x \ T^a T^a \chi_R(U) \tag{2.110}$$

We recognize immediately $\sum_a T^a T^a$, evaluated in representation R, as the eigenvalue of the quadratic Casimir $c_2(R)$ in representation R, where we used $\oint \mathrm{d}x = 1$. Therefore

$$E_R = \frac{g_{\mathrm{ym}}^2}{2} \ c_2(R) \tag{2.111}$$

[18]The Lagrangian is directly written as the Legendre transform of the Hamiltonian.

[19]The are class functions because they must be invariant under conjugation $\psi(g) = \psi\left(g^{-1} U g\right)$

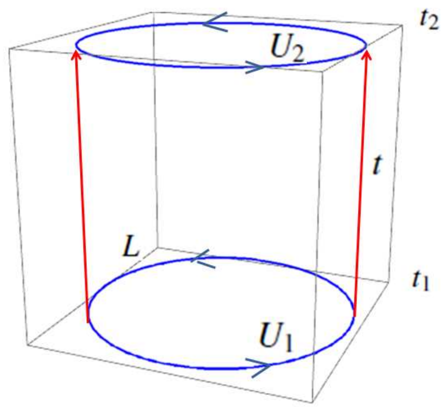


Figure 2.5: Transition amplitude between an initial holonomy U_1 and a final holonomy U_2, along time t in the vertical direction.

Transition amplitude It is now convenient to exploit the area-preserving diffeomorphisms to deform our surface Σ into a two-holed sphere, or more simply a cylinder, with an initial state circle C_1 and a final state circle C_2. The euclidean path integral for this manifold can be rewritten in the Hamiltonian approach as a transition amplitude

$$Z\left(U_1, U_2, a\right) = \langle U_1 | \, e^{-Ha} \, | U_2 \rangle \tag{2.112}$$

between the initial state U_1 and the final state U_2, corrisponding respectevely to the holonomy around the initial circle at time t_1 and final circle at time t_2, with $t = t_2 - t_1$, as shown in figure 2.5. Since the length of the circles is 1, we can set $t = a$, the area enclosed by the cylinder.

Having diagonalized the Hamiltonian, we can immediately write the propagator corresponding to this transition amplitude, by inserting a completeness on the representation basis.

$$\begin{aligned} Z\left(U_1, U_2, a\right) = \langle U_1 | \, e^{-Ha} \, | U_2 \rangle &= \sum_R \langle U_1 | R \rangle \, e^{-Ha} \, \langle R | U_2 \rangle \\ &= \sum_R \chi_R(U_1) \chi_R(U_2^\dagger) \; e^{-\frac{g_{\text{ym}}^2}{2} c_2(R) a} \end{aligned} \tag{2.113}$$

From the cylinder to the disk At area $a = 0$ the amplitude can be calculated in the topological theory, in fact the small a limit is identical to the small g_{ym}^2 limit in which the action (2.100) becomes topological. In this limit, we have seen that the integral over ϕ imposes the condition that A is a flat connection, which renders the holonomy $U = \mathbb{1}$. This means that the wavefunction $\psi(U)$ is completely concentrated in $U = \mathbb{1}$ and null for any other value of the holonomy, i.e

$$\psi(U) = \delta\left(U, \mathbb{1}\right) \tag{2.114}$$

where δ is the delta function in the Haar measure

$$\int \mathrm{d}U \; x(U) \delta\left(U, \mathbb{1}\right) = x(\mathbb{1}) \tag{2.115}$$

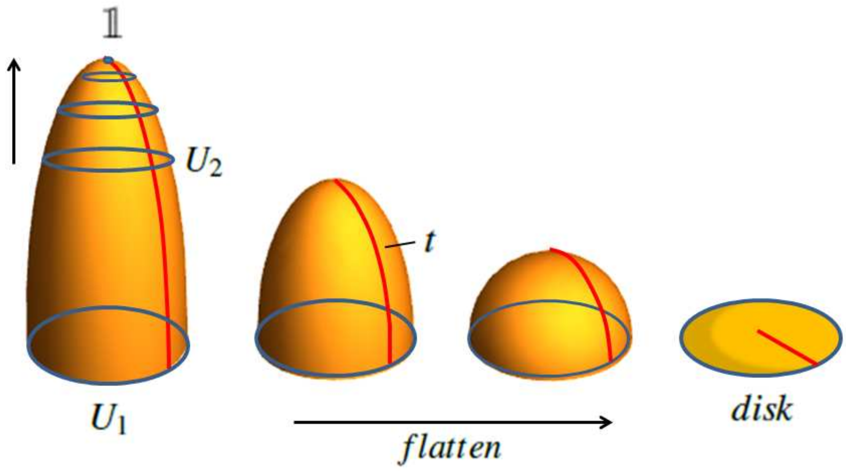


Figure 2.6: Following the VERTICAL ARROW: the final holonomy U_2(at final time t_2) becomes $\mathbb{1}$ in the limit in which we contract the area of the circle (along which the holonomy is defined) to zero. Following the HORIZONTAL ARROW: we gradually flatten the shape on the left to obtain a disk and we note that the red line, which indicates the time interval t, becomes a ray of the disk. In this way we get the radial quantization pattern in which time flows along the radial direction.

To find the partition function of the object shown on the left in figure 2.6, we should glue the partition function of a cylinder with the one just obtained in the limit $a \to 0$, i.e.

$$Z(U,a) = \int \mathrm{d}U_1 \; Z(U,U_1,a)\,\delta(U_1,\mathbb{1}) = \sum_R \chi_R(U)\chi_R(\mathbb{1}) \; e^{-\frac{g_{\mathrm{ym}}^2}{2}c_2(R)a} \tag{2.116}$$

where in the second step (2.115) and (2.113) have been used. Now we take advantage of $\chi_R(\mathbb{1}) = \mathrm{tr}_R(\mathbb{1}) = \dim R$ to yield

$$Z(U,a)_{\text{disk}} = \sum_R \dim R \; \chi_R(U) \; e^{-\frac{g_{\mathrm{ym}}^2}{2}c_2(R)a} \tag{2.117}$$

Using the area preserving diffeomorphism invariance, we may flatten out the shape we are considering (see figure 2.6) and regard (2.117) as an amplitude for a disk (or one-holed sphere).

The sphere To get the partition function for the sphere, we just need to set $U = \mathbb{1}$ in (2.117), which just corresponds to closing the remaining hole by setting the holonomy to the identity. We obtain a new factor of $\dim R$:

$$Z(a)_{\text{sphere}} = \sum_R (\dim R)^2 \; e^{-\frac{g_{\mathrm{ym}}^2}{2}c_2(R)a} \tag{2.118}$$

2.2.2 The instanton expansion

2dYM also admits an equivalent formulation with a fermionic symmetry. Let ψ be a Grassmann valued, and adjoint valued, one-form field. The action

$$S_{\text{susy}}(A,\psi,\phi) = -\frac{g_{\text{ym}}^2}{2}\int_{\Sigma_g}\eta\text{Tr}\phi^2 - i\int_{\Sigma_g}\left(\phi F + \frac{1}{2}\psi\wedge\psi\right) \tag{2.119}$$

yields a theory which is equivalent to 2dYM in the first order formulation (2.100), up to an overall renormalization from integrating out ψ. Indeed the path integral over ψ plays the role of inducing the correct symplectic measure on $\mathcal{A}$ [20], the space of gauge connections on Σ_g. S_{susy} is also invariant under the following BRST odd symmetry transformations:

$$\begin{aligned}\delta A_\mu &= i\psi_\mu\\ \delta\psi_\mu &= -D_\mu\phi\\ \delta\phi &= 0\end{aligned} \tag{2.120}$$

The transformation δ squares to an infinitesimal gauge transformation with parameter ϕ

$$\delta^2 = -\mathrm{i}\delta_\phi \tag{2.121}$$

In [130], Witten showed that the cohomologial Yang-Mills action (2.119) admits a δ-exact deformation which can be used to localize the theory. The specific deformation used by Witten, which requires the introduction of two additional supermultiplets, was shown to localize the theory to the moduli space of solutions to the Yang-Mills equations on Σ:

$$D^i F_{ij} = 0 \tag{2.122}$$

where D is the derivative which is covariant with respect to both the Levi-Civita connection and A. The connected components of this moduli space, which deserve the name instanton sectors, are labeled by Γ: the magnetic weight lattice of G, which is dual to Γ^*: the usual weight lattice of G, modulo the action of the Weyl group W [4]. Recall that the latter is equivalent to the set of irreducible representations of the gauge group G.

The localization procedure used by Witten, which differs from standard localization computations, was given the name "non-abelian localization". See Appendix A.2 for a brief review on non-abelian localization. The value of the partition function derived using non-abelian localization is, schematically,

$$Z = \sum_{\mathfrak{m}\in\Gamma/W} q(\mathfrak{m},\alpha,g)\,e^{-S_{\text{ym}}(\mathfrak{m})} \tag{2.123}$$

here $S_{\text{ym}}(\mathfrak{m})$ is the value of the action (2.98) on any solution of the Yang-Mills equations in the component labeled by $\mathfrak{m}$, and $q(\mathfrak{m},\alpha,g)$ is a function which was not explicitly computed in [130]. Witten argues that when the moduli space of flat connections, which we denote $\mathcal{M}_{\text{flat}}$, is smooth, the part of (2.123) which is not exponentially suppressed in the limit $\alpha\to 0$ is in fact a polynomial in the area α, which we call $p(\alpha,g)$. It was further argued by Witten that

[20] with symplectic form $\omega = \int \operatorname{tr}\delta A\wedge\delta A$

$p(\alpha,g)$ contains all of the information required in order to compute the intersection theory on $\mathcal{M}_{\text{flat}}$. More explicitly:

$$\int_{\mathcal{M}_{\text{flat}}} \exp(\omega+\alpha\Theta) = p(\alpha,g) \tag{2.124}$$

where Θ is the BRST invariant observable $\operatorname{tr}\phi^2$ and the left hand side expresses the intersection numbers on $\mathcal{M}_{\text{flat}}$. [21]

The expression (2.123), known as instanton expansion, and the Migdal representation (2.102) are indeed equivalent, and Witten argues that they are related by Poisson summation [130]. In the following, we will specify the gauge group to be $U(N)$ and the instanton expansion will be more manifest. Moreover, we will mainly focus on the sphere ($g=0$) where we can basically forget issues about flat connections (there no flat connections wrapping the nontrivial cycles of Σ) and intersection theory, since in that case $\mathcal{M}_{\text{flat}}$ becomes trivial.

2.2.3 The U(N) theory

Below, we will regard the partition function as a function of the rank N and written in terms of the effective adimensional coupling $\alpha=\lambda a$, where $\lambda=g_{\text{YM}}^2 N$ is the usual 't Hooft coupling. An irreducible representation R of U(N) is labeled by its highest weight vector $\boldsymbol{\lambda}\in\mathbb{Z}^N$ of ordered integers

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N\,. \tag{2.125}$$

The dimension of R and the eigenvalue of its quadratic Casimir are given by

$$\dim R = \prod_{i<j}\left(1-\frac{\lambda_i-\lambda_j}{i-j}\right)\,, \tag{2.126}$$

$$C_2(R) = \sum_{i=1}^{N}\lambda_i(\lambda_i-2i+N+1)\,. \tag{2.127}$$

For the purpose of applying this to (2.102), it is useful to rewrite the above in terms of new variables $\ell_i=-\lambda_i-i+N$. The constraint (2.125) restricting to the fundamental Weyl chamber now reads

$$\ell_1<\ell_2<\ldots<\ell_N\,. \tag{2.128}$$

With the new variables, the dimension

$$\dim R = \frac{\Delta(\ell_1,\ldots,\ell_N)}{G(N+1)}\,, \tag{2.129}$$

is expressed in terms of the Vandermonde determinant, defined as

$$\Delta(\ell_1,\ldots,\ell_N) = \det\begin{pmatrix} \ell_1^0 & \ell_1^1 & \cdots & \ell_1^{N-1} \\ \ell_2^0 & \ell_2^1 & \cdots & \ell_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_N^0 & \ell_N^1 & \cdots & \ell_N^{N-1}\end{pmatrix} = \prod_{i<j}(\ell_j-\ell_i)\,. \tag{2.130}$$

[21] Witten denotes $\mathcal{M}_{\text{flat}}$ with $\mu^{-1}(0)$, the inverse of the moment map $\mu(A)=-\frac{F}{4\pi^2}$.

Here, G is the Barnes function. The eigenvalue of the quadratic Casimir reads

$$C_2(R) = \frac{N(1-N^2)}{12} + \langle \boldsymbol{\ell} \rangle \,, \tag{2.131}$$

where we have introduced the shorthand

$$\langle \boldsymbol{\ell} \rangle = \sum_{i=1}^{N} \left(\ell_i - \frac{N-1}{2} \right)^2 \,. \tag{2.132}$$

Focusing on the sphere topology (i.e. on $\mathbf{g} = 0$), since both $\dim R$ and $C_2(R)$ are invariant under permutations of the ℓ_i's, and since $\dim R$ vanishes whenever two of these coincide, we can simply lift the constraint (2.128) and normalize appropriately. This gives

$$\begin{aligned} Z(\alpha) &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^N} \hat{z}_{\boldsymbol{\ell}}(\alpha) \\ &= \sum_{\boldsymbol{\ell} \in \mathbb{Z}^N} \frac{e^{\alpha(N^2-1)/24}}{N!\, G^2(N+1)} \Delta^2(\ell_1, \ldots, \ell_N)\, e^{-\frac{\alpha}{2N} \langle \boldsymbol{\ell} \rangle} \,. \end{aligned} \tag{2.133}$$

Through the Poisson summation formula, the partition function (2.133) can be recast in terms of a dual representation

$$Z(\alpha) = \sum_{\mathfrak{m} \in \mathbb{Z}^N} z_{\mathfrak{m}}(\alpha) \,, \tag{2.134}$$

where

$$z_{\mathfrak{m}}(\alpha) = \int_{\mathbb{R}^N} \mathrm{d}\ell_1 \ldots \mathrm{d}\ell_N \, e^{-2\pi \mathrm{i} \mathfrak{m} \cdot \boldsymbol{\ell}} \, \hat{z}_{\boldsymbol{\ell}}(\alpha) \,. \tag{2.135}$$

This is nothing but the instanton representation mentioned earlier, where now we regard $\mathfrak{m}$ as a set of N integers through the natural isomorphism $\Gamma \simeq \mathbb{Z}^{\mathrm{rk}\, G}$. The physical interpretation as a sum over classical configurations becomes manifest upon performing the Fourier transform above. In fact, one finds that each term in (2.134) has the form

$$z_{\mathfrak{m}}(\alpha) = w_{\mathfrak{m}}(\alpha)\, e^{-S_{\mathrm{cl}}(\mathfrak{m})} \,, \tag{2.136}$$

where the function [97]

$$\begin{aligned} w_{\mathfrak{m}}(\alpha) = (-1)^m & \frac{e^{\alpha(N^2-1)/24}}{N!\, G^2(N+1)} \left(\frac{2\pi N}{\alpha} \right)^{N^2} \\ & \times \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \, e^{-S_{\mathrm{cl}}(\boldsymbol{x})} \prod_{i<j} \left[(x_i - x_j)^2 - (\mathfrak{m}_i - \mathfrak{m}_j)^2 \right] \end{aligned} \tag{2.137}$$

captures the quantum fluctuations about the classical configuration. By introducing the differential operator

$$\mathbf{V} = (-4\pi^2)^{-N(N-1)/2} \prod_{i<j} (\partial_{\mathfrak{m}_i} - \partial_{\mathfrak{m}_j})^2 \,, \tag{2.138}$$

we can write

$$\begin{aligned} z_{\mathfrak{m}}(\alpha) &= \frac{e^{\alpha(N^2-1)/24}}{N!\,G^2(N+1)}\,(-1)^m \int_{\mathbb{R}^N} \mathrm{d}h_1 \ldots \mathrm{d}h_N\, \mathbf{V}\, e^{-\frac{\alpha}{2N}|\boldsymbol{h}|^2 - 2\pi \mathrm{i} \mathfrak{m}\cdot \boldsymbol{h}} \\ &= z_{\mathbf{0}}(\alpha)\, \frac{(\alpha/N)^{\nu}}{N!\,G(N+1)}\,(-1)^m\, \mathbf{V}\, e^{-2\pi^2 N |\mathfrak{m}|^2/\alpha}\,, \end{aligned} \tag{2.139}$$

where $\nu = N(N-1)/2$, and $m = (N-1)(\mathfrak{m}_1 + \ldots + \mathfrak{m}_N)$.

The result for the zero-flux and the unit-flux sectors read [64]

$$z_{\mathbf{0}}(\alpha) = C_N\, e^{\alpha(N^2-1)/24}\, \alpha^{-N^2/2}\,, \tag{2.140}$$

$$z_{\mathbf{1}}(\alpha) = (-1)^{N-1} N^{-1}\, e^{-2\pi^2 N/\alpha}\, L^1_{N-1}(4\pi^2 N/\alpha)\; z_{\mathbf{0}}(\alpha) \tag{2.141}$$

where we denoted with $\mathbf{1} \in \mathbb{Z}^N$ a generic unit vector, and we defined

$$C_N = \frac{(2\pi)^{N/2} N^{N^2/2}}{G(N+1)}\,. \tag{2.142}$$

The former corresponds to the contribution coming from the vacuum sector and describes the perturbative regime of the theory. The latter captures the contribution of the first nontrivial solution associated with a monopole configuration of unit flux and classical action $2\pi^2 N/\alpha$.

2.2.4 The large N limit

When N becomes large, 2d Yang-Mills theory is conjectured to be dual to some string theory with target space Σ [61, 66, 29]. Taking advantage of the solvability of two dimensional $U(N)$ pure Yang-Mills theory on the torus topology, Gross and Taylor in fact identified the free-energy of the theory at weak coupling (in the t'Hooft limit) as a string expansion. Specifically, the Yang–Mills free energy should compute the partition function of a string winding on Σ with coupling $g_{\mathrm{s}} = 1/N$ and tension λ. Evidence for the duality is given by the fact that the $1/N$-expansion of the free energy takes the form

$$\begin{aligned} F(\alpha) &= \log Z(\alpha) \\ &= \sum_{\ell=\mathbf{g}}^{\infty} N^{2-2\ell}\, F_\ell(\alpha)\,. \end{aligned} \tag{2.143}$$

Gross and Taylor have shown that the coefficients of the free energy in the expansion can be interpreted as maps between branched covers of Riemann surfaces, with different genus. This is consistent with the fact that, according to the Riemann–Hurwitz formula, there are no covering maps between a worldsheet of genus ℓ and a two-dimensional target space of genus $\mathbf{g}$, if $\ell < \mathbf{g}$. Moreover, the coefficients of the Gross-Taylor string expansion were shown to have a deep geometrical interpretation in terms of modular functions.

The case of the sphere In the context of large N, the topology of the sphere is peculiar. In fact, due to presence of the Vandermonde determinant at the numerator, as in (2.133), the theory undergoes a phase transition, known as Douglas-Kazakov phase transition. Below a certain threshold in the value of the area α, the system is dominated by a weak-phase regime where the full Gross-Taylor expansion (which remains unaltered in the strong regime) becomes trivial. This phenomenon due to the fact the in the weak-coupling phase only the zero-instanton sector dominates, with all higher instantons being suppressed. Below, we will explore this points in more detail.

At leading order, the large-N limit analysis can be efficiently tackled by approximating the sum in (2.133) through the functional integral [35]

$$Z = \int [\mathrm{d}h]\, e^{-N^2 S_{\text{eff}}[h]} \,, \tag{2.144}$$

$$S_{\text{eff}}[h] = -\frac{\alpha}{24} - \frac{3}{2} + \frac{\alpha}{2}\int_0^1 \mathrm{d}x\, h^2(x) - \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y\, \log|h(x)-h(y)| \,, \tag{2.145}$$

where the integral is performed over the function $h:[0,1]\to\mathbb{R}$ obeying the constraint $h' \geq 1$. Interestingly, the saddle-point approximation of the above is analogous to that of a Gaussian matrix model. In fact, the density $\rho(h) = \partial x/\partial h$ obeys the saddle-point equation

$$\frac{\alpha}{2} h = \fint \frac{\rho(s)}{h-s}\mathrm{d}s \,. \tag{2.146}$$

What makes this model nontrivial, however, is the presence of the constraint on h'. This implies that a general solution of the above should be of the form

$$\rho(s) = \begin{cases} 1 & \text{for } |s| < b \,, \\ u(s) & \text{for } b \leq |s| < a \,, \\ 0 & \text{for } a \leq |s| \,. \end{cases} \tag{2.147}$$

For $\alpha < \pi^2$, one finds that $b = 0$, while ρ obeys the typical Wigner semicircle law. For $\alpha > \pi^2$, instead, $b > 0$ and to find the density ρ one should solve

$$\frac{\alpha}{2} h - \log\frac{h-b}{h+b} = \fint_{-a}^{-b} \frac{u(s)}{h-s}\mathrm{d}u + \fint_{+b}^{+a} \frac{u(s)}{h-s}\mathrm{d}u \,. \tag{2.148}$$

The saturation of the constraint on h' is responsible for a third-order phase transition at $\alpha = \pi^2$ that the theory undergoes in the large-N limit, first observed by Douglas and Kazakov [35]. Later, in [64], it was shown that the transition is induced by instantons. By evaluating the ratio between the unit-flux and the zero-flux partition functions, one can see that for small values of the effective 't Hooft coupling, the former is exponentially suppressed in N/α only for $\alpha < \pi^2$. Specifically, by taking the large-N limit of (2.140) and (2.141) below the critical point, one finds

$$\log\frac{\boldsymbol{z_1}(\alpha)}{\boldsymbol{z_0}(\alpha)} \sim -\frac{2\pi^2 N}{\alpha}\gamma(\alpha/\pi^2) \,, \tag{2.149}$$

where

$$\gamma(z) = \sqrt{1-z} - \frac{z}{2}\log\frac{1+\sqrt{1-z}}{1-\sqrt{1-z}} \,. \tag{2.150}$$

The function $\gamma(z)$ is positive for $z < 1$, i.e. in the weak phase, but vanishes as its argument reaches the critical value $z = 1$.

The large-N limit of the theory is characterized by the leading order of the free energy in the $1/N$ expansion, which we can write as

$$F_0(\alpha) = \frac{3}{4} + \frac{\alpha}{24} - \frac{\log\alpha}{2} + \Theta(\alpha-\pi^2)\,\Delta F_0(\alpha)\,, \tag{2.151}$$

where Θ denotes the Heaviside step function. The function ΔF_0 captures the behavior above the transition. Its derivative reads [35]

$$\partial_\alpha \Delta F_0(\alpha) = \frac{1}{2\alpha} - \frac{4(k(\alpha)+1)\,K^2(k(\alpha))}{3\alpha^2} - \frac{8(k(\alpha)-1)^2\,K^4(k(\alpha))}{3\alpha^3}\,, \tag{2.152}$$

where $k(\alpha)$ is obtained by inverting

$$\alpha = 4K(k)\,(2E(k) + (k-1)\,K(k))\,. \tag{2.153}$$

Here, K and E denote elliptic integrals of the first and second kind, respectively. Near the transition point,

$$\Delta F_0(\alpha) = -\frac{(\alpha-\pi^2)^3}{3\pi^6} + O((\alpha-\pi^2)^4)\,, \tag{2.154}$$

which shows, indeed, that the transition is of the third order.

For large values of α, the free energy is given by the expansion [35]

$$F_0(\alpha) = 2e^{-\alpha/2} + (\alpha^2/2 - 2\alpha - 1)e^{-\alpha} + (\alpha^4/3 - 8\alpha^3/3 + 4\alpha^2 + 8/3)e^{-3\alpha/2} + \dots\,, \tag{2.155}$$

that is perfectly consistent with the Gross–Taylor string expansion.

2.2.5 Quantization of JT gravity as a BF theory

In this section we prove that JT gravity can be equivalently written in the first-order formulation, which involves the frame and spin-connection of the manifold Σ, as a two-dimensional BF theory with $\mathrm{SL}(2,\mathbb{R})$ gauge group. The close analogy with 2d Yang-Mills theory will lead us to borrow some results shown above to rederive the JT partition function in this framework.

We start by reviewing how the correspondence works at the classical level. Let us consider a BF theory with action

$$S_{\mathrm{BF}} = -i\int_\Sigma \mathrm{tr}\,(\phi F) \tag{2.156}$$

This is just a purely topological theory [22] and corresponds to the $g_{\mathrm{ym}} \to 0$ limit of (2.100). We choose a two-dimensional real representation $\mathrm{P}_0, \mathrm{P}_1, \mathrm{P}_2$ for the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$\mathrm{P}_0 = i\frac{\sigma_2}{2} \qquad \mathrm{P}_1 = \frac{\sigma_1}{2} \qquad \mathrm{P}_2 = \frac{\sigma_3}{2} \tag{2.157}$$

One can check they satisfy the commutation relations of the algebra of $\mathfrak{sl}(2,\mathbb{R})$:

$$[\mathrm{P}_0, \mathrm{P}_1] = \mathrm{P}_2 \qquad [\mathrm{P}_0, \mathrm{P}_2] = -\mathrm{P}_1 \qquad [\mathrm{P}_1, \mathrm{P}_2] = -\mathrm{P}_0 \tag{2.158}$$

[22] The action is completely written in terms of forms and does not require a metric on Σ.

We then denote the adjoint fields A and ϕ as

$$A(x) = \sqrt{\frac{\Lambda}{2}}e^a(x)\mathrm{P}_a + \omega(x)\mathrm{P}_0 \qquad \phi(x) = \phi^a(x)\mathrm{P}_a + \phi^0(x)\mathrm{P}_0 \tag{2.159}$$

where $\phi^a(x)$ and $\phi_0(x)$ are scalar functions while $e^a(x) = e^a_\mu \mathrm{d}x^\mu$ and $\omega(x) = \omega_\mu \mathrm{d}x^\mu$ are one forms. Exploiting the definition $F = \mathrm{d}A + A \wedge A$ and the commutation relations (2.158), we calculate the non-abelian field strenght

$$\begin{aligned} F &= F^a\mathrm{P}_a + F^0\mathrm{P}_0 \\ &= \sqrt{\frac{\Lambda}{2}}\left(\mathrm{d}e^1 + \omega\wedge e^2\right)\mathrm{P}_1 + \sqrt{\frac{\Lambda}{2}}\left(\mathrm{d}e^2 - \omega\wedge e^1\right)\mathrm{P}_2 + \left(\mathrm{d}\omega + \frac{\Lambda}{2}e^1\wedge e^2\right)\mathrm{P}_0 \end{aligned} \tag{2.160}$$

Plugging it in the action (2.156) and using normalization $\mathrm{tr}\left(\phi_i \mathrm{P}^i \; F_j \mathrm{P}^j\right) = \phi_i F_j \eta^{ij}/2$ we get

$$S_{\mathrm{BF}} = -\frac{i}{2}\int \sqrt{\frac{\Lambda}{2}}\left[\phi_1\left(\mathrm{d}e^1 + \omega\wedge e^2\right) + \phi_2\left(\mathrm{d}e^2 - \omega\wedge e^1\right)\right] - \phi_0\left(\mathrm{d}\omega + \frac{\Lambda}{2}e^1\wedge e^2\right) \tag{2.161}$$

The variation of ϕ_1 and ϕ_2 yields the equation of motion

$$\mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0 \qquad \omega^1{}_2 = -\omega^2{}_1 = \omega \to \omega = \frac{1}{2}\epsilon^b{}_a \omega^a{}_b \tag{2.162}$$

which are precisely the zero torsion conditions for the frame e^a with spin connection $\omega^a{}_b$. We then evaluate the action on shell plugging these equations back into (2.161)

$$S_{\mathrm{BF}} = \frac{i}{2}\int \phi_0\left(\mathrm{d}\omega + \frac{\Lambda}{2}e^1\wedge e^2\right) \tag{2.163}$$

Moreover we have the following relations in 2d

$$\mathrm{d}^2x\sqrt{g} = e_1\wedge e_2 \qquad \mathrm{d}^2x\sqrt{g}R = 2d\omega \tag{2.164}$$

such that the BF action will then finally read as

$$S_{\mathrm{BF}} = \frac{i}{4}\int \mathrm{d}^2x\sqrt{g}\; \phi_0\left(R + \Lambda\right) \tag{2.165}$$

which is precisely the bulk part of the JT action if we identify the dilaton $\Phi = -i\;\phi_0/4$.

Recovering the boundary dynamics Let's specify Σ to be the surface of a disk, since this is the classical topology in JT gravity. If we vary the action (2.165) with respect to the gauge field, we obtain

$$\delta S_{\mathrm{BF}} = (\text{bulk equations of motion}) - i\int_{\partial\Sigma} \mathrm{d}\tau \;\; \mathrm{tr}\left(\phi\delta A_\tau\right) \tag{2.166}$$

where boundary term has been produced by integration by parts. As a consequence the BF theory has a well-defined variational principle only when fixing the gauge field A_τ along the boundary. This is in contrast with the second-order formulation of JT gravity (2.4), when fixing the metric and the dilaton along the boundary is necessary.

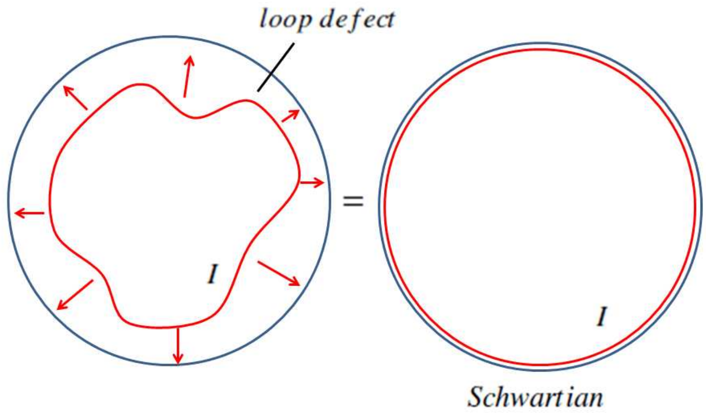


Figure 2.7: The resulting theory (2.167) is invariant under perimeter preserving defect diffeomorphisms and thus the defect can be brought arbitrarily close to the boundary of the manifold. Furthermore, the degrees of freedom of the gauge theory defect can be captured by those in the Schwarzian theory.

However, solely fixing the gauge field around the boundary yields a trivial topological theory. Of course, such a theory cannot be dual to the Schwarzian. In order to effectively modify the dynamics of the theory we consider a defect along a loop [74], i.e we add a term S_{I} to the BF action

$$S_{\mathrm{BF+def}} = S_{\mathrm{BF}} + S_{\mathrm{I}} \qquad S_{\mathrm{I}} = e\int_0^\beta \mathrm{d}u\ \sqrt{\gamma_{uu}}\ V\left(\phi(u)\right) \tag{2.167}$$

where e is a coupling constant (related to the gravitational coupling constant through $e = 2/C$) and u is the proper length parametrization of the loop I, whose embedding coordinates are given by $x_{\mathrm{I}}(u)$ and whose total length is β, measured with the induced background metric γ_{uu} from the disk. [23] V is a potential depending on the value of ϕ on such a loop, but needs to be of trace class in order to be gauge invariant. Our guess is

$$V(\phi) = -\frac{1}{4}\,\mathrm{tr}\left(\phi^2\right) \tag{2.168}$$

with the trace in the fundamental representation of $\mathrm{SL}(2,\mathbb{R})$. Due to the appearance of the length form in (2.167) the action is invariant under diffeomorphisms which preserve the local length element on I, i.e. the perimeter of the defect. As shown in figure 2.7, this means that the defect can be brought arbitrarily close to the boundary of the manifold by modifying the metric through a local diffeomorphisms away from I. In A.3 we prove that this form of the potential is suitable to recover the Schwarzian dynamics at the boundary of the disk [74].

Quantization Once we write down the complete action

$$S_{\mathrm{BF+def}}\left(\phi, A\right) = -i\int_\Sigma \mathrm{tr}\left(\phi F\right) - \frac{e}{4}\int_I \mathrm{d}u\,\mathrm{Tr}\left(\phi^2\right) \tag{2.169}$$

[23]Consequently, since the defect depends on the metric, it is not topological.

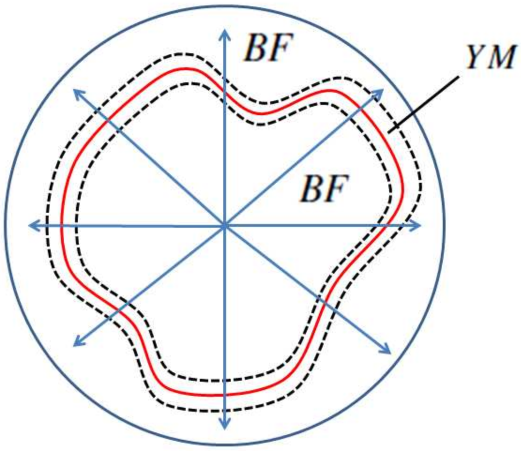


Figure 2.8: Blue arrows indicate time propagation flow, the loop defect is represented in red. The theory is topological, i.e not dynamical ($H = 0$), inside and outside the region delimited by dotted lines, while a YM theory is defined within this area. Actually the external region vanishes in the limit in which we push the loop defect arbitrarly close to the boundary in order to recover the Schwarzian theory.

we notice a clear similarity with the 2d Yang-Mills action (2.100): the only difference is that the potential term is supported only along the path I of length β and not over the entire disk surface. The Hamiltonian quantization we performed for Yang-Mills on the disk is not affected by this slight modification. This means that we can deduce the partition function for the theory (2.169) from (2.117)

$$Z_{\text{BF+def}}(U,\beta) = \sum_R \dim R \; \chi_R(U) \; e^{-e\beta c_2(R)} \tag{2.170}$$

and immediately write down the answer, where U is again the boundary holonomy of the gauge field.

One way to motivate this is considering radial quantization and choosing time-slices to be concentric to the loop. However, instead of having an hamiltonian defined on all disk, now it is non-vanishing only in corrispondence of the loop defect and null everywhere. This means that the hamiltonian (2.104) becomes time dipendent, because it turns active only for an infinitesimal time around the loop defect. In this way, the propagation factor becomes the dimensionless quantity $e\beta$. Alternatively, one can consider the gluing of a totally topological theory with $e = 0$ in the regions inside and outside I, and a Yang-Mills theory of type (2.100) in a fattened region around I with infinitesimal width, as shown in Figure 2.8.

Non compact gauge group G So far we have assumed that the group G is compact, and thus the spectrum of unitary irreps is discrete. The only modification required in the case of a non-compact gauge group, which is indeed the case of $\mathrm{SL}(2,\mathbb{R})$, is that the irreducible irreps are in general part of a continuous spectrum, so one should replace a discrete summation with an integral $\sum_R \to \int \mathrm{d}R$. Then (2.170) simply becomes

$$Z_{\mathrm{BF+def}}(U,\beta) = \int \mathrm{d}R\ \rho(R)\ \chi_R(U)\ e^{-e\beta c_2(R)} \tag{2.171}$$

where $\rho(R)$ is the Plancherel measure that generalizes $\dim R$ in the continuos case. In particular, we should pick as our non-compact group $\mathrm{SL}(2,\mathbb{R})$, the gauge group relevant for JT gravity. The unitary irreducible representations of $\mathrm{SL}(2,\mathbb{R})$ have all been classified: there are both continuos and discrete series representations [24]. However, we know from the result deduced from the Schwarzian theory that only the continuos ones should contribute to the partition function. Moreover, the Plancherel measure for the principal continuos series of $\mathrm{SL}(2,\mathbb{R})$ is $\rho(k) = k\tanh(\pi k)$: this would imply a density of states $\propto \tanh \pi\sqrt{E}$. The latter does not display a Cardy rise at large energies, consistent with the semiclassical Bekenstein-Hawking entropy formula. So an $\mathrm{SL}(2,\mathbb{R})$ BF theory will not result in a correct calculation of the black hole entropy, as there are simply not enough states [12].

There are two possible resolutions for these puzzling features in the literature:

- the first proposal is based on an analytic continuation of the universal cover of $\mathrm{SL}(2,\mathbb{R})$ [74]. The parameter μ governing the extension is taken to infinity: in this regime we both recover the right Plancherel measure for the continuos series representations and the discrete series contributions gets exponentially suppressed. From a physical point of view, in the boundary particle approach to JT gravity (see A.6), this is equivalent to having a particle propagating in the hyperbolic plane under the effect of an infinite imaginary magnetic field.
- consider the positive semigroup $\mathrm{SL}^+(2,\mathbb{R})$, where one can find directly the correct Plancherel measure $\rho(k) = k\sinh(2\pi k)$ for the continuos series representation and there are no discrete ones appearing [12].

In this dissertation, we choose the follow the second root, because it provides an interesting physical interpretation. We explain in A.7 how restricting to the positive semigroup corresponds to projecting out all non-physical geometries with conical singularities in the bulk: these represent replicated geometry, with the spacetime boundary allowed to wind multiple times and self-intersect. This physical picture is very clear in the derivation of the JT gravity partition function through the boundary particle approach, in the limit of vanishing magnetic field, that we review in A.6.

The $\mathrm{SL}^+(2,\mathbb{R})$ quadratic Casimir evaluated on a spin $j = -\frac{1}{2} + ik, k \in \mathbb{R}$ representation is given by $c_2(j) = -j(j+1) = k^2 + \frac{1}{4}$, so it is convenient to perform a shift in the ground state of the Hamiltonian $c_2(R) \to c_2(R) - \frac{1}{4}$ in such a way that k^2 is directly the energy eigenvalue [25]. Inserting the $\mathrm{SL}^+(2,\mathbb{R})$ Plancherel measure in (2.171), we obtain

$$Z_{\mathrm{BF+def}}(U,\beta) = \int_0^{+\infty} \mathrm{d}k\ k\sinh(2\pi k)\ \chi_k(U_\lambda)\ e^{-e\beta k^2} \tag{2.172}$$

where we parametrized the holonomy as $U = e^{-2\pi\lambda H}$ with H is the Cartan generator of $\mathrm{SL}^+(2,\mathbb{R})$[26].

[24] In (A.7) we give a brief review of $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}^+(2,\mathbb{R})$ representation theory.
[25] The same shift is performed in the Hamiltonian of the boundary particle in the magnetic field in A.6.
[26] In our case $H = \mathrm{P}_2$

The $\mathrm{SL}^+(2,\mathbb{R})$ character for the representation labeled by k is given by

$$\chi_k(U_\lambda) = \frac{\cos(k\lambda)}{\sinh\lambda}$$

We are interested in considering the limit in which the total boundary holonomy $U \to \mathbb{1}$: this is achieved by sending $\lambda \to 0$. In this limit the character $\chi_k(U_\lambda)$, i.e. the sum over all states in each continuous series irrep labeled by k, becomes divergent but yet the divergence is indipendent of the representation [74]. Therefore, indicating with $\Xi = \lim_{\lambda\to 0}\chi_k(U_\lambda)$, this divergent factor Ξ can be brought outside the integral and be reabsorbed inside the overall proportionality constant. We are left with

$$Z_{\mathrm{BF+def}}(\beta) = \Xi \int_0^{+\infty} \mathrm{d}k\; k \sinh(2\pi k)\; e^{-e\beta k^2} \tag{2.173}$$

Performing the last integral, we finally obtain the exact partition function

$$Z_{\mathrm{BF+def}}(\beta) \propto \left(\frac{2\pi}{e\beta}\right)^{3/2} e^{\frac{2\pi^2}{e\beta}} \tag{2.174}$$

which, up to β-indipendent normalization factors, exactly matches with the result (2.42) obtained through quantization of the Schwarzian theory.

2.2.6 Wilson lines in JT gravity

The gauge reformulation of JT gravity is remarkable because it allows to compute amplitudes and correlators in the bulk in an exact way. Specifically, in this dissertation we are interested in particular observables on the gauge theory side, Wilson lines, that will prove to be equivalent to bilocal correlators in the bulk. Let use describe these objects.

When placing the theory on a topologically trivial manifold all Wilson lines and loops that do not intersect the defect are contractible and have trivial expectation values: in fact they are not dynamical because the hamiltonian of the system lives on the defect. We are thus only considering boundary anchored Wilson lines, that intersect the defect in two points τ_1 and τ_2, i.e by definition

$$\mathcal{W}_R(C_{\tau_1\tau_2}) = \chi_R\left(\mathcal{P}\ \exp \int_{C_{\tau_1\tau_2}} A\right) \tag{2.175}$$

where $C_{\tau_1\tau_2}$ is the underlying path along which the connection is parallel transported.

These boundary anchored Wilson lines have a precise gravitational interpretation: in fact, in the semiclassical limit, they represent the equivalent of the geodesics of a mass particle in the bulk. Furthermore, they are intrinsically quantum objects, because they do not take into account only the classical geodesics of the particle, but the full Feynman path integral as a sum over all possible paths weighted by the action of the particle on that trajectory [74], i.e.

$$\mathcal{W}_R(C_{\tau_1\tau_2}) = \int_{\mathrm{paths}\sim C_{\tau_1\tau_2}} [\mathcal{D}x]\, e^{-m\int_{C_{\tau_1\tau_2}} \mathrm{d}s\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} \tag{2.176}$$

Moreover, the mass m of the particle is determined by the representation R of the Wilson line, in particular it is related to its Casimir

$$m^2 = c_2(\ell) = \ell(1-\ell) \tag{2.177}$$

where we assume the Wilson line to be in the positive or negative discrete series irreducible representation of $\mathrm{SL}(2,\mathbb{R})$. This statement will be justified later.

The gravitational interpretation To compute the Wilson line expectation value we need to perform a path integral over A and ϕ in the theory (2.169), i.e.

$$\langle \mathcal{W}_R(C_{\tau_1\tau_2})\rangle = \int \mathcal{D}\phi\mathcal{D}A \; \mathcal{W}_R(C_{\tau_1\tau_2}) \; e^{-S_{\text{BF+def}}(\phi,A)} \tag{2.178}$$

By virtue of the equivalence between the second order metric formulation and the first order gauge formulation of JT gravity, the above should be equivalent to the gravitational path integral

$$\langle \mathcal{W}_R(C_{\tau_1\tau_2})\rangle = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\Phi \int_{\text{paths}\sim C_{\tau_1\tau_2}} [\mathcal{D}x]\, e^{-S_{\text{JT}}(g,\Phi)-m\int_{C_{\tau_1\tau_2}}\mathrm{d}s\sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} \tag{2.179}$$

This means that determining the expectation value of the boundary anchored Wilson line offers the possibility to compute the exact coupling to probe matter in JT gravity, including the interaction of the particle with a fluctuating quantum spacetime background.

Wilson line computation To compute the Wilson line expectation value, we can exploit the almost topological nature of the theory we are considering and employ a gluing procedure, very similar to the one used in [130] to compute the partition function of higher genus topologies.

In fact a boundary anchored Wilson line splits our original disk in two patches, which are in turn diffeomorphic to a disk. If no operator is inserted on the common arc shared by the two regions, integrating over the link leaves the partition function invariant:

$$\int \mathrm{d}U \; Z_{\text{BF+def}}(VU,\beta_1)\; Z_{\text{BF+def}}(U^\dagger W, el_2) = Z_{\text{BF+def}}\,(VW, e(\beta_1+\beta_2)) \tag{2.180}$$

where U is the holonomy running on the common interface and $\beta_1+\beta_2$ is just the total boundary length of the glued surface.

If instead a non-trivial character $\chi_R(h)$ is inserted on the Wilson line support, then the non-normalized expectation value of the observable $\mathcal{W}_R$ becomes [74]:

$$\langle \mathcal{W}_R(C_{\tau_1\tau_2})\rangle = \int \mathrm{d}h \; Z\,(g_1 h, e\tau_{21})\; \chi_R(h)\; Z\left(h^{-1}g_2, e\tau_{12}\right) \tag{2.181}$$

where $h = \mathcal{P}\ \exp \int_{C_{\tau_1\tau_2}} A$ is the holonomy along $C_{\tau_1\tau_2}$ and we have labeled with $\tau_{21} = \tau_2 - \tau_1$ the length of I enclosed by the first patch of the disk and $\tau_{21} = \beta - \tau_2 + \tau_1$ as the complementary length enclosed by the other patch. The total length of I is set to β: see Figure 2.9. Now, we want to set the total holonomy around the boundary to be trivial, $g_1 g_2 = \mathbb{1}$, and so for semplicity we impose $g_1 = g_2 = \mathbb{1}$. To obtain the normalize expectation value of the Wilson line we should also divide by the total partition function of the disk, i.e.

$$\langle \mathcal{W}_R(C_{\tau_1\tau_2})\rangle = \left(\frac{e\beta}{2\pi}\right)^{3/2} e^{-\frac{2\pi^2}{e\beta}} \int \mathrm{d}h \; Z\,(h, e\tau_{21})\; \chi_R(h)\; Z\left(h^{-1}, e\tau_{12}\right) \tag{2.182}$$

where we have plugged (2.174) in the second step. We assume the partition function of the two patches of the disk to be characterized by the principal unitary series representations of $\mathrm{SL}^+(2,\mathbb{R})$, while we consider the Wilson line defined on the discrete series representations of highest weight λ. We will shortly see this is necessary to reproduce the correct form of the

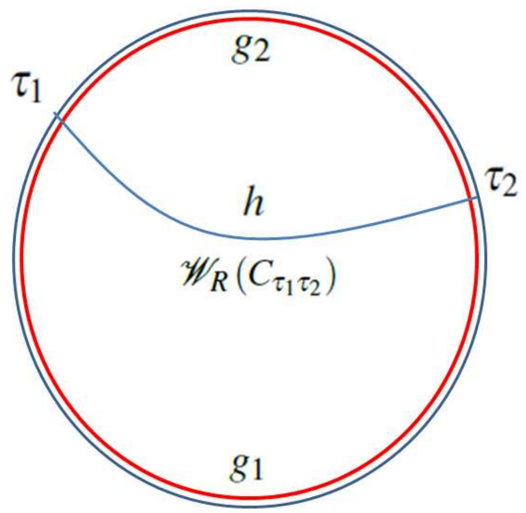


Figure 2.9: Red line corresponds to the loop defect of total length β which has been pushed infinitely close to the boundary of the Poincarè disk. The blue line crossing the disk corresponds to the boundary anchored Wilson line $\mathcal{W}_R(C_{\tau_1\tau_2})$.

bilocal correlator in the Schwarzian theory. Therefore sobstituting the expression (2.172) and indicating $\chi_R(h) \equiv \chi_\ell(h)$ we have

$$\langle \mathcal{W}_\ell(C_{\tau_1\tau_2})\rangle = \left(\frac{e\beta}{2\pi}\right)^{3/2} e^{-\frac{2\pi^2}{e\beta}} \int_0^{+\infty} ds_1 s_1 \sinh(2\pi s_1)\, e^{-e\tau s_1^2} \int_0^{+\infty} ds_2 s_2 \sinh(2\pi s_2)\, e^{-e(\beta-\tau)s_2^2} \times$$
$$\times \int \mathrm{d}h\ \chi_{s_1}(h)\chi_\ell(h)\chi_{s_2}(h^{-1}) \tag{2.183}$$

where we have denoted $\tau_{21} = \tau$ for semplicity. Let us concentrate on the last term. We enunciate the following definition

$$\int \mathrm{d}U\ \chi_{R_1}(U)\chi_R(U)\chi_{R_2}(U^{-1}) \equiv N^{R_2}_{R,R_1} \tag{2.184}$$

where $N^{R_2}_{R,R_1}$ are called the "fusion numbers", i.e. some coefficients given by the decomposition of a tensor product of two representations R and R_1 into a direct sum of irreducible representations R_2.

$$|R\rangle \otimes |R_1\rangle = \oplus_{R_2} N^{R_2}_{R,R_1}\ |R_2\rangle \tag{2.185}$$

The group integral gives a Clebsch-Gordan coefficient for R_2 in $R_1 \otimes R$. One can show that if R_1 and R_2 are principal unitary series representations of $\mathrm{SL}^+(2,\mathbb{R})$ labeled by $\lambda_1 = \frac{1}{2} + is_1$ and $\lambda_2 = \frac{1}{2} + is_2$ respectevely, and R is a discrete representation labeled by ℓ, then the correspondent fusion number is given by [27]

$$N^{R_2}_{R,R_1} \equiv N^{s_2}_{s_1,\ell} = \frac{\Gamma(\ell \pm is_1 \pm is_2)}{\Gamma(2\ell)} \tag{2.186}$$

The relevant group integral to be evaluated is [11]

$$\int_{-\infty}^{+\infty} \mathrm{d}\phi e^{2\phi} R_{s_1}(\phi) R_{\ell,0,0}(\phi) R_{s_2}(\phi) \tag{2.187}$$

where $\mathrm{d}\phi e^{2\phi}$ comes from the Haar measure, $R_{s_1}(\phi) \equiv R_{s_1,\lambda,\nu}(\phi)$ corresponds to the mixed parabolic matrix element of $\mathrm{SL}^+(2,\mathbb{R})$ (see (A.63)), subjected to the Schwarzian contraint $\nu = \lambda = \sqrt{\mu}$. The latter leads to restrict the Wilson line endpoints on the boundary to $J^+ = 0$, meaning that J^+ is conserved. Inserting so the relevant eigenfunctions, we get [28]

$$\int_{-\infty}^{+\infty} \mathrm{d}\phi e^{2\ell\phi} K_{2is_1}\left(\sqrt{\mu}e^{\phi}\right) K_{2is_2}\left(\sqrt{\mu}e^{\phi}\right) = \frac{\Gamma(\lambda \pm is_1 \pm is_2)}{\Gamma(2\lambda)} \tag{2.188}$$

Performing the group integral, exactly gives the correct 3j symbol presented above. We notice that this construction fails for continuous irrep Wilson lines: the limit $\lambda,\nu \to 0$ is ill-defined for $j = -\frac{1}{2} + ik$, because the Bessel function oscillates erratically near the origin [11]. As such, the integral (2.188) only converges for $j = \ell$ integer, justifying why we are restricted to considering discrete rep Wilson lines as operators in JT gravity. [29]

So the exact evaluation of the normalized Wilson line $\langle \mathcal{W}_\ell(C_{0,\tau})\rangle \equiv \langle \mathcal{W}_\ell \rangle$ finally yields

$$\langle \mathcal{W}_\ell \rangle = \left(\frac{e\beta}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{2\pi^2}{e\beta}} \int_0^{+\infty} \mathrm{d}s_1\, \mathrm{d}s_2\; s_1 \sinh(2\pi s_1)\, s_2 \sinh(2\pi s_2)\; \frac{\Gamma(\ell \pm is_1 \pm is_2)}{\Gamma(2\ell)}\; e^{-e\tau s_1^2 - e(\beta-\tau)s_2^2} \tag{2.189}$$

Using the correspondence $e = 2/C$ and setting $C = 1$, the result agrees precisely with the computation performed in [95] through conformal bootstrap.

Wilson lines as bilocal correlators in the Schwarzian theory As a final statement, in this section we will probe the Wilson line definition exactly reproduces the form of a bilocal correlator in the Schwarzian theory [11]. In particular this is a Wilson line in the lowest weight state of a discrete $j = l$ representation of $\mathrm{SL}^+(2,\mathbb{R})$. [30]

Since the disk is a simply connected manifold, every flat gauge connection, solution to $F = 0$ [31], is gauge-equivalent to $A = 0$. This means that every flat connection can be parametrized as a pure-gauge as

$$A_\tau = g^{-1}\mathrm{d}g \tag{2.190}$$

with g a element of $\mathrm{SL}(2,\mathbb{R})$, $g(\tau+\beta) = g(\tau)$. The boundary value of the guage field is constrained by the gravitational boundary conditions

$$A_\tau|_{\partial\Sigma} = \begin{pmatrix} 0 & -\frac{1}{2}T(\tau) \\ 1 & 0 \end{pmatrix} \tag{2.191}$$

[27] This is known as $3j$ symbol and is equal to $\Gamma(x \pm y \pm z) = \Gamma(x+y+z)\Gamma(x+y-z)\Gamma(x-y+z)\Gamma(x-y-z)$

[28] $R_{\ell,0,0}$ can be computed as $R_{\ell,0,0} = \lim_{\epsilon\to 0} e^{-\phi} J_{2\ell-1}\left(\epsilon e^{-\phi}\right)$ starting from the $\mathrm{SL}^+(2,\mathbb{R})$ discrete series matrix element of mixed parabolic type.

[29] This separation of states (continuous irreps) and operators (discrete reps) happens naturally for the Schwarzian once we impose the relevant boundary conditions.

[30] This representation is identical to a discrete representation of $\mathrm{SL}(2,\mathbb{R})$.

[31] after integrating out ϕ.

which descend from the Brown-Henneaux boundary conditions in 3d-gravity via dimensional reduction. We use the following parametrization for the boundary $\mathrm{SL}(2,\mathbb{R})$ group element and the gravitational boundary conditions:

$$g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}T(\tau) \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \tag{2.192}$$

from which we obtain the following equations

$$\begin{aligned} A'' + \frac{1}{2}T(\tau)A = 0 \qquad B = A' \\ C'' + \frac{1}{2}T(\tau)C = 0 \qquad D = C' \end{aligned} \tag{2.193}$$

together with the $\mathrm{SL}(2,\mathbb{R})$ constraint $AC' - A'C = 1$. Up to permutations, there is a unique solution to this system

$$A = \frac{1}{\sqrt{f'}} \qquad C = \frac{f}{\sqrt{f'}} \tag{2.194}$$

with f the solution of $\{f,\tau\} = T(\tau)$. We now write g^{-1} using the Gauss decomposition

$$g^{-1} = \begin{pmatrix} \frac{1}{\sqrt{f'}} & -\frac{1}{2\sqrt{f'}}\frac{f''}{f'} \\ \frac{f}{\sqrt{f'}} & -\frac{f}{2\sqrt{f'}}\frac{f''}{f'} + \sqrt{f'} \end{pmatrix} = e^{\gamma_- J_-} e^{2\phi J_0} e^{i\gamma_+ J_+} \tag{2.195}$$

and we hence identify

$$\gamma_- = f \qquad e^{-\phi} = \frac{1}{\sqrt{f'}} \qquad \gamma_+ = -\frac{1}{2}\frac{f''}{f'} \tag{2.196}$$

In the finite-dimensional spin-j representation, with states $|\ell\rangle$, $|l-1\rangle, \cdots, |-l\rangle$, we have the following action of the generators:

$$J_0\,|m\rangle = m\,|m\rangle \qquad J_\pm\,|m\rangle = (\ell \pm m + 1)\,|m \pm 1\rangle \tag{2.197}$$

We now evaluate explicitly the Wilson line in terms of the group element g

$$\mathcal{P}\exp\int_{\tau_1}^{\tau_2} A(z)\mathrm{d}z = g(\tau_2)g^{-1}(\tau_1) \tag{2.198}$$

One can then compute the following matrix element $\langle -\ell|\, g(\tau_2)g^{-1}(\tau_1)\,|\ell\rangle$. We first compute directly [32]

$$g^{-1}(\tau_1)\,|\ell\rangle = e^{\gamma_-(\tau_1)J_-} e^{2\phi(\tau_1)J_0} e^{i\gamma_+(\tau_1)J_+}\,|\ell\rangle = e^{2\ell\phi(\tau_1)} \sum_{n=0}^{2\ell} \gamma_-(\tau_1)^n\,|\ell - n\rangle \tag{2.199}$$

while

$$\langle -\ell|\, g(\tau_2) = e^{2\ell\phi(\tau_2)} \sum_{n=0}^{2\ell} \binom{2\ell}{n} (-\gamma_-(\tau_2))^n \,\langle -\ell + n| \tag{2.200}$$

[32]Using that $J_\pm^n\,|m\rangle = \frac{(\ell\pm m+n)!}{(\ell\pm m)}\,|m \pm n\rangle$.

Exploiting orthogonality, we finally obtain

$$\begin{aligned}\langle -\ell | \, g(\tau_2) g^{-1}(\tau_1) \, |\ell\rangle &= \left[e^{\phi(\tau_1)+\phi(\tau_2)} \left(\gamma_-(\tau_1) - \gamma_-(\tau_2)\right)\right]^{2\ell} \\ &= \left[\frac{\left(f(\tau_1) - f(\tau_2)\right)^2}{f'(\tau_1) f'(\tau_2)} \right]^{\ell}\end{aligned} \tag{2.201}$$

where we used (2.196). We now generalize to the infinite-dimensional lowest/highest-weight representations by setting $\ell = -h$ where $h > 0$. The state $|\ell\rangle_\ell$ is still annihilated by the corresponding J_+, but is never annihilated by powers of J_-. The Wilson line in terms of h is therefore

$$\mathcal{W}_h(\tau_1, \tau_2) = \left[\frac{f'(\tau_1) f'(\tau_2)}{\left(f(\tau_1) - f(\tau_2)\right)^2} \right]^h \tag{2.202}$$

which exactly corresponds to the bilocal correlator in the Schwarzian theory we computed in (2.58) when performing a double derivative with respect to the source and setting $h = \Delta$.

Wilson line on the trumpet When exposing higher topologies in JT gravity, we have already described the trumpet configuration and argued its partition function is still one-loop exact. The additional geodesic boundary can be taken into account by considering a disk with a hyperbolic defect in the bulk [92]. The partition function of JT gravity on the trumpet is given by the integral of a slightly modified spectral density, namely

$$\mathcal{Z}_{\text{trump.}}(\beta) = \int_0^{+\infty} \mathrm{d}s \, \cos(2\pi b s) \, e^{-\kappa\beta s^2} = \left(\frac{\pi}{\kappa\beta}\right)^{\frac{1}{2}} e^{-\frac{b^2\pi^2}{\beta\kappa}} \tag{2.203}$$

Now, one can ask what is the exact form of a Wilson line on the trumpet configuration. Following the same logic that led to (2.189) in the case of the disk, we can write down the expectation value of a boundary anchored Wilson line also in this case. Since this type of path will split the trumpet into two regions, homeomorphic to a disk and a trumpet, we can easily show that

$$\langle \mathcal{W}_\lambda(\tau) \rangle_{\text{trump.}} = \mathcal{N}_t \int_0^\infty \int_0^\infty \mathrm{d}s_1 \mathrm{d}s_2 \, s_1 \sinh(2\pi s_1) \cos(2\pi b s_2) \, \frac{\Gamma(\lambda \pm i s_1 \pm i s_2)}{\Gamma(2\lambda)} \; e^{-\kappa\tau s_1^2 - \kappa(\beta-\tau) s_2^2}. \tag{2.204}$$

with normalization $\mathcal{N}_t \equiv \frac{\kappa^{2\lambda}}{2\pi} \mathcal{Z}^{-1}_{\text{trumpet}} = \kappa^{2\lambda} \left(\frac{\kappa\beta}{\pi}\right)^{\frac{1}{2}} e^{\frac{b^2\pi^2}{\beta\kappa}}$. From the point of view of anchored Wilson loops, this correlator describes bi-local lines not winding around the defects. It was observed in [92] that this observable could not arise from free matter in the bulk since it does not satisfy the KMS condition (which is equivalent to periodicity around the boundary circle). Therefore they generalized the bi-local operator to satisfy the KMS condition, including an explicit sum over integers in its definition. Computing the correlators with the improved operator is equivalent to sum over Wilson lines encircling the defect, with fixed anchored points. Self-intersections naturally appear for non-trivial windings with the associated $6j$-symbols, complicating the evaluation of the two-point function.

2.3 $T\overline{T}$ deformation of QFTs/CFTs

In order to properly introduce the concept of $T\overline{T}$-deformation, we make a brief digression to motivate the importante of this particular type of irrelevant deformation.

2.3.1 The relevance of being irrelevant

In the context of Wilson's Renormalization Group Theory, the usual renormalization prescription consists in starting from an high energy Lagrangian and integrate out high energies degrees of freedom in the Fourier space, rescaling the momenta and formally rewriting the theory in the new set of variables. As a result, we can define a flow in the space of theories, which starts from an high energy Lagrangian and flows to a lower energy formulation. At some point, the theory stabilizes in a fixed point. The simpler type of fixed point is the gaussian one, which corresponds to a non-interacting theory. We know that we can add linear perturbations around a gaussian fixed point. Such perturbations are described by local operators which interact with the Lagrangian at the fixed point with a given coupling constant. It is important to remember that the operators are classified in this way:

- Relevant operators flow away from the fixed point, since their coupling constant diverges at low energy scales.
- Irrelevant operators do not usually induce modifications to the flow, since their coupling constant is suppressed at low energy scales.
- Marginal operators have a coupling constant which does not scale at the first order approximation, therefore a better analysis is needed to understand their behavior.

If we follow the usual RG flow, it is customary to focus on relevant operators, indeed they modify the IR theory and the renormalization procedure is mathematically consistent. On the other hand, irrelevant operators are more problematic. Firstly, we notice that they give physical effects only in the UV regime. If we try to follow the irrelevant deformation at high energies we usually run into a Landau pole (i.e. a divergence). Therefore, in order to study an irrelevant operator at high energies, it is usually required to define an infinite amount of counter-terms when we apply the renormalization procedure. This also implies that we should fix an infinite amount of experimental data which are not at our disposal. These problems may be interpreted in a physical intuitive motivation: It is unlikely that we can recover the physical description of an high energy system starting from a system with less degrees of freedom.

Now an important question is whether all irrelevant deformations[33] are ill-defined. At this stage the concept of $T\overline{T}$-deformation enters in the story, indeed this theory was historically introduced by Zamolodchikov [135] as an example of integrable irrelevant deformation. In a certain sense, exploiting integrability, we can define and derive an infinite tower of conserved quantities, which are used to fix the experimental data along the flow. This suggests that a $T\overline{T}$-deformed theory is UV complete, allowing to move consistently against the RG flow and explore non-trivial UV dynamics (see Figure 2.10).

Beyond the possibility of having a better understanding of the space of integrable field theories [118], the topic has a wide range of physical properties as we will see.

[33]The coupling constant is of dimension m^{-2}

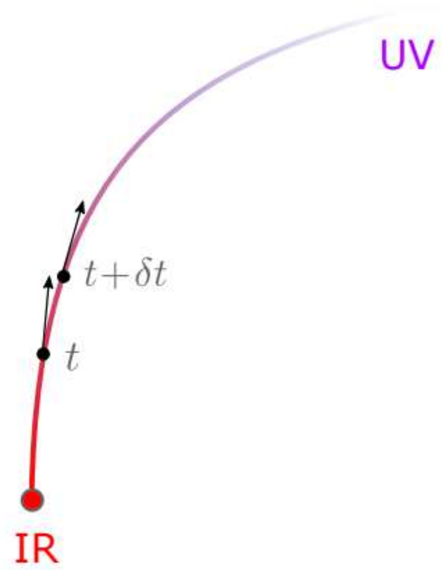


Figure 2.10: Thanks to integrability, the $T\bar{T}$ allows to move in the opposite direction, starting from a IR fixed point and flowing to UV.

Therefore, now that we have hopefully understood why $T\overline{T}$-deformation is so special, we can give its formal definition at first order around the fixed point:

$$\mathscr{L}^{(\mu+\delta\mu)} - \mathscr{L}^{(\mu)} = -\delta\mu \, T\overline{T}^{(\mu)} \tag{2.205}$$

where μ is the deformation coupling constant. Clearly, we can iteratively apply (2.205) and give a formulation ([19], [25]) at the level of Lagrangian or Hamiltonian flow:

$$\partial_\mu \mathscr{L}^{(\mu)} = T\overline{T}^{(\mu)} \quad , \quad T^{(\mu)}_{\alpha\beta} = -\frac{2}{\sqrt{g}} \frac{\delta S^{(\mu)}}{\delta g^{\alpha\beta}} \tag{2.206}$$

Where $\mathscr{L}$ is the Lagrangian, and $g^{\alpha\beta}$ it the two-dimensional metric and S is the action. For simple theories ([8], [13]) it is possible directly apply (2.206) and to study the effects of $T\overline{T}$-deformation[34].

2.3.2 Definition of the $T\overline{T}$ operator

In this section we properly define the $T\overline{T}$ operator, mainly following [135], [118], ,[77].

Hypothesis and basic definitions

Let's consider a generic two dimensional Quantum Field Theory or Conformal Field Theory, defined on an Euclidean space $\mathbb{R}^2$. Let's assume that:

- 1) The theory is local. Then we can describe the theory in terms of a Lagrangian made of local operators.

[34] As an example, if we start from a 2d non-interacting free boson and follow the lagrangian flow driven by $T\overline{T}$, we are surprisingly lead to the Nambu-Goto string action. This is a first suggestion that $T\overline{T}$ is somehow linked to a non-local UV completion of IR theories, indeed string theory is intrinsically non-local.

- 2) The theory has global translational symmetry. In other words, we assume that, for each local operator $\mathcal{O}_i(z)$, the expectation value does not depend on spatial coordinates: $\partial_z \langle \mathcal{O}_i(z) \rangle = 0$. This also has implications at the level of two point correlation functions: $\langle \mathcal{O}_i(z)\mathcal{O}_j(z') \rangle = G_{ij}(z-z')$. Where $G_{ij}(z-z')$ depends only on the distance between points.

- 3) Moreover, if local translational symmetry holds, we can always define a conserved Noether current: the energy-momentum tensor $T_{\alpha,\beta}$, satisfying the property: $\partial^\alpha T_{\alpha,\beta} = 0$. Clearly, $T_{\alpha,\beta}$ is a 2×2 matrix whose entries are operators, made of the fields of the theory.

- 4) Finally, if Lorentz invariance holds, the stress energy tensor is symmetric: $T_{\alpha,\beta} = T_{\beta,\alpha}$. The above conditions are needed to build the formal construction of the $T\overline{T}$ operator and the corresponding deformation.

We denote x,y the euclidean coordinates and then we introduce the complex conjugated variables: $(\mathrm{z},\overline{\mathrm{z}}) = (\mathrm{x}+i\mathrm{y},\mathrm{x}-i\mathrm{y})$. The components of the stress-energy tensor are related, in the two basis, in the following way:

$$T_{\mathrm{zz}} = \frac{1}{4}\left(T_{xx} - T_{yy} - 2\mathrm{i}T_{xy}\right), \tag{2.207}$$

$$T_{\overline{\mathrm{z}}\overline{\mathrm{z}}} = \frac{1}{4}\left(T_{xx} - T_{yy} + 2\mathrm{i}T_{xy}\right), \tag{2.208}$$

$$T_{\mathrm{z}\overline{\mathrm{z}}} = \frac{1}{4}\left(T_{xx} + T_{yy}\right). \tag{2.209}$$

We can also introduce the chiral components of the 2d stress-energy tensor, according to CFT conventions:

$$T = -2\pi T_{\mathrm{zz}}, \quad \bar{T} = -2\pi T_{\overline{\mathrm{z}}\overline{\mathrm{z}}}, \quad \Theta = 2\pi T_{\mathrm{z}\overline{\mathrm{z}}}$$

The continuity equation for the energy-momentum tensor in this basis reads:

$$\partial_{\overline{\mathrm{z}}} T(z) = \partial_{\mathrm{z}} \Theta(z) \tag{2.210}$$

$$\partial_{\mathrm{z}} \bar{T}(z) = \partial_{\overline{\mathrm{z}}} \Theta(z) \tag{2.211}$$

The $\hat{\mathcal{C}}_{T\overline{T}}$ operator

Now we focus on the following product of operators denoted as $\hat{\mathcal{C}}_{T\overline{T}}$ and depending on two different position z and z':

$$\hat{\mathcal{C}}_{T\overline{T}}(z,z') := T(z)\bar{T}\left(z'\right) - \Theta(z)\Theta\left(z'\right)$$

Actually $\hat{\mathcal{C}}_{T\overline{T}}(z,z')$ is a composite operator and it can be pictorially remembered as the determinant of the energy momentum tensor in complex basis, up to a multiplication factor[35]. The remarkable property of $\hat{\mathcal{C}}_{T\overline{T}}$ is that it has a nice operator product expansion (OPE):

$$\hat{\mathcal{C}}_{T\overline{T}}(z,z') = \mathcal{O}_{T\bar{T}}\left(z'\right) + \text{ derivative terms} \tag{2.212}$$

[35] However, this definition is clearly not rigorous, since we are not giving indications on the spatial dependence z and z'. The right statement is the following: $\hat{\mathcal{C}}(z,z') \propto -\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}T_{\alpha\gamma}(z)T_{\beta\delta}(z')$. Where $T_{\alpha\gamma}$ is the energy-momentum tensor in the complex basis.

Where $\mathcal{O}_{T\bar{T}}(z')$ is a local operator. Let us prove this property starting from the general hypothesis. We start by computing the derivatives of the $\hat{\mathcal{C}}_{T\overline{T}}$ operator, using properties (2.210) and (2.211):

$$\partial_{\bar{z}}\hat{\mathcal{C}}_{T\overline{T}}(z,z') = \partial_{\bar{z}}\left(T(z)\bar{T}\left(z'\right) - \Theta(z)\Theta\left(z'\right)\right) = \tag{2.213}$$
$$= \quad (\partial_{\mathrm{z}} + \partial_{\mathrm{z}'})\,\Theta(z)\bar{T}\left(z'\right) - (\partial_{\overline{\mathrm{z}}} + \partial_{\overline{\mathrm{z}}'})\,\Theta(z)\Theta\left(z'\right)$$

$$\partial_{z}\hat{\mathcal{C}}_{T\overline{T}}(z,z') = \partial_{z}\left(T(z)\bar{T}\left(z'\right) - \Theta(z)\Theta\left(z'\right)\right) = \tag{2.214}$$
$$= (\partial_{\mathrm{z}} + \partial_{\mathrm{z}'})\,T(z)\bar{T}\left(z'\right) - (\partial_{\overline{\mathrm{z}}} + \partial_{\bar{z}'})\,T(z)\Theta\left(z'\right)$$

In order to have a better intuition of what is going on, we can introduce the operator product expansion, in the following way:

$$\Theta(z)\bar{T}\left(z'\right) = \sum_i B_i\left(z-z'\right)\mathcal{O}_i\left(z'\right)$$
$$T(z)\Theta\left(z'\right) = \sum_i A_i\left(z-z'\right)\mathcal{O}_i\left(z'\right)$$
$$T(z)\bar{T}\left(z'\right) = \sum_i D_i\left(z-z'\right)\mathcal{O}_i\left(z'\right)$$
$$\Theta(z)\Theta\left(z'\right) = \sum_i C_i\left(z-z'\right)\mathcal{O}_i\left(z'\right)$$

The sum involves generic local fields. We can also write the product operator expansion for $\hat{\mathcal{C}}_{T\overline{T}}(z,z')$, in the following way:

$$\hat{\mathcal{C}}_{T\overline{T}}(z,z') = T(z)\bar{T}\left(z'\right) - \Theta(z)\Theta\left(z'\right) = \sum_i F_i\left(z-z'\right)\mathcal{O}_i\left(z'\right)$$

where:

$$F_i\left(z-z'\right) = D_i\left(z-z'\right) - C_i\left(z-z'\right)$$

Then, the derivatives (2.213) (2.214) can be written as:

$$\partial_{\bar{z}}\hat{\mathcal{C}}_{T\overline{T}}(z,z') = \sum_i \partial_{\overline{\mathrm{z}}}F_i\left(z-z'\right)\mathcal{O}_i\left(z'\right) = \tag{2.215}$$
$$= \sum_i\left(B_i\left(z-z'\right)\partial_{\mathrm{z}'}\mathcal{O}_i\left(z'\right) - C_i\left(z-z'\right)\partial_{\bar{z}'}\mathcal{O}_i\left(z'\right)\right)$$

$$\partial_{z}\hat{\mathcal{C}}_{T\overline{T}}(z,z') = \sum_i \partial_{\mathrm{z}}F_i\left(z-z'\right)\mathcal{O}_i\left(z'\right) = \tag{2.216}$$
$$= \sum_i\left(D_i\left(z-z'\right)\partial_{\mathrm{z}'}\mathcal{O}_i\left(z'\right) - A_i\left(z-z'\right)\partial_{\bar{z}'}\mathcal{O}_i\left(z'\right)\right)$$

Therefore, we can state that a generic derivative of $\hat{\mathcal{C}}_{T\overline{T}}$ has an OPE which is only made of derivatives operators. This statement is needed to prove that the OPE of $\hat{\mathcal{C}}_{T\overline{T}}$ contains terms that must obey one of the following conditions:

- 1) $\mathcal{O}_i(z)$ is a local operator dependent on z, and it multiplies coefficient $F_i(z-z')$ which actually does not depend on z nor z'.

- 2) $\mathcal{O}_i(z)$ is a derivative operator.

We can prove the previous statement by contradiction [?]: let's assume that in the OPE of $\hat{\mathcal{C}}_{T\overline{T}}$ there is a term $\tilde{F}(z-z')\ \tilde{\mathcal{O}}(z)$ where $\mathcal{O}$ is not a derivative and $\tilde{F}$ is non vanishing. Then it follows that:

$$\partial_z \hat{\mathcal{C}}_{T\overline{T}}(z,z') = \partial_z \left[\tilde{F}\left(z-z'\right)\ \tilde{\mathcal{O}}(z)\right] + \dots = \partial_z \left[\tilde{F}\left(z-z'\right)\right]\ \tilde{\mathcal{O}}(z) + \dots \tag{2.217}$$

Clearly the previous expression (2.217) is not compatible with the general expression (2.216) for the derivative of $\hat{\mathcal{C}}_{T\overline{T}}$, since it contains an operator which is not a derivative. Clearly this is a contradiction, and the only non-derivative local operator which appears in the OPE should respect condition 1). Therefore, as promised, we can conclude that the $\hat{\mathcal{C}}_{T\overline{T}}$ operator has the particular OPE shown in (2.212).

The $T\overline{T}$ operator

Now we can define the $T\overline{T}$ operator taking advantage of (2.212), which can be schematically written as:

$$T\overline{T}(z) = \lim_{z'\to z}\left(\hat{\mathcal{C}}_{T\overline{T}}(z,z')\right) - \text{derivative operators} \tag{2.218}$$

This was the so called point-splitting definition of $T\overline{T}$: instead of starting from an ill-defined determinant of the stress-energy tensor $\det T_{\alpha\beta}(z)$, we start from a product of its components ($\equiv$ local fields) at different spatial coordinates z and z'. Then, morally speaking, we can state that $T\overline{T}$ is obtained from the determinant of the stress-energy tensor, applying a limiting procedure $z \to z'$ on the spatial coordinates and omitting the derivative operators[36]. Such operators are not important, since in QFTs we are interested in expectations values and the derivative of expectation values must vanish due to the translational invariance of the theory. Therefore the physical content is only located in $T\overline{T}$, which is a non-vanishing observable.
Now we can also make a comment on the historical origin of the name $T\overline{T}$. Originally, the $T\overline{T}$-deformation was studied by Zamolodchikov in the context of CFTs. As we know, in such theories the stress-energy tensor is traceless, due to conformal invariance. Therefore $\Theta \propto T_{z\bar{z}} = 0$ and the operator $\hat{\mathcal{C}}_{T\overline{T}}$ reduces to $T(z)\bar{T}(z')$; this means that in CFTs:

$$T\overline{T}(z) = \lim_{z'\to z}\left(T(z)\bar{T}\left(z'\right)\right) - \text{derivative operators} \tag{2.219}$$

[36]However, as we will show, in the case of two dimensional Yang-Mills theory, the energy-momentum tensor does not depend on the position, therefore, there are not divergencies in the derivative operators appearing in the OPE of $\hat{\mathcal{C}}_{T\overline{T}}$.

The Burger's equation

A very convenient framework to study $T\overline{T}$-deformation consists in considering finite-volume theories. For example, we can define an (undeformed) QFT on a finite cylinder[37] with radius R. It is possible to study the spectrum of the corresponding $T\overline{T}$ starting from the Zamolodchikov's factorization formula [135]:

$$\langle n|T\bar{T}|n\rangle = \langle n|T|n\rangle\langle n|\bar{T}|n\rangle - \langle n|\Theta|n\rangle\langle n|\Theta|n\rangle \tag{2.220}$$

where $|n\rangle$ is an eigenvalue of the Hamiltonian of the theory. The previous equation will lead to a useful formulation of the $T\overline{T}$-deformed theory in terms of a flow equation involving the observables of the undeformed theory. The proof of (2.220) consists in three steps:

- 1) Prove that the expectation value $\langle\hat{\mathcal{C}}_{T\overline{T}}(z,w)\rangle$ is a constant.
- 2) Factorize the products of operators.
- 3) Compute the pinching limit of the factorized expression.

In order to prove 1) we can take a derivative of the expectation value and use the continuity equation (2.210) and translational invariance:

$$\begin{aligned}\partial_{\overline{z}}\langle\hat{\mathcal{C}}_{T\overline{T}}(z,w)\rangle &= \left\langle\partial_{\overline{z}}T(z)\bar{T}(w)\right\rangle - \langle\partial_{\overline{z}}\Theta(z)\Theta(w)\rangle = \\ &= \left\langle\partial_{z}\Theta(z)\bar{T}(w)\right\rangle + \langle\Theta(z)\partial_{\overline{w}}\Theta(w)\rangle = \\ &= -\left\langle\Theta(z)\partial_{w}\bar{T}(w)\right\rangle + \langle\Theta(z)\partial_{\overline{w}}\Theta(w)\rangle = 0\end{aligned}$$

A similar result holds for $\bar{z}$, and the statement is proven.
In order to prove 2) it is sufficient to notice that the expectation value does not depend on the distance between z and w, therefore, we can send one of the two point at infinite distance and apply the factorization theorem, then we have that:

$$\langle\hat{\mathcal{C}}_{T\overline{T}}(z,w)\rangle = \langle T(z)\rangle\left\langle\overline{T}(w)\right\rangle - \langle\Theta(z)\rangle\langle\Theta(w)\rangle$$

The computation 3) consist simply in substituting $w \to z$, using again the fact that the expectation value does not depend on the positions, we have therefore proven[38] (2.220).

Since the result (2.220) is based on Lorentz invariance and conservation of the stress-energy tensor, it is clear that it holds along all the flow. Therefore, we can try to translate the formula in the language of a differential equation which relates the deformed solution to the undeformed one. We start from the third formula in (2.206) and we introduce the eigenvalues of the Hamiltonian by inserting the operators in a bracket:

$$\langle n|\partial_\mu\mathscr{H}^{(\mu)}|n\rangle = \langle n|T\overline{T}^{(\mu)}|n\rangle \tag{2.221}$$

Moreover, we can apply the factorization formula (2.220):

[37]In the case of 2d Yang-Mills it is possible to extend this property to other topologies, by the procedure of gluing cylinders.

[38]Actually the proof restricts to the vacuum expectation value, but can be generalized to generic states.

$$\langle n|\partial_\mu \mathscr{H}^{(\mu)}|n\rangle = \langle n|T|n\rangle\langle n|\bar{T}|n\rangle - \langle n|\Theta|n\rangle\langle n|\Theta|n\rangle \tag{2.222}$$

In a QFT on a cylinder of radius R the (adimensional) eigenvalues of energy ($\mathcal{E}_n$) and momentum (P_n) are related to the stress energy tensor in the following way:

$$\frac{\mathcal{E}_n(R,t)}{R} = -\langle n|T|n\rangle, \quad \partial_R \mathcal{E}_n(R,t) = -\left\langle n\middle|\overline{T}\middle|n\right\rangle, \quad P_n = -\mathrm{i}R\langle n|\Theta|n\rangle. \tag{2.223}$$

Therefore we conclude that the following flow equation holds[39]:

$$\partial_\mu \mathcal{E}_n(R,\mu) = \mathcal{E}_n(R,\mu)\partial_R \mathcal{E}_n(R,\mu) + \frac{1}{R}P_n^2(R) \tag{2.224}$$

Clearly, this is a nonlinear partial differential equation and it is known in literature as inviscid Burgers equation.

2.3.3 $T\bar{T}$ deformation of JT gravity

One of the most interasting aspects of the $T\bar{T}$-deformation is its deep connection with gravity. For instance, it was shown that deforming a theory with the $T\bar{T}$ operator is equivalent to coupling it with random geometries [17] of with JT gravity its self [37]. However most interesting connection with gravity, for the purposes of this dissertation, is indeed the intepretation of $T\bar{T}$ in the context of the holographic principle. We will review it in this section.

Deformation of $\mathrm{AdS}_3/\mathrm{CFT}_2$ holographic duality

In [89] it was claimed that the dual of a finite patch of an asymptotically-AdS_3 spacetime is given by a $T\bar{T}$-deformed CFT_2. Three-dimensional gravity action with a boundary condition is obtained by coupling the standard Eistein-Hilbert term to a Gibbons-Hawking-York term:

$$S = -\frac{1}{16\pi G}\int_M \mathrm{d}^3x\sqrt{g}\left(R+2\ell^{-2}\right) - \frac{1}{8\pi G}\int_{\partial M} \mathrm{d}^2x\sqrt{h}\left(K-\ell^{-1}\right) \tag{2.225}$$

It is always possible to "diagonalize" the metric and consider a set (r,x_i,x_j) [40] of coordinates in the 3d space, such that:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + g_{ij}(x,r)\mathrm{d}x^i\,\mathrm{d}x^j$$

Therefore, at the boundary r is fixed and we can write the extrinsic curvature as a derivative of g_{ij}, i.e. $K_{ij} = \frac{1}{2}\partial_r g_{ij}$. If we set $l=1$, the Einstein equation in the 3d space, is just

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - g_{\mu\nu} = 0$$

[39]Here we are omitting the fact that, in principle, the deformation includes also contributions coming from derivatives of operators. However, also a more rigorous derivation is possible: it consists in computing the partition function on the torus and then extract the spectrum from it.

[40]We will use (r,ϕ,τ) coordinates in the following.

and can then be rewritten in terms of the extrinsic curvature.

It is also possible to reexpress the 3d action in terms of K:

$$S = -\frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R^{(2)} + K^2 - K^{ij}K_{ij} + 2\right) + \frac{1}{8\pi G}\int_{\partial M} d^2x\sqrt{h} \tag{2.226}$$

This expression is useful to compute the variation of the quasi local action on the boundary. We need to vary the action (2.226) on a shell of fixed radius: instead of the usual practice in AdS/CFT to send the boundary to asymptotic infinity, we push it at a finite distance $r = r_c$. On the other hand, we can write the same variation in terms of the stress-energy tensor at the boundary:

$$\delta S = \frac{1}{4\pi}\int d^2x\sqrt{h}T^{ij}\delta h_{ij} \tag{2.227}$$

Combining (2.227) and the variation of (2.226), we obtain the expression of the 2D stress-energy tensor:

$$T_{ij} = \frac{1}{4G}\left(K_{ij} - Kg_{ij} + g_{ij}\right) \tag{2.228}$$

Now the important point is that in AdS/CFT it is customary to identify the quasi-local stress energy tensor to the stress energy tensor of the corresponding CFT. Clearly T_{ij} can be used to construct the $T\overline{T}$ operator, as usual: $T\cdot\overline{T} - \Theta^2$.
Specifically, the Θ term reads as:

$$\Theta = T_{ij}g^{ij} = \frac{1}{4G}(K - 2K + 2) = \frac{1}{4G}(2 - K) \tag{2.229}$$

Once we impose the $E^r_r = 0$ Einstein equation, the $T\overline{T}$ operator reduces instead to:

$$T\overline{T} = -\frac{1}{64G^2}(2 - K) - \frac{R^{(2)}}{128G^2} \tag{2.230}$$

If we consider a flat metric at the boundary, the curvature $R^{(2)}$ is null, therefore comparing (2.230) and (2.229), we can state that the Einstein equations imply the following identity, which is known as trace flow equation:

$$\Theta = -16GT\overline{T} \tag{2.231}$$

Now we want to prove that the same equation arises in the rather different context of $T\overline{T}$ deformed CFTs. As we know, at the level of action, the flow acts as:

$$\frac{dS(\mu)}{d\mu} = \int d^2x\sqrt{g}T\overline{T} \tag{2.232}$$

The ingredient of the proof is to exploit the fact that the theory is conformal, therefore, if we rescale the action the only contribution comes from the trace of the energy-momentum tensor, implying once again the trace flow equation, provided that we identify $\mu = 8G$. We should interpret the previous result as a strong indication that the underlying physics is the same, and the deformation preserves the duality. Although this is not a rigorous proof, at least at a classical level the Einstein equations of motion imply the same physics which arises in the context of $T\overline{T}$.

$T\bar{T}$ deformation of the Schwarzian theory

We now proceed by applying the same idea to JT gravity by dimensional reduction, showing that JT at finite cutoff is described by a deformation of the Schwarzian action, which is the dimensionally reduced $T\bar{T}$ deformation [63].

By begin by rewriting the flow equation (2.206) in an equivalent form. For a CFT flow [41]

$$\partial_\lambda S = 8\int \mathrm{d}^2x\sqrt{\gamma}T\bar{T} \tag{2.233}$$

one has, because of the trace flow equation

$$T^\mu_\mu = -16\lambda T\bar{T} = -2\lambda\left(T_{ij}T^{ij} - (T^i_i)^2\right) \tag{2.234}$$

From this equation we can extrapolate T^ϕ_ϕ:

$$T^\phi_\phi = \frac{T^\tau_\tau + 4\lambda T_{\tau\phi}T^{\tau\phi}}{4\lambda T^\tau_\tau - 1} \tag{2.235}$$

We plug it back into the flow equation and we rewrite it as

$$\partial_\lambda S = \int \mathrm{d}^2x\sqrt{\gamma}\left[\frac{(T^\tau_\tau)^2 + T_{\tau\phi}T^{\tau\phi}}{\frac{1}{2} - 2\lambda T^\tau_\tau}\right] \tag{2.236}$$

If we identify $\langle T^\tau_\tau \equiv E\rangle$ the energy and $\langle T^\tau_\tau \equiv iJ\rangle$ the angular momentum, the above implies the following flow equation for the spectrum

$$\partial_\lambda E = \frac{E^2 - J^2}{\frac{1}{2} - 2\lambda E} \tag{2.237}$$

which is solved by

$$E(\lambda) = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda E_0 + 16\lambda^2 J^2}\right) \tag{2.238}$$

An important point is that, after a careful identification of the physical parameters, (2.238) exactly represents the spectrum of 3d dimensional black hole. In fact, if we insert a BTZ black hole[42] in the bulk [7], which induces zero curvature at the boundary, with metric [43]

$$\mathrm{d}s^2 = \left(r^2 - r_+^2\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{r^2 - r_+^2} + r^2\mathrm{d}\phi^2 \tag{2.239}$$

we can compute the black hole spectrum, by using the definition in general relativity for the proper energy:

$$\mathcal{E} = \int \mathrm{d}\phi\sqrt{g_{\phi\phi}}u^i u^j T_{ij}$$

where u is a normal vector to the radial cutoff surface at $r = r_c$. By doing so, we find exactly the form (2.238), one we identify the radial cutoff and the $T\bar{T}$ deformation parameter. In particular $E_0 \simeq M$, i.e. the mass of the black hole, and J represents its angular momentum.

[41] To follow [63] here we denote with λ the deformation parameter that we previously called μ. It easy to map the two conventions.

[42] It is the AdS_3 variant of the Kerr rotating black-hole, in three dimensions.

[43] We write down the metric of a spin-less black-hole for semplicity.

Dimensional reduction We can now perfom dimensional reduction to go from the BTZ black hole to the JT gravity black hole, by dimensionally reducing the action above along the ϕ circle. We get

$$\mathrm{d}s^2 = \left(r^2 - r_+^2\right)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{r^2 - r_+^2} \tag{2.240}$$

which is exactly the form (2.11) we obtained by extremizing the JT gravity action.

Furthermore, by setting $T^{\tau\phi} = 0$, the flow equation modifies accordingly, with its related solution:

$$\partial_\lambda E = \frac{E^2}{\frac{1}{2} - 2\lambda E} \qquad \rightarrow E(\lambda) = \frac{1}{4\lambda}\left(1 - \sqrt{1 - 8\lambda E_0}\right) \tag{2.241}$$

which agrees precisely with the energy of the two-dimensional black holes at finite cutoff upon the identification $\lambda = \frac{2\pi G}{r_c}$, where r_c labales the cutoff radial surface. Accordingly, (2.241) represents the $T\bar{T}$-deformed spectrum of the Schwarzian theory.

An important remark is that the holographic correspondence holds only in the negative deformation parameter regime ($\lambda < 0$). This regime is known in the literature to be rather problematic, since it exhibits some pathologies in the high energy regime. As we can directly see from (2.241), for high values of M (i.e. of the energy eigenvalue), the argument of the square root becomes negative and the spectrum becomes complex. This could be interpreted as a breakdown of unitarity. However we should also notice that our analysis holds only at the perturbative level in μ. As proposed by Aharony et al. [1], a possible solution to the pathologies in the spectrum could come from non-trivial nonperturbative effects, such as instantons depending on the deformation parameter λ, which arise as non-trivial solutions of the Burgers equation (2.224). One of the main results of this dissertation is indeed to clarify this point, showing how the presence of these nonperturbative terms naturally emerges to cure the problem. This is the content of Chapter 4.

2.3.4 $T\bar{T}$ deformation of Yang–Mills theory

In this section, instead of applying the $T\bar{T}$ operator to a CFT, we study its effect on a QFT in two dimensions, i.e. Yang-Mills, a model that we have extensively studied in 2.2.1. In the first part, we explicitly solve the flow equation for the action, finding its exact $T\bar{T}$-deformed version. Secondly, starting from the flow equation for the action, we derive an associated flow equation for the partition function its self, which will be a necessary tool in order to fully quantize the theory. This is the content of Chapters 5.1, 5 and 6.

The deformed classical action

Our goal is to construct the $T\bar{T}$ operator associated to Yang-Mills action in (2.99). We first consider the abelian case. Since f is the only local gauge-invariant scalar degree of freedom of the theory, one can assume that the deformed Lagrangian density will be some function of f. In fact, we can equivalently define it as $\mathscr{L}(u,\mu)$, i.e. as a function of the deformation parameter μ and of

$$u = \frac{f^2}{2g_{\mathrm{YM}}^2}\,, \tag{2.242}$$

which is the undeformed Lagrangian density itself. With this choice, we have $\mathscr{L}(u,0)=u$. We can compute the stress-tensor of the deformed theory by varying the action with respect to the metric. Since f is defined as the Hodge dual of the field strength, it carries a dependence on the metric and contributes to the variation

$$\begin{aligned}\delta S_{\text{YM}} &= \delta \int_\Sigma \eta\, \mathscr{L}(u,\mu) \\ &= \int_\Sigma \eta \left(\frac{1}{2}\mathscr{L}(u,\mu) - u\,\partial_u \mathscr{L}(u,\mu)\right) g^{\kappa\lambda}\,\delta g_{\kappa\lambda}\,. \end{aligned} \tag{2.243}$$

From the above, we can easily read off the expression of $T^{\kappa\lambda}$ that, in turn, can be plugged into the flow equation (2.206). This produces an equation for the Lagrangian density,

$$\begin{aligned}\partial_\mu \mathscr{L} &= \mathcal{O}_{T\bar{T}} \\ &= 2(\mathscr{L} - 2u\,\partial_u \mathscr{L})^2\,, \end{aligned} \tag{2.244}$$

that we solve using the ansatz

$$\mathscr{L}(u,\mu) = \sum_{n=0}^{\infty} \mu^n \mathscr{L}_n(u)\,, \tag{2.245}$$

with $\mathscr{L}_0(u)=u$, as mentioned above. We find

$$\mathscr{L}_n(u) = \frac{3\,(4n+1)!}{n!\,(3n+3)!}\,(2u)^{n+1}\,, \tag{2.246}$$

which, upon summation, gives[44]

$$\mathscr{L} = \frac{3}{8\mu}\left({}_3F_2\Big(-\frac{1}{2},-\frac{1}{4},\frac{1}{4};\frac{1}{3},\frac{2}{3};\frac{512}{27}\,\mu\mathscr{L}_0\Big) - 1\right)\,. \tag{2.247}$$

We can repeat the analysis for the nonabelian theory. In principle, one is now faced with the choice of which trace structure to include in the deformed action. However, since the undeformed theory, and therefore its stress-tensor, only contain $\operatorname{tr} f^2$, one can safely assume that no other term could appear in the deformed Lagrangian density. With this in mind, we simply redefine

$$u = \frac{\operatorname{tr} f^2}{2g_{\text{YM}}^2}\,, \tag{2.248}$$

and repeat the steps above to find that (2.247) holds for the nonabelian theory as well.

Notice that (2.247) has a branch cut for $\mu\mathscr{L}_0 > 27/512$. This feature is not entirely unexpected, as it appears in other instances of $T\bar{T}$-deformed Lagrangians [13, 14], but poses a problem if one tries to quantize the deformed theory by starting from (2.247). Since we will take a different route to the quantum theory, we will defer this discussion to Section 5.5, which is devoted to the semiclassical limit.

[44]The deformed Lagrangian density (2.247) was obtained for the first time in [25].

The deformed partition function

We now allow for a general gauge group G and start with an ansatz for a deformed action which is a generalization of (2.100),

$$S_{\text{top}} = \mathrm{i}\int_\Sigma \operatorname{tr}(\phi F) + \frac{g_{\text{YM}}^2}{2}\int_\Sigma \eta\,\mathscr{U}(v,\mu)\,, \tag{2.249}$$

where $v = \operatorname{tr}\phi^2$. The undeformed Yang–Mills action is recovered with $\mathscr{U}(v,0) = v$. Again, note that the one defined above is not the most general potential that can be considered, since one could in principle involve other invariant polynomials in $\mathfrak{g}$. However, because of the initial condition, no other term can enter the deformed action. From the variation

$$\delta S_{\text{top}} = \frac{g_{\text{YM}}^2}{4}\int_\Sigma \eta\,\mathscr{U}(v,\mu)\,g^{\kappa\lambda}\,\delta g_{\kappa\lambda} \tag{2.250}$$

one can read off the expression for the stress-energy tensor and plug it in (2.206) to obtain an equation for the deformed potential $\mathscr{U}$,

$$\begin{aligned}\partial_\mu \mathscr{U}(v,\mu) &= 2\mathcal{O}_{T\bar{T}}/g_{\text{YM}}^2 \\ &= g_{\text{YM}}^2\,\mathscr{U}^2(v,\mu)\,.\end{aligned} \tag{2.251}$$

Let us now proceed in analogy with [130] and repeat the argument 2.2.1 exposed above to quantize Yang-Mills theory through an Hamiltonian approach. We consider as usual an initial-value circle $C \subset \Sigma$. As before, in a neighborhood of C, we write the volume form in terms of local coordinates as $\eta = \mathrm{d}s \wedge \mathrm{d}t$, where C corresponds to $t = 0$ and s is a coordinate along C such that $\oint \mathrm{d}s = 1$. Since the action (2.249) is linear in F, the Hamiltonian reads

$$H = \frac{g_{\text{YM}}^2}{2}\oint_C \mathrm{d}s\,\mathscr{U}(v,\mu)\,, \tag{2.252}$$

and generates translations along t. When acting on the representation basis, as in (2.103), the Hamiltonian is diagonal and takes the simple form $H = g_{\text{YM}}^2/2\,\mathscr{U}(C_2(R),\mu)$.

If we now consider a finite cylinder spanned by the range $t \in [0,a]$, the associated partition function will depend on the relevant couplings as $e^{-aH(\mu)}$, where a is the area of the cylinder. As a consequence, one concludes that the deformed partition function obeys the flow equation [19, 75, 115]

$$\frac{\partial Z}{\partial \mu} + 2a\,\frac{\partial^2 Z}{\partial a^2} = 0\,. \tag{2.253}$$

We remark that the differential equation above is fully general, since there is nothing special about the chosen topology. In fact, it still applies if we consider, for instance, a disk or a sphere partition function. We simply need to shrink the boundary circles to points, and in doing so, impose trivial holonomies on them. Arbitrary topologies can be further obtained through gluing, thus exploiting the quasi-topological character of the theory.

Before moving on, let us also mention that, while one can safely employ the flow equation for the deformation of the classical action, at the quantum level things are more subtle as one needs to deal with potential ambiguities associated with the UV behavior of composite operators. More precisely, the deformation operator $\mathcal{O}_{T\bar{T}}$ is typically only defined on flat backgrounds where one employs point-split regularization and shows that the pinching limit is actually regular, up to derivative terms. However, in quantum theories described by the action (2.249), correlators of gauge-invariant local operators are topological, i.e. do not depend on the position of the operator insertions. As a consequence, the regularity of the pinching limit of such operators is trivially guaranteed.

Part II

JT gravity

3 Bi-local correlators in JT gravity

An appealing aspect of JT gravity is the existence of a particular class of n-point functions that we can compute exactly, the so-called bi-local correlators [86, 134]. Specifically, these observables can be viewed either as n-point vacuum expectation value for bi-local operators evaluated at the boundary of the AdS_2 space-time or as $2n$-point correlators of some 1D 'matter CFT' at finite temperature coupled to the Schwarzian theory on the boundary [86]. Their general structure on disk and trumpet topologies has been studied by exploiting different techniques, and their explicit form can be systematically obtained for any n as an integral of momentum space amplitudes, using a simple set of diagrammatic rules. Originally the derivation relied on the precise equivalence between the 1D Schwarzian theory and a certain large central charge limit of 2D Virasoro CFT [95].

More recently, taking advantage of the $SL(2,\mathbb{R})$ gauge theory formulation [1], the correlation functions of bi-local operators have been computed as correlators of Wilson lines anchored at two points on the boundary [74]. We have indeed reviewed the gauge reformulation of JT gravity in 2.2.5 and computed in 2.2.6 a boudnary anchored Wilson line in this framework. [2] Anchored Wilson lines also have a gravitational interpretation, representing the sum over all possible world-line paths for a particle moving between two fixed points on the boundary of the AdS_2 patch [74, 11]. The computation of bi-local correlators have been later extended in the presence of defects [92], and the inclusion of higher-genus corrections was also considered [111], with particular attention to their late time behavior and non-perturbative properties. On the other hand, correlation functions on the disk can also be studied through a perturbative expansion in the Schwarzian coupling constant [86, 117], as we reviewed in 2.1.3. In this approach, one directly computes Feynman diagrams for boundary gravitons, i.e., the quantum mechanical degrees of freedom associated with the fluctuations of the wiggle AdS_2 boundary. [3]

Quite surprisingly, the consistency of the exact results obtained through CFT and gauge theoretical techniques with the Schwarzian perturbative expressions has never been checked or discussed in details until recently[4] [90]. More generally, the structure of the perturbative series and its convergence properties have been somehow overlooked despite certain interesting pieces of information that could be directly extracted from it, as the relation with the gravitational S-matrix or the trigger of the full quantum regime at large time with respect to Schwarzian coupling constant. A particularly intriguing point concerns the convergence itself

[1] see [42] for an exhaustive analysis of the subject

[2] In particular, the time ordering is encoded into the intersection of the Wilson lines in the bulk, resulting in the appearance of momentum-dependent fusion coefficients and 6-j symbols inside the integrated amplitudes.

[3] The semiclassical limit and the first quantum correction to two-point and four-point functions were studied in [82]. Schwarzian perturbation theory has also found applications for higher-point functions [69], while higher loop corrections were analyzed in [108].

[4] The semiclassical limit for the two-point and the four-point functions has been checked in [82]

of the perturbative series and the presence of non-perturbative contributions inside the exact expressions derived in [95, 74]. A first attempt to answer this question has been taken in [90]: the exact two-point correlator on the disk for a bi-local operator of conformal weight $\lambda \in -\mathbb{N}/2$ have been expanded for a small value of κ, the Schwarzian coupling, and confronted successfully with the perturbative result beyond the semiclassical regime. Moreover, exploiting the simplicity of the cases $\lambda = -1/2$ and $\lambda = -1$ and the limit of zero temperature, it was argued that for generic λ the series is asymptotic, implying the presence of non-perturbative contributions. The asymptotic character for generic conformal weights was taken as a signal of non-perturbative contributions of order $e^{-\frac{1}{\kappa}}$ inside these correlators, competing therefore with the higher-genus corrections of order $e^{-\frac{1}{G_N}}$, derived by matrix-model techniques [113, 111], because κ is proportional to gravitational Newton constant G_N.

In this chapter, which is mainly based on the content of our paper [59], we obtain some progress in these directions, performing explicit computations and elucidating the analytical structure of the bi-local correlators in the case of general positive conformal weight and general temperature.

We also slightly extend our analysis beyond the disk topology by considering the trumpet configuration: this could be relevant in view of further studies on higher-genus topologies [90, 93, 79] or for investigating one-point functions in the presence of defects [92]. Our first aim is to recover the Schwarzian perturbative result beyond the leading semiclassical order in the case of positive λ, on the disk and trumpet topologies. We have obtained a perfect agreement by evaluating the exact expression through a saddle-point approximation: the computation heavily relies on the relevant amplitudes' analytical properties. In the general case, it reduces to an integral around branch-cuts determined by the conformal weight of the operators involved. The result is obtained for finite boundary separations; it exhibits the correct time periodicity and, as expected in this case, the bi-local correlator is singular at coincident points. The outcome completes and generalizes the analysis of [90], performed for negative semi-integer weights λ and in the particular case of zero temperature, and strengths our trust in the analytical approach. Actually, in the case of $\lambda \in \mathbb{N}/2$ we can go well beyond the first subleading quantum correction; the branch-cut singularity of the two-point function reduces to a pole, and by carefully computing the residue, we obtain an all-order expansion in the Schwarzian coupling constant κ. Finally we have also examined the zero-temperature case, in order to recover the results of [90] in this limit: although being potentially singular, as seen from the previous expansions, we have obtained a nice and compact expression in terms of Bernoulli polynomials, consistent with the general result. We can draw from this limit some conclusions on the convergence properties of the perturbative series, confirming its asymptotic character for positive semi-integer weights. Moreover, the alternate sign of the perturbative orders points towards a possible Borel summability for the full series. As a final observation, we point out that the exact expression for the bi-local correlator can also be written in terms of Mordell integrals [101], suggesting a link with the world of Mock-modular forms [136, 23].

The chapter's structure is the following: we first present the perturbative computation of the bi-local correlator on the trumpet, generalizing the previous calculation for the disk topology, reviewed in 2.1.3. Then, we perform the saddle-point analysis of the exact expressions on the disk and the trumpet, successfully recovering the first subleading correction to the semiclassical result. Section 3.3 is devoted to the all-order expansion in the case $\lambda \in \mathbb{N}/2$. We give the explicit (although a little cumbersome) general expression and examine in more detail the weights $\lambda = 1/2$ and $\lambda = 1$. The zero-temperature limit is instead the subject of Section 3.4.

Subsequently, in 3.5, we illustrate how the bi-local correlators for integer 2λ can be written in closed form in terms of Mordell integrals.

3.1 Perturbation theory in the Schwarzian

In [74] [11] it was proposed that the bulk Wilson line anchored to the points τ_1 and τ_2 of the boundary is dual to the bi-local correlator of conformal dimension λ:

$$\mathcal{O}(\tau \equiv \tau_1 - \tau_2) = \left[\frac{t'(\tau_1)t'(\tau_2)}{(t(\tau_1)-t(\tau_2))^2}\right]^\lambda \tag{3.1}$$

computed in the Schwarzian theory, whose action we have already encountered in (2.26) and reads as [5]

$$S_{\text{Sch}}[t] = -\frac{\phi_r}{16\pi G_N}\int_{\partial\mathcal{M}} d\tau\,\{t(\tau),\tau\} = -\frac{1}{2\kappa}\int_{\partial\mathcal{M}} d\tau\,\{t(\tau),\tau\}\,. \tag{3.2}$$

As we have seen in 2.1.2, here the fundamental field $t(\tau)$ plays the role of a reparameterization mode, or boundary graviton, capturing the boundary excitations over AdS_2 in JT gravity. The expectation value $\langle\mathcal{O}(\tau)\rangle$ is found by inserting (3.1) inside the path integral over the boundary mode t weighted by the Schwarzian action[6]. In 2.2.6 we have demonstrated the equivalence for a Wilson line in the lowest weight state of a discrete $j=l$ representation of $\text{SL}^+(2,\mathbb{R})$. We can use this representation to compute these observables perturbatively.

This result indirectly provides a check for the exact formulae (2.189) and (2.204), that we derived in 2.2.6 in the context of the gauge formulation of JT gravity as a BF theory. Below, we shall briefly describe how to do the perturbative computation in the trumpet's less trivial case. The analysis is very similar to the one performed in 2.1.3 for the disk.[7].

The classical equations of motion for (3.2) are solved by a field $t(\tau)$ with a constant Schwarzian derivative. In the trumpet case, the classical saddle can be parameterized as

$$t(\tau) = e^{-\vartheta(\tau)} \qquad \vartheta(\tau) = \frac{2\pi b}{\beta}\left(\tau + \varepsilon(\tau)\right), \tag{3.3}$$

where $\varepsilon(\tau)$ is a small fluctuation over the classical background[8]. Plugging eq. (3.3) into eq.

[5]We implement a slight change of notation, indicating with τ the proper boundary time, instead of u. We also map $C \to 1/2\kappa$.

[6]On the disk the Schwarzian path integral is

$$\langle\mathcal{O}(\tau)\rangle = \int \frac{\mathcal{D}t}{\text{SL}(2,\mathbb{R})}\; e^{-S_{\text{Sch}}[t]}\;\mathcal{O}(\tau)$$

where $\text{SL}\,(2,\mathbb{R})$ are gauge redundancies of the Schwarzian action. When a hole is inserted, this breaks the gauge group to U(1).

[7]The perturbative computation for the disk is shown in detail in [86, 117]

[8]This parametrization can be justified by looking at the metric solution for the disk and the trumpet in Rindler coordinates, which are respectevely

$$\mathrm{d}s^2_{\text{disk}} = \mathrm{d}\varrho^2 + \sinh^2\varrho\;\mathrm{d}\tau^2 \qquad \mathrm{d}s^2_{\text{trumpet}} = \mathrm{d}\sigma^2 + \cosh^2\sigma\;\mathrm{d}\vartheta^2$$

where the coordinate ϑ obeys the twisted periodicty $\vartheta \sim \vartheta + b$. The relation between the τ and ϑ coordinates at the boundary of the regular hyperbolic disk is $\cos\tau = \tanh\vartheta$ and therefore this implies $t = \tan\frac{\tau}{2} = e^{-\vartheta}$.

(3.1), we find at zero order in ε

$$\langle \mathcal{O}(\tau)\rangle^{\text{tr.}}_{\text{tree}} = \left(\frac{\pi b}{\beta \sinh\left(\frac{\pi b}{\beta}\tau\right)}\right)^{2\lambda}, \tag{3.4}$$

which is the tree level amplitude for the correlator on the trumpet geometry. To compute the quantum correction to the tree-level result, we must determine the propagator for the field ε. Expanding the action (3.2) to order ϵ^2 around the saddle (3.3) we find

$$S_\varepsilon = -\frac{1}{2\kappa}\int_0^\beta \mathrm{d}\tau \left[\varepsilon''(\tau)^2 + \left(\frac{2\pi b}{\beta}\right)^2 \varepsilon'(\tau)^2\right] = -\frac{\beta}{2\kappa}\left(\frac{2\pi}{\beta}\right)^4 \sum_{n\in\mathbb{Z}} \varepsilon_n \varepsilon_{-n}\, n^2\left(n^2+b^2\right), \tag{3.5}$$

where we have Fourier-expanded the fluctuation as $\varepsilon(\tau) = \sum_{n\in\mathbb{Z}} \varepsilon_n\, e^{\frac{2\pi i n\tau}{\beta}}$. We recognize the presence of a zero mode ($n=0$) associated with the residual U(1) gauge redundancy present in the trumpet geometry. The propagator can be found by inverting the quadratic action and we get

$$\begin{aligned}\langle \varepsilon(0)\varepsilon(\tau)\rangle =& \frac{\kappa\beta^3}{8\pi^4}\sum_{n\neq 0}\frac{e^{\frac{2\pi i n\tau}{\beta}}}{n^2\left(n^2+b^2\right)} = \\ =& \frac{\beta\kappa\left(\pi^2 b^2\left(\beta^2-6\beta\tau+6\tau^2\right)+3\beta^2-3\pi\beta^2 b\ \text{csch}(\pi b)\cosh\left(\frac{\pi b(\beta-2\tau)}{\beta}\right)\right)}{24\pi^4 b^4}.\end{aligned} \tag{3.6}$$

where the sum over negative and positive integers has been computed in terms of elementary functions by exploiting standard complex analysis techniques [117].

To obtain the correction of order κ to this observable, we do not need to proceed further in expanding the action. We have instead to expand the bi-local correlator (3.1) around the saddle (3.3) up to order ε^2. The $\mathcal{O}(\varepsilon)$ has vanishing expectation value since the one-point function is zero for the quadratic action (3.5). Normalizing with respect to the tree level (3.4), we get

$$\begin{aligned}&\frac{\lambda}{2\beta^2}\Bigg\{4b^2\pi^2\left(\lambda\coth^2\frac{\pi b\tau}{\beta}+\frac{1}{2}\text{csch}^2\frac{\pi b\tau}{\beta}\right)(\varepsilon(\tau_1)-\varepsilon(\tau_2))^2+\beta^2\left[\lambda\left(\varepsilon'(\tau_1)+\varepsilon'(\tau_2)\right)^2-\right.\\ &\left.-\varepsilon'(\tau_1)^2-\varepsilon'(\tau_2)^2\right]+4\pi b\lambda\beta\coth\left(\frac{\pi b\tau}{\beta}\right)\left[(\varepsilon(\tau_2)-\varepsilon(\tau_1))\left(\varepsilon'(\tau_1)+\varepsilon'(\tau_2)\right)\right]\Bigg\}\end{aligned} \tag{3.7}$$

We now substitute every appearance of ε^2-combination with their expectation value at this order, i.e. with the propagator (3.6) $\langle\varepsilon(0)\varepsilon(\tau)\rangle \equiv \mathcal{G}(\tau)$ or its derivatives. Introducing the auxiliary combination $\xi = \frac{\tau}{\beta}$, the first perturbative term finally reads as

$$\begin{aligned}\frac{\langle \mathcal{O}(\tau)\rangle^{\text{tr.}}}{\langle \mathcal{O}(\tau)\rangle^{\text{tr.}}_{\text{tree}}} =& 1+\frac{\beta\kappa\lambda}{4\pi^2 b^2}\text{csch}^2(\pi b\xi)\left[2\lambda-1-2\pi^2 b^2(\lambda+1)(\xi-1)\xi+\right.\\ &\left.+b\pi(\lambda(4\xi-2)-1)\sinh(2\pi b\xi)+\left(1-2\lambda\left(\pi^2 b^2(\xi-1)\xi+1\right)\right)\cosh(2\pi b\xi)\right]+O(\kappa^2).\end{aligned} \tag{3.8}$$

We will reproduce the above expression from the exact formula (2.204) in the next section.

3.2 Recovering the perturbative expansion

3.2.1 Bi-local correlator on the disk

In the following our goal is to illustrate how the perturbative results for the bi-local correlator $\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}}$ on the disk can be recovered from its exact integral representation (2.189), that we rewrite here for clarity [9]:

$$\langle \mathcal{W}_\lambda(\tau)\rangle_{\text{disk}} = \mathcal{N}_d \int_0^\infty\int_0^\infty \mathrm{d}s_1 \mathrm{d}s_2 s_1\ s_2 \sinh(2\pi s_1)\sinh(2\pi s_2)\frac{\Gamma(\lambda \pm i s_1 \pm i s_2)}{\Gamma(2\lambda)}\ e^{-\kappa\tau s_1^2-\kappa(\beta-\tau)s_2^2}. \tag{3.9}$$

where we choose the normalization constant $\mathcal{N}_d = \frac{\kappa^{2\lambda}}{2\pi}\mathcal{Z}_{\text{disk}}^{-1} = \frac{\kappa^{2\lambda}}{2\pi}\left(\frac{\kappa\beta}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{\pi^2}{\beta\kappa}}$.

A simple-minded Taylor-expansion of the integrand would lead to divergent expressions since the representation (3.9) is naturally suited to derive the large κ expansion for the two-point function.

To obtain the perturbative series in κ we have to rearrange the κ dependence of (3.9) and we start by using the following identity, first derived by Ramanujan:

$$\int_{-\infty}^{\infty} \mathrm{d}p\ \mathrm{sech}^{2a}\left(\frac{p}{2}\right) e^{ipx} = \frac{2^{2a-1}}{\Gamma(2a)}\Gamma(a+ix)\Gamma(a-ix). \tag{3.10}$$

which holds for $\mathrm{Re}(a) > 0$. It is not the first time this identity appears in the context of JT gravity [82, 134]. In both cases, it was used to probe the leading saddle of the momentum integrals.

In this way we can express the 3-j symbol in Fourier space and we get

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} = \frac{\mathcal{N}_d\Gamma(2\lambda)}{2^{4\lambda-2}}\int_0^\infty\int_0^\infty \mathrm{d}s_1\mathrm{d}s_2\ s_1 s_2\ \sinh 2\pi s_1 \sinh 2\pi s_2\ e^{-\kappa(\beta-\tau)s_1^2-\kappa\tau s_2^2}\times \\ \times\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq\ e^{ip(s_1+s_2)+iq(s_1-s_2)}\ \mathrm{sech}^{2\lambda}\frac{p}{2}\,\mathrm{sech}^{2\lambda}\frac{q}{2}\end{aligned} \tag{3.11}$$

Since the integrand is even in s_1 and s_2, we can extend the region of integration to the entire real line and subsequently perform the gaussian integration over s_1 and s_2. We are left with a double integral over p and q:

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} = \frac{\pi\mathcal{N}_d\Gamma(2\lambda)}{2^{4\lambda+2}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}}\int_{-\infty}^{\infty} dpdq\ \frac{q^2-(p-2i\pi)^2}{\cosh^{2\lambda}\frac{p}{2}\cosh^{2\lambda}\frac{q}{2}} e^{-\frac{(p+q-2i\pi)^2}{4\kappa(\beta-\tau)}-\frac{(p-q-2i\pi)^2}{4\kappa\tau}} \tag{3.12}$$

The original symmetry in the exchange $\tau \leftrightarrow \beta-\tau$ in (3.9) is now realized by the change of variables $q \leftrightarrow -q$. Next we perform the shift $p \to p+2\pi i$ by considering the contour displayed in fig. 3.1 in the complex $p-$plane. In the following we shall assume that $2\lambda \notin \mathbb{N}$[10]. Then the contour encircles the branch cut, due to $(\cosh\frac{p}{2})^{-2\lambda}$, that has been chosen to run from $p = \pi i$ to $p = \infty + \pi i$[11]. Moreover, we take $0 < 2\lambda < 1$ so that the contribution of the semicircle (IV) is finite and vanishes when we shrink its radius to zero. At the end we will recover the perturbative result for $2\lambda > 1$ by extending analytically the final expression.

[9] We slightly change notation, from e to κ.

[10] The case $2\lambda \in \mathbb{N}$, where the cut is replaced by a pole, will be discussed in detail in sec. 3.3.

[11] This position of the cut is obtained by choosing the phase around the branch point between $(-\frac{3\pi}{2}, \frac{\pi}{2}]$.

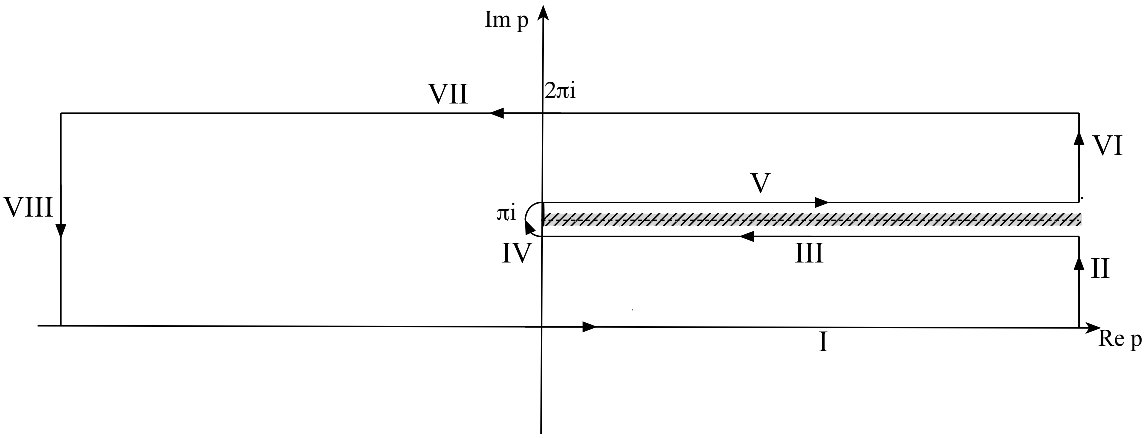


Figure 3.1: Contour in the complex p-plane used to perform the shift $p \to p + 2\pi i$.

The contributions of the vertical edges (II, VI and VIII) of the contour vanish when we approach infinity and thus the original integral (edge I) can be replaced by the two terms coming respectively from the horizontal edge (VII) and the discontinuity around the cut

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} = \frac{\pi \mathcal{N}_d e^{-2\pi i\lambda}\Gamma(2\lambda)}{2^{4\lambda+2}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}}\int_{-\infty}^{\infty} dpdq\ \frac{q^2-p^2}{\cosh^{2\lambda}\frac{p}{2}\cosh^{2\lambda}\frac{q}{2}} e^{-\frac{(p+q)^2}{4\kappa(\beta-\tau)}-\frac{(p-q)^2}{4\kappa\tau}} - \\ - \frac{\pi i \mathcal{N}_d e^{\pi i\lambda}\sin(2\pi\lambda)\Gamma(2\lambda)}{2^{4\lambda+1}\kappa^3\tau^{\frac{3}{2}}(\beta-\tau)^{\frac{3}{2}}}\int_{-\infty}^{\infty} dq\int_0^{\infty} dt\ \frac{q^2-(t-i\pi)^2}{\cosh^{2\lambda}\frac{q}{2}\sinh^{2\lambda}\frac{t}{2}} e^{-\frac{(t+q-i\pi)^2}{4\kappa(\beta-\tau)}-\frac{(t-q-i\pi)^2}{4\kappa\tau}} \tag{3.13}$$

The first integral in (3.13) vanishes because of the antisymmetry in the exchange $p \leftrightarrow q$. In the second one we can safely perform the following shift

$$q \to q + \frac{(t-i\pi)(\beta-2\tau)}{\beta} \tag{3.14}$$

since we do not encounter any branch cut or singularity of the integrand during this process (at least for generic values of β and τ). This shift centers the integral at $q = 0$ and we obtain

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} = -\frac{\pi i \mathcal{N}_d e^{\pi i\lambda}\sin(2\pi\lambda)\Gamma(2\lambda)}{2^{4\lambda+1}\kappa^3\beta^3\xi^{\frac{3}{2}}(1-\xi)^{\frac{3}{2}}}\times \\ \times\int_{-\infty}^{\infty} dq\int_0^{\infty} dt\ \frac{(q+2(t-\pi i)(1-\xi))(q-2(t-\pi i)\xi)}{\cosh^{2\lambda}\frac{q+(t-\pi i)(1-2\xi)}{2}\sinh^{2\lambda}\frac{t}{2}} e^{-\frac{q^2}{4\kappa\beta\xi(1-\xi)}-\frac{(t-\pi i)^2}{\beta\kappa}} \tag{3.15}$$

where we have found it convenient to introduce the auxiliary combination $\xi \equiv \frac{\tau}{\beta}$. The form (3.15) of the integral representation is suited to identify the origin of the dominant contributions in the limit $\kappa \to 0$. A neighborhood around $q = 0$ dominates the integration over q due to the integrand's gaussian weight. For the same reason, one might assume that the integration over t is also primarily controlled by a small interval around $t = \pi i$. On the other hand, since t spans the semi-infinite interval $[0, +\infty]$, we have to consider a second candidate, namely the neighbourhood around $t = 0$ (see [103] for the general theory). Comparing the two possibilities,

we find that the integral (3.15) in the limit $\kappa \to 0$ is dominated by the second one since the gaussian weight scales as $e^{\frac{\pi^2}{\kappa\beta}}$.
The simplest way to construct systematically the asymptotic series in the limit $\kappa \to 0$ is to perform the following rescaling of variables

$$q \mapsto \sqrt{\kappa} q \qquad t \mapsto \kappa t. \tag{3.16}$$

The different scaling of the variable t takes into account that the leading contribution comes from the lower extremum of the integral and not from a saddle-point. Using the explicit form of the normalization $\mathcal{N}_d$ we get

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} &= -\frac{i e^{\pi i \lambda}\kappa^{2\lambda}\sin(2\pi\lambda)\Gamma(2\lambda)}{2^{4\lambda+2}\pi^{5/2}\beta^{3/2}(1-\xi)^{3/2}\xi^{3/2}}\times \\ &\times \int_{-\infty}^{\infty} dq \int_0^\infty dt\, \frac{(\sqrt{\kappa}q + 2(\kappa t - \pi i)(1-\xi))(\sqrt{\kappa}q - 2(\kappa t - \pi i)\xi)}{\cosh^{2\lambda}\frac{\sqrt{\kappa}q + (\kappa t - \pi i)(1-2\xi)}{2}\sinh^{2\lambda}\frac{\kappa t}{2}} e^{-\frac{q^2}{4\beta\xi(1-\xi)} - \frac{\kappa t^2}{\beta} - \frac{2\pi i t}{\beta}}\end{aligned} \tag{3.17}$$

The non-analytic factor $e^{-\frac{\pi^2}{\beta\kappa}}$ present in $\mathcal{N}_d$ cancels exactly against the constant term in the gaussian weight for t and we can Taylor-expand the integrand (3.17) around $\kappa = 0$, obtaining a series with both integer and semi-integer powers of κ. The latter is always proportional to an odd power of q and vanish when the integral is performed. Moreover, the expansion generates integrals over t, which are divergent for real β. We can take care of this issue by rotating our path of integration in t of a small positive angle α before expanding. Once we have integrated over t, the final result does not depend on α. Alternatively, we could assume that β has a small imaginary part and then analytically continue to real values.
After integrating over t and q term by term, we find

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} &= \frac{\pi^{2\lambda}}{\beta^{2\lambda}\sin^{2\lambda}(\pi\xi)}\Big[1 + \frac{\kappa\beta\lambda}{4\pi^2\sin^2(\pi\xi)}(2\pi^2(\lambda+1)\xi^2 - 2\pi^2(\lambda+1)\xi - \\ &- \pi(2\lambda+1)(2\xi-1)\sin(2\pi\xi) + (2\lambda(\pi^2(\xi-1)\xi-1)-1)\cos(2\pi\xi) + 2\lambda+1) + O(\kappa^2)\Big]\end{aligned} \tag{3.18}$$

In this expansion, we recognize the classical term and the one-loop contribution obtained by a direct diagrammatic computation in [117], whose final result we explicitly derived and shown in (2.71). A systematic all order expansion can also be obtained by expanding the integrand in terms of generalized Apostol-Eulerian and Bernoulli polynomials. However, the final expression is not particularly appealing, and we will concentrate on the particular case $2\lambda \in \mathbb{N}$.

3.2.2 Bi-local correlator on the trumpet

The structure of the bi-local operator on the trumpet (2.204) is quite similar to the case of the disk and if we use the symmetry of the integrand, we can rearrange it in the following form:

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{tr.}} &= \frac{\mathcal{N}_t}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}s_2\mathrm{d}s_2\; s_1\; e^{2\pi(s_1 + i b s_2) - \kappa(\beta-\tau)s_2^2 - \kappa\tau s_1^2}\times \\ &\times \frac{\Gamma(\lambda - i s_1 - i s_2)\Gamma(\lambda + i s_1 + i s_2)\Gamma(\lambda + i s_1 - i s_2)\Gamma(\lambda - i s_1 + i s_2)}{\Gamma(2\lambda)}.\end{aligned} \tag{3.19}$$

As in the case of the disk, we can use the identity (3.10) to eliminate the Gamma function and perform the gaussian integration over s_1 and s_2. We find this new integral representation for the bi-local correlator (2.204):

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{tr.}} = \frac{i\pi \mathcal{N}_t \Gamma(2\lambda)}{2^{4\lambda+1}\kappa^2\sqrt{\tau}(\beta-\tau)^{\frac{3}{2}}}\int_{-\infty}^{\infty} dpdq\ \frac{(p+q-2\pi i)}{\cosh^{2\lambda}\left(\frac{p}{2}\right)\cosh^{2\lambda}\left(\frac{q}{2}\right)} e^{-\frac{(2\pi b+p-q)^2}{4\kappa(\beta-\tau)}-\frac{(p+q-2i\pi)^2}{4\kappa\tau}}. \tag{3.20}$$

Next we shift the variables of integration as follows $p\mapsto p-\pi b$ $q\mapsto q+\pi b$ and we get

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{tr.}} = \frac{i\pi \mathcal{N}_t \Gamma(2\lambda)}{2^{4\lambda+1}\kappa^2\sqrt{\tau}(\beta-\tau)^{\frac{3}{2}}}\int_{-\infty}^{\infty} dpdq\ \frac{(p+q-2\pi i)e^{-\frac{(p-q)^2}{4\kappa(\beta-\tau)}-\frac{(p+q-2\pi i)^2}{4\kappa\tau}}}{\cosh^{2\lambda}\left(\frac{p-\pi b}{2}\right)\cosh^{2\lambda}\left(\frac{q+\pi b}{2}\right)}. \tag{3.21}$$

Again we perform the shift $p\to p+2\pi i$ by considering a contour similar to the one displayed in fig. 3.1. The only difference is the position of the branch cut that now runs $p=\pi b+\pi i$ to $p=\infty+\pi i$. As in the case of the disk we assume that $2\lambda\notin\mathbb{N}$ and $0<2\lambda<1$. We get

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{tr.}} =& \frac{i\pi e^{-2\pi i\lambda}\mathcal{N}_t \Gamma(2\lambda)}{2^{4\lambda+1}\kappa^2\sqrt{\tau}(\beta-\tau)^{\frac{3}{2}}}\int_{-\infty}^{\infty} dpdq\ \frac{(p+q)e^{-\frac{(p-q+2\pi i)^2}{4\kappa(\beta-\tau)}-\frac{(p+q)^2}{4\kappa\tau}}}{\cosh^{2\lambda}\left(\frac{p-\pi b}{2}\right)\cosh^{2\lambda}\left(\frac{q+\pi b}{2}\right)}+ \\ &+\frac{2\pi\mathcal{N}_t\Gamma(2\lambda)e^{\pi i\lambda}\sin(2\pi\lambda)}{2^{4\lambda+1}\kappa^2\sqrt{\tau}(\beta-\tau)^{\frac{3}{2}}}\int_{-\infty}^{\infty}dq\int_0^{\infty}dt\ \frac{(t+q-\pi i+\pi\beta)e^{-\frac{(t-q+\pi b+\pi i)^2}{4\kappa(\beta-\tau)}-\frac{(t+q-\pi i+\pi b)^2}{4\kappa\tau}}}{\sinh^{2\lambda}\left(\frac{t}{2}\right)\cosh^{2\lambda}\left(\frac{q+\pi b}{2}\right)}.\end{aligned} \tag{3.22}$$

The first integral vanishes because it is odd under the transformations $p\mapsto -q$ and $q\mapsto -p$. In the second integral we perform a shift in q to center the gaussian weight around $q=0$ and we obtain the analog of (3.15):

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{tr.}} =& \frac{\pi\mathcal{N}_t\Gamma(2\lambda)\sin(2\pi\lambda)}{16^\lambda\beta^2\kappa^2\sqrt{1-\xi}\xi^{3/2}}\times \\ &\times\int_{-\infty}^{\infty}dq\int_0^{\infty}dt\ \frac{2\xi(\pi b+t)+q}{\sinh^{2\lambda}\left(\frac{t}{2}\right)\sinh^{2\lambda}\left(\frac{1}{2}(2\xi(\pi b+t)+q-t)\right)}e^{-\frac{(\pi b+t)^2}{\beta\kappa}-\frac{q^2}{4\beta\kappa\xi(1-\xi)}}\end{aligned} \tag{3.23}$$

where we have again introduced the auxiliary combination $\xi\equiv\frac{\tau}{\beta}$. As in the previous case, the integration over q is again dominated by a neighbourhood around $q=0$ due to the gaussian weight in the integrand. Since t spans the semi-infinite interval $[0,+\infty]$ and in this interval the gaussian weight is monotonic (for $b>0$), we find that the integral over t in the limit $\kappa\to 0$ is controlled by the lower bound of the integration interval, $t=0$.

Next we scale the variables t and q as in (3.16) and expand the integrand around $\kappa=0$. Performing the two integrations term by term we find

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{tr.}} =& \frac{\pi^{2\lambda}b^{2\lambda}}{\beta^{2\lambda}\sinh^{2\lambda}(\pi b\xi)}\Bigg[1+\frac{\beta\kappa\lambda}{4\pi^2b^2\sinh^2(\pi b\xi)}\Big(-2\pi^2b^2(\lambda+1)\xi^2+2\pi^2b^2(\lambda+1)\xi+ \\ &+\Big(1-2\lambda\left(\pi^2b^2(\xi-1)\xi+1\right)\Big)\cosh(2\pi b\xi)+\pi b(\lambda(4\xi-2)-1)\sinh(2\pi b\xi)+2\lambda-1\Big)+O(\kappa^2)\Bigg]\end{aligned} \tag{3.24}$$

In this expansion we recognize the classical term and the one-loop contribution obtained by a direct diagrammatic computation in subsec. 3.1.

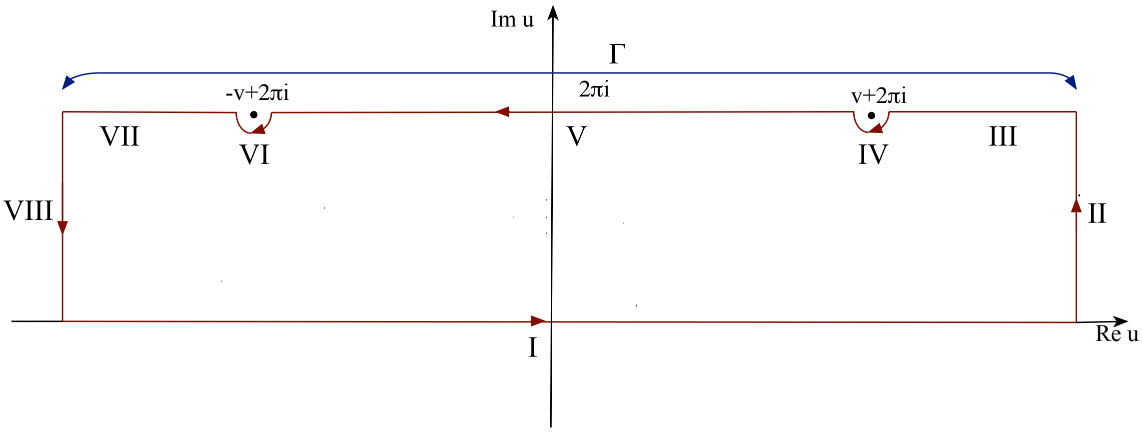


Figure 3.2: The red contour C used to perform the integration over u

3.3 All order expansion: the case $2\lambda \in \mathbb{N}$

In this section we focus our attention on a particular but very interesting case, namely $2\lambda \in \mathbb{N}$. For semi-integer values of λ, the cut present in fig. 3.1 is replaced by a pole of order 2λ. For this reason it is convenient to start over our analysis from the integral representation (3.12) and use as new variables of integration

$$p = \frac{u+v}{2} \qquad q = \frac{u-v}{2}. \tag{3.25}$$

We get a nice and symmetric representation for the bi-local correlator on the disk:

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} = -\frac{\pi \mathcal{N}_d \Gamma(2\lambda)}{2^{2\lambda+3}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}} \int_{-\infty}^{\infty} dudv \, \frac{(u-2\pi i)(v-2\pi i)}{(\cosh\frac{u}{2}+\cosh\frac{v}{2})^{2\lambda}} e^{-\frac{(u-2i\pi)^2}{4\kappa(\beta-\tau)}-\frac{(v-2i\pi)^2}{4\kappa\tau}} \tag{3.26}$$

Next we evaluate the integral over u in (3.26) using residues. Consider the closed red contour C depicted in fig. 3.2. Along this path the integral of the function

$$f(u,v) = -\frac{\pi \mathcal{N}_d \Gamma(2\lambda)}{2^{2\lambda+3}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}} \frac{(u-2\pi i)(v-2\pi i)}{(\cosh\frac{u}{2}+\cosh\frac{v}{2})^{2\lambda}} e^{-\frac{(u-2i\pi)^2}{4\kappa(\beta-\tau)}-\frac{(v-2i\pi)^2}{4\kappa\tau}} \tag{3.27}$$

is identically zero as (3.27) defines a holomorphic function in the enclosed region. Since the contributions of the two vertical edges II and VIII vanish when they approach infinity, the integral along the entire real $u-$axis, i.e. the original integral, is equal to minus the integral of $f(u,v)$ along Γ (see fig. 3.2):

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} = \frac{\pi \mathcal{N}_d \Gamma(2\lambda)}{2^{2\lambda+3}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}} \int_{-\infty}^{\infty} dv \int_\Gamma \mathrm{d}u \frac{(u-2\pi i)(v-2\pi i)}{\left(\cosh\frac{v}{2}+\cosh\frac{u}{2}\right)^{2\lambda}} e^{-\frac{(v-2i\pi)^2}{4\kappa\tau}-\frac{(u-2i\pi)^2}{4\kappa(\beta-\tau)}} \tag{3.28}$$

The path Γ is composed by three straight segments (III,V and VII) and two semi-circumferences (IV and VI). The former three contributions either cancel or vanish because the resulting integrand is an odd function under reflection with respect to the axis Im u. Instead the latter two (i.e. IV and VI) yield

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{disk}} &= -\frac{\pi^2 i \mathcal{N}_d \Gamma(2\lambda)}{2^{2\lambda+4}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}}\int_{-\infty}^{\infty} dv\; e^{-\frac{(v-2i\pi)^2}{4\kappa\tau}}(v-2\pi i)\times\\ &\times\left(\text{Res}_{u=v+2\pi i}\left[\frac{(u-2\pi i)e^{-\frac{(u-2i\pi)^2}{4\kappa(\beta-\tau)}}}{\left(\cosh\frac{v}{2}+\cosh\frac{u}{2}\right)^{2\lambda}}\right]+\text{Res}_{u=-v+2\pi i}\left[\frac{(u-2\pi i)e^{-\frac{(u-2i\pi)^2}{4\kappa(\beta-\tau)}}}{\left(\cosh\frac{v}{2}+\cosh\frac{u}{2}\right)^{2\lambda}}\right]\right)=\\ &= -\frac{\pi^2 i \mathcal{N}_d \Gamma(2\lambda)}{2^{2\lambda+3}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}}\int_{-\infty}^{\infty} dv\;(v-2\pi i)e^{-\frac{(v-2i\pi)^2}{4\kappa\tau}}\text{Res}_{u=v}\left[\frac{u e^{-\frac{u^2}{4\kappa(\beta-\tau)}}}{\left(\cosh\frac{v}{2}-\cosh\frac{u}{2}\right)^{2\lambda}}\right],\end{aligned} \tag{3.29}$$

where we used the symmetry of the integrand to show that the two residues are equal.

3.3.1 Some interesting examples: small λ values

The representation (3.29) is very efficient in reconstructing the perturbative series at all orders. To illustrate how we can recover the series for small κ, we first focus on the case $\lambda = \frac{1}{2}$. Then the residue in (3.29) can be easily evaluated and is given by

$$\text{Res}_{u=v}\left(\frac{u\; e^{-\frac{u^2}{4\kappa(\beta-\tau)}}}{\cosh\frac{v}{2}-\cosh\frac{u}{2}}\right) = -2v\;\text{csch}\frac{v}{2}e^{-\frac{v^2}{4\kappa(\beta-\tau)}}. \tag{3.30}$$

Next we can recast the integral (3.29) as follows

$$\langle \mathcal{O}^{(\frac{1}{2})}(\tau)\rangle_\beta^{\text{disk}} = \frac{i\pi^2\mathcal{N}_d e^{\frac{\pi^2}{\beta\kappa}}}{2^{2\lambda+1}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}}\int_{-\infty}^{+\infty}\text{d}v\frac{v\,(v-2i\pi)}{\sinh\frac{v}{2}}\; e^{-\frac{\beta\left(v-\frac{2i\pi(\beta-\tau)}{\beta}\right)^2}{4\kappa\tau(\beta-\tau)}}. \tag{3.31}$$

It is convenient to center the gaussian weight in (3.31) $v=0$ through the shift $v\to v+\frac{2i\pi(\beta-\tau)}{\beta}$;

$$\langle \mathcal{O}^{(\frac{1}{2})}(\tau)\rangle_\beta^{\text{disk}} = -\frac{i\kappa^{-\frac{1}{2}}\beta^{\frac{3}{2}}}{8\pi^{\frac{3}{2}}\tau^{3/2}(\beta-\tau)^{3/2}}\int_{-\infty}^{+\infty}\text{d}v\left(v^2+\frac{2i\pi(\beta-2\tau)}{\beta}v+\frac{4\pi^2\tau(\beta-\tau)}{\beta^2}\right)\frac{e^{-\frac{\beta v^2}{4\kappa\tau(\beta-\tau)}}}{\sinh\left(\frac{v}{2}-\frac{i\pi\tau}{\beta}\right)} \tag{3.32}$$

Now we replace $1/\sinh(\cdots)$ with its representation in terms of exponentials

$$\text{csch}\left(\frac{v}{2}-\frac{\pi i\tau}{\beta}\right) = \frac{2}{v}e^{-\frac{\pi i\tau}{\beta}}\left(\frac{v\;e^{\frac{v}{2}}}{e^{v-\frac{2\pi i\tau}{\beta}}-1}\right), \tag{3.33}$$

and recognize that the quantity between parenthesis is the generating functional of the so-called generalized Apostol-Bernoulli polynomials of degree one. The definition of these polynomials for general degree and some of their properties are briefly discussed in app. A.8. Therefore we directly write

$$\text{csch}\left(\frac{v}{2}-\frac{\pi i\tau}{\beta}\right) = 2e^{-\frac{\pi i\tau}{\beta}}\sum_{n=0}^{\infty}\mathcal{B}_n^{(1)}\left(\frac{1}{2},e^{-\frac{2\pi i\tau}{\beta}}\right)\frac{v^{n-1}}{n!}. \tag{3.34}$$

If we integrate in v term by term using the expansion (3.34), we encounter only powers of v averaged over a gaussian weight. We find convenient to treat separately the even and the odd powers in (3.34). Reordering the powers in κ produced by the gaussian integrations, we obtain the following perturbative series for the expectation value of the bi-local operator with $\lambda = \frac{1}{2}$:

$$\begin{aligned}\langle \mathcal{O}^{(\frac{1}{2})}(\tau)\rangle_\beta^{\text{disk}} =& \frac{e^{-\frac{\pi i \tau}{\beta}}}{\pi^{\frac{1}{2}}}\sum_{p=0}^{\infty}\frac{2^{2p}\kappa^p\tau^p(\beta-\tau)^p\Gamma\left[p+\frac{1}{2}\right]}{(2p)!\beta^p}\left[\frac{(\beta-2\tau)}{\tau(\beta-\tau)}\mathcal{B}_{2p}^{(1)}\left(\frac{1}{2},e^{-\frac{2\pi i\tau}{\beta}}\right)-\right.\\ &\left.-\frac{i}{(2p+1)}\mathcal{B}_{2p+1}^{(1)}\left(\frac{1}{2},e^{-\frac{2\pi i\tau}{\beta}}\right)-\frac{ip\beta}{\pi\tau(\beta-\tau)}\mathcal{B}_{2p-1}^{(1)}\left(\frac{1}{2},e^{-\frac{2\pi i\tau}{\beta}}\right)\right].\end{aligned} \tag{3.35}$$

The case $\lambda = 1$ is slightly more involved: the explicit form of the residue is

$$\text{Res}_{u=v}\left(\frac{u\ e^{-\frac{u^2}{4\kappa(\beta-\tau)}}}{(\cosh\frac{v}{2}-\cosh\frac{u}{2})^2}\right) = -\frac{2e^{-\frac{v^2}{4\kappa(\beta-\tau)}}}{\sinh^2\frac{v}{2}}\left(\frac{v^2}{\kappa(\beta-\tau)}+v\coth\left(\frac{v}{2}\right)-2\right), \tag{3.36}$$

and we can perform again the previous analysis. This time the dependence on $1/\sinh(\cdots)$ is accompanied by higher powers, namely $1/\sinh^2(\cdots)$ and $1/\sinh^3(\cdots)$. With computations similar to the case $\lambda = 1/2$, we can also obtain with little effort the all order expansion

$$\begin{aligned}\langle \mathcal{O}^{(1)}(\tau)\rangle_\beta^{\text{disk}} =& \frac{e^{-\frac{2i\pi\tau}{\beta}}}{\pi^{\frac{1}{2}}}\sum_{p=0}^{\infty}\frac{2^{2p}\kappa^p\tau^p(\beta-\tau)^p\Gamma\left(p+\frac{1}{2}\right)}{(2p)!\beta^p}\left[\left(\frac{p-1}{2p-1}\frac{\beta^2}{\tau^2(\beta-\tau)^2}-\frac{3}{\tau(\beta-\tau)}\right)\mathcal{B}_{2p}^{(2)}\left(1,e^{-\frac{2\pi i\tau}{\beta}}\right)\right.\\ &-\frac{3i\pi}{(2p+1)}\frac{\beta-2\tau}{\beta\tau(\beta-\tau)}\mathcal{B}_{2p+1}^{(2)}\left(1,e^{-\frac{2\pi i\tau}{\beta}}\right)-\frac{4\pi^2}{\beta^2(2p+1)(2p+2)}\mathcal{B}_{2p+2}^{(2)}\left(1,e^{-\frac{2\pi i\tau}{\beta}}\right)\\ &\left.-\frac{ip}{2\pi}\frac{\beta(\beta-2\tau)}{\tau^2(\beta-\tau)^2}\mathcal{B}_{2p-1}^{(2)}\left(1,e^{-\frac{2\pi i\tau}{\beta}}\right)\right]\end{aligned} \tag{3.37}$$

where also generalized Apostol-Bernoulli polynomials of degree 2 appears in the expansion.

The same analysis can be easily carried out for the trumpet. Here the starting point is

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_\beta^{\text{tr.}} = -\frac{\pi^2\Gamma(2\lambda)\mathcal{N}}{2^{2\lambda+1}\kappa^2(\beta-\tau)^{\frac{1}{2}}\tau^{\frac{3}{2}}}\int_{-\infty}^{+\infty}dv\ \text{Res}_{u=v}\left(\frac{u\ e^{-\frac{u^2}{4\kappa(\beta-\tau)}}}{\left(\cosh\frac{v}{2}-\cosh\frac{u}{2}\right)^{2\lambda}}\right)e^{-\frac{(v+2b\pi)^2}{4\kappa\tau}} \tag{3.38}$$

The case $\lambda = 1/2$ and $\lambda = 1$ are again obtained along the same lines discussed above and one gets

$$\langle \mathcal{O}^{(\frac{1}{2})}(\tau)\rangle_\beta^{\text{tr.}} = \frac{e^{-\frac{b\pi\tau}{\beta}}}{\pi^{\frac{1}{2}}}\sum_{p=0}^{\infty}\frac{2^{2p}\kappa^p\tau^p(\beta-\tau)^p\Gamma\left[p+\frac{1}{2}\right]}{(2p!)\beta^p}\left(\frac{1}{\tau}\mathcal{B}_{2p}^{(1)}\left(\frac{1}{2},e^{-\frac{2b\pi\tau}{\beta}}\right)-\frac{2b\pi}{(2p+1)\beta}\mathcal{B}_{2p+1}^{(1)}\left(\frac{1}{2},e^{-\frac{2b\pi\tau}{\beta}}\right)\right) \tag{3.39}$$

and

$$\begin{aligned}\langle \mathcal{O}^{(1)}(\tau)\rangle_\beta^{\text{tr.}} =& \frac{e^{-\frac{2b\pi\tau}{\beta}}}{\pi^{\frac{1}{2}}}\sum_{p=0}^{\infty}\frac{2^{2p}\kappa^p\tau^p(\beta-\tau)^p\Gamma\left[p+\frac{1}{2}\right]}{(2p!)\beta^p}\left[\left(\frac{1}{\tau^2}-\frac{\beta+b\pi\tau}{\tau^2(\beta-\tau)(2p-1)}\right)\mathcal{B}_{2p}^{(2)}\left(1,e^{-\frac{2b\pi\tau}{\beta}}\right)\right.\\ &-\frac{4b\pi}{\beta\tau(2p+1)}\mathcal{B}_{2p+1}^{(2)}(1,e^{-\frac{2\pi b\tau}{\beta}})+\frac{4b^2\pi^2}{\beta^2(2p+1)(2p+2)}\mathcal{B}_{2p+2}^{(2)}\left(1,e^{-\frac{2b\pi\tau}{\beta}}\right)+\frac{p\beta}{\tau^2(\beta-\tau)(2p-1)}\mathcal{B}_{2p-1}^{(2)}\left(1,e^{-\frac{2\pi b\tau}{\beta}}\right)\\ &\left.+\frac{1}{\tau^2(\beta-\tau)(2p-1)}\left(\beta\mathcal{B}_{2p}^{(3)}\left(1,e^{-\frac{2\pi b\tau}{\beta}}\right)-\frac{2b\pi\tau}{(2p+1)}\mathcal{B}_{2p+1}^{(3)}\left(1,e^{-\frac{2\pi b\tau}{\beta}}\right)\right)\right]\end{aligned} \tag{3.40}$$

The trumpet expressions become a little bit more cumbersome since we lose the symmetry $\tau \to \beta - \tau$, as expected since we are working in the zero winding sector. As a consequence, we also notice the presence of generalized Apostol-Bernoulli polynomials of degree 3.

3.3.2 The case of generic n

Having trained with the simplest cases, we are ready now to examine the case of generic semi-integer λ. The details of the computation are presented in app. A.8 and app. A.10. Here we summarise the mains steps of our analysis. To begin with, we have to compute the residue for generic n. The structure of the answer is

$$\mathrm{Res}_{u=v}\left(\frac{u\ e^{-\alpha u^2}}{\left(\cosh\frac{v}{2}-\cosh\frac{u}{2}\right)^{2\lambda}}\right)=e^{-\alpha v^2}f(v) \tag{3.41}$$

where $\alpha=\frac{1}{4\kappa(\beta-\tau)}$ and $f(v)$ can be written as

$$f(v)=e^{\lambda v}\sum_{\ell=0}^{2\lambda-1}\frac{(-2)^{4\lambda-\ell-1}\alpha^{\frac{2\lambda-\ell-2}{2}}\mathcal{H}_{2\lambda-\ell}(\sqrt{\alpha}v)}{(2\lambda-\ell-1)!}\sum_{j=0}^{\ell}\frac{\mathrm{B}^{(2\lambda)}_{\ell-j}(\lambda)\mathcal{B}^{(2\lambda)}_{j+2\lambda}(\lambda,e^v)}{(\ell-j)!(2\lambda+j)!}. \tag{3.42}$$

In (3.42) $\mathcal{H}_n(x)$ stands for the usual Hermite polynomials, while $\mathcal{B}^n_k(x,y)$ and $\mathrm{B}^n_k(x)$ are respectively generalized Apostol-Bernoulli of degree n and generalized Bernoulli polynomials. Their definition and some of their properties are discussed in app. A.8 together with the details of the computation. Then the remaining integral in v takes the following form:

$$\langle\mathcal{O}^{(\lambda)}(\tau)\rangle^{\rm disk}_\beta=-\frac{i\pi^2\Gamma(2\lambda)\mathcal{N}_d}{2^{2\lambda+2}\kappa^3\tau^{3/2}(\beta-\tau)^{3/2}}\int_{-\infty}^{+\infty}\mathrm{d}v\ (v-2i\pi)f(v)\ e^{-\frac{(v-2i\pi)^2}{4\kappa\tau}-\frac{v^2}{4\kappa(\beta-\tau)}}. \tag{3.43}$$

Next we perform the shift $v\mapsto v+\frac{2\pi i(\beta-\tau)}{\beta}$ to move the gaussian center around $v=0$ and we expand the f around $v=0$. Subsequently we reorganize the result exploiting the properties of Hermite polynomials and perform the integration over v to get

$$\begin{aligned}\langle\mathcal{O}^{(\lambda)}(\tau)\rangle^{\rm disk}_\beta=&-\frac{i\pi^2(-1)^{2\lambda}\Gamma(2\lambda)\kappa^{2\lambda-\frac{3}{2}}\beta^{\frac{3}{2}}}{2^{2\lambda+3}\pi^{\frac{7}{2}}\tau^{\frac{3}{2}}(\beta-\tau)^{\frac{3}{2}}}\sum_{m=0}^{\infty}\frac{2^{m+1}(\kappa\tau(\beta-\tau))^{\frac{m+1}{2}}}{\beta^{\frac{m+1}{2}}m!}\sum_{\ell=0}^{2\lambda-1}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}c^{(\lambda)}_{\ell,m}(\beta,\tau)\times\\&\times\left(\frac{1}{4\kappa(\beta-\tau)}\right)^{\frac{2\lambda-\ell-2}{2}}\sum_{k=0}^{2\lambda-\ell}\binom{2\lambda-\ell}{k}\left(\frac{2\pi i\sqrt{\beta-\tau}}{\beta\sqrt{\kappa}}\right)^{2\lambda-\ell-k}\left(\sqrt{\frac{4\kappa\tau(\beta-\tau)}{\beta}}\mathrm{P}_{k,m+1}-\frac{2\pi i\tau}{\beta}\mathrm{P}_{k,m}\right),\end{aligned} \tag{3.44}$$

where

$$\mathrm{P}_{k,m}=\int_{-\infty}^{\infty}dv\ v^m\mathcal{H}_k\left(\sqrt{\frac{\tau}{\beta}}v\right)e^{-v^2}. \tag{3.45}$$

and

$$c^{(\lambda)}_{\ell,m}(\beta,\tau)=e^{-\frac{2\pi i\lambda\tau}{\beta}}\sum_{j=0}^{\ell}\frac{(j+m)!}{(\ell-j)!(2\lambda+j+m)!j!}\ \mathrm{B}^{(2\lambda)}_{\ell-j}(\lambda)\mathcal{B}^{(2\lambda)}_{j+2\lambda+m}\left(\lambda;e^{-\frac{2\pi i\tau}{\beta}}\right). \tag{3.46}$$

The integral $\mathrm{P}_{k,m}$ yields a polynomial of order k in $\sqrt{\frac{\tau}{\beta}}$ and its explicit expression in terms of the associated Legendre function is given in (A.104). Obviously $\mathrm{P}_{k,m}$ is different from zero only when $m+k$ is an even number. We use this selection rule to rearrange the our expression and to arrive to the final expansion

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta}^{\text{disk}} = \frac{\pi^{2\lambda}}{\beta^{2\lambda}\sin^{2\lambda}\frac{\pi\tau}{\beta}} + \frac{(-\pi i)^{2\lambda+1}\Gamma(2\lambda)}{(2\beta)^{2\lambda}\pi^{\frac{5}{2}}}\sum_{n=1}^{\infty}\left(\frac{4\kappa\tau(\beta-\tau)}{\beta}\right)^{n}\sum_{\ell=0}^{\min(n,2\lambda-1)}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}\times$$
$$\times\left[\sum_{k=0}^{\min(2n-2\ell-1,2\lambda-\ell)}\frac{c^{(\lambda)}_{\ell,2n-2\ell-k-1}(\beta,\tau)}{(2n-2\ell-k-1)!}\binom{2\lambda-\ell}{k}\left(\frac{\beta^2}{4\pi i\tau(\beta-\tau)}\right)^{\ell+k}\left(\frac{\tau}{\beta}\right)^{\frac{k}{2}-1}\mathrm{P}_{k,2n-2\ell-k}-\right.$$
$$\left.-2\pi i\sum_{k=0}^{\min(2n-2\ell,2\lambda-\ell)}\frac{c^{(\lambda)}_{\ell,2n-2\ell-k}(\beta,\tau)}{(2n-2\ell-k)!}\binom{2\lambda-\ell}{k}\left(\frac{\beta^2}{4\pi i\tau(\beta-\tau)}\right)^{\ell+k}\left(\frac{\tau}{\beta}\right)^{\frac{k}{2}}\mathrm{P}_{k,2n-2\ell-k}\right]. \tag{3.47}$$

We make a couple of observations on the above expression: first of all, we notice that at sufficiently large order in κ the non-trigonometric dependence on τ cannot grow arbitrarily, being a polynomial bounded by the weight of the bi-local operators itself. Moreover we expect the original symmetry $\tau \to \beta - \tau$ to be preserved by the expansion: looking at the structure of the coefficients it is not manifest but we checked its presence till the order κ^6. Making explicit this symmetry should probably simplify the final formula. A second remark concerns the trigonometric dependence of the generic perturbative term and its singularity properties as $\tau \to 0$. The trigonometric dependence is completely encoded into the coefficients $c^{(\lambda)}_{\ell,m}(\beta,\tau)$: we expect the presence of negative powers of $\sin(\tau/\beta)$, generating a singular behavior at small τ. This fact is also evident from the singularity appearing in this limit for the generalized Apostol-Bernoulli polynomials.

3.4 Expansion for $\beta \to \infty$

It is now interesting to concentrate on the zero temperature limit of the bi-local correlator to check explicitly the agreement with [90]. The structure of our integrals simplifies significantly as $\beta \to \infty$: moreover we observe that both the disk and the trumpet share the same behavior in this regime, since we expected that as the total boundary length diverges (while keeping b fixed) the presence of an hole in the interior becomes negligible.

However this limit cannot be directly extracted from the final result of subsec. 3.3.2 since the limit of generalized Apostol-Bernoulli polynomial $\mathcal{B}^{(\ell)}_n(x,\mu)$ is discontinuous when μ approaches one. Therefore it is convenient to go back to eq. (3.26) and take the limit $\beta \to \infty$ at this level. The limit of the integrand and of its normalization is smooth and we get

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = \frac{\kappa^{2\lambda}\Gamma(2\lambda)}{2^{2\lambda+4}\pi^{\frac{5}{2}}\kappa^{\frac{3}{2}}\tau^{\frac{3}{2}}}\int_{-\infty}^{\infty} du dv\, \frac{(u-2\pi i)(v-2\pi i)}{(\cosh\frac{u}{2}+\cosh\frac{v}{2})^{2\lambda}}e^{-\frac{(v-2i\pi)^2}{4\kappa\tau}} \tag{3.48}$$

The integral over u can be now evaluated in closed form. The linear term in u vanishes since it is odd, while the contribution proportional to $2\pi i$ yields

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = \frac{8\pi i\kappa^{2\lambda}\Gamma(2\lambda)}{2^{2\lambda+4}\pi^{\frac{5}{2}}\kappa^{\frac{3}{2}}\tau^{\frac{3}{2}}}\int_{-\infty}^{\infty} dv\, \frac{(v-2\pi i)}{\sinh^{2\lambda}\frac{v}{2}}e^{-\frac{(v-2i\pi)^2}{4\kappa\tau}}\mathcal{Q}_{2\lambda-1}\left(\coth\frac{v}{2}\right), \tag{3.49}$$

where we have used that

$$\frac{\mathcal{Q}_n(\coth\frac{v}{2})}{\sinh^{n+1}\frac{v}{2}} = \frac{1}{4}\int_{-\infty}^{+\infty}\frac{\mathrm{d}u}{\left(\cosh\frac{v}{2}+\cosh\frac{u}{2}\right)^{n+1}}. \tag{3.50}$$

In eq. (3.50) $\mathcal{Q}_n(z)$ stands for the so-called Legendre function of the second kind. Next we eliminate the dependence on the linear factor $(v-2\pi i)$ by integrating by parts and we write

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = \frac{16\pi i \kappa^{2\lambda}\Gamma(2\lambda)\kappa\tau}{2^{2\lambda+4}\pi^{\frac{5}{2}}\kappa^{\frac{3}{2}}\tau^{\frac{3}{2}}}\int_{-\infty}^{\infty} dv\; e^{-\frac{(v-2i\pi)^2}{4\kappa\tau}} \frac{d}{dv}\left(\frac{\mathcal{Q}_{2\lambda-1}\left(\coth\frac{v}{2}\right)}{\sinh^{2\lambda}\frac{v}{2}}\right), \tag{3.51}$$

Exploiting the recurrence relation for the Legendre function of the second kind and its derivatives for integer indices, it is straightforward to show that

$$\frac{\mathrm{d}}{\mathrm{d}v}\left[\frac{\mathcal{Q}_{2\lambda-1}\left(\coth\frac{v}{2}\right)}{\sinh^{2\lambda}\frac{v}{2}}\right] = -\lambda\frac{\mathcal{Q}_{2\lambda}\left(\coth\frac{v}{2}\right)}{\sinh^{2\lambda}\frac{v}{2}} \tag{3.52}$$

Thus

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = -\frac{16\pi i\lambda \kappa^{2\lambda}\Gamma(2\lambda)\kappa\tau}{2^{2\lambda+4}\pi^{\frac{5}{2}}\kappa^{\frac{3}{2}}\tau^{\frac{3}{2}}}\int_{-\infty}^{\infty} dv\; e^{-\frac{(v-2i\pi)^2}{4\kappa\tau}} \frac{\mathcal{Q}_{2\lambda}\left(\coth\frac{v}{2}\right)}{\sinh^{2\lambda}\frac{v}{2}}. \tag{3.53}$$

Although the structure of the integrand suggests the possible presence of a singularity at $v=0$, it is not difficult to check this singularity is only apparent. In fact a careful analysis of the integrand shows that it is completely regular at $v=0$. When 2λ is an integer $\mathcal{Q}_{2\lambda}$ can be expressed in terms of the Legendre polynomials. Specifically the following identity holds

$$\mathcal{Q}_{2\lambda}\left(\coth\frac{v}{2}\right) = \frac{1}{2}\mathcal{P}_{2\lambda}\left(\coth\frac{v}{2}\right)v - \mathcal{W}_{2\lambda-1}\left(\coth\frac{v}{2}\right) \tag{3.54}$$

with

$$\mathcal{W}_{2\lambda-1}\left(\coth\frac{v}{2}\right) = \sum_{k=1}^{2\lambda}\frac{1}{k}\mathcal{P}_{k-1}\left(\coth\frac{v}{2}\right)\mathcal{P}_{2\lambda-k}\left(\coth\frac{v}{2}\right) \tag{3.55}$$

Therefore the Legendre function of the second kind is not periodic under the shift $v\to v+2i\pi$, but we have

$$\mathcal{Q}_{2\lambda}\left(\coth\frac{v}{2}\right) \to \mathcal{Q}_{2\lambda}\left(\coth\frac{v}{2}\right) + i\pi\mathcal{P}_{2\lambda}\left(\coth\frac{v}{2}\right). \tag{3.56}$$

If we perform this shift in our integral we find

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = -\frac{16\pi i(-1)^{2\lambda}\lambda \kappa^{2\lambda}\Gamma(2\lambda)\kappa\tau}{2^{2\lambda+4}\pi^{\frac{5}{2}}\kappa^{\frac{3}{2}}\tau^{\frac{3}{2}}}\int_{-\infty}^{\infty} dv\; e^{-\frac{v^2}{4\kappa\tau}} \frac{\left[\mathcal{Q}_{2\lambda}\left(\coth\frac{v}{2}\right)+\pi i\mathcal{P}_{2\lambda}\left(\coth\frac{v}{2}\right)\right]}{\sinh^{2\lambda}\frac{v}{2}}. \tag{3.57}$$

The combination $\sinh^{-2\lambda}\left(\frac{v}{2}\right)\mathcal{Q}_{2\lambda}\left(\coth\frac{v}{2}\right)$ is an odd function and so its contribution to the integral (3.57) identically vanishes. So we are left with the term proportional to $\mathcal{P}_{2\lambda}$ only, i.e.

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = \frac{16\pi^2(-1)^{2\lambda}\lambda \kappa^{2\lambda}\Gamma(2\lambda)\kappa\tau}{2^{2\lambda+4}\pi^{\frac{5}{2}}\kappa^{\frac{3}{2}}\tau^{\frac{3}{2}}}\int_{-\infty}^{\infty} dv\; e^{-\frac{v^2}{4\kappa\tau}} \frac{\mathcal{P}_{2\lambda}\left(\coth\frac{v}{2}\right)}{\sinh^{2\lambda}\frac{v}{2}}. \tag{3.58}$$

A remark is now in order. The final integral is singular at $v=0$. If we perform, as we should, the translation $v\to v+2i\pi$ as a change of path in the complex $v-$plane, we have to deform a little bit the contour to avoid precisely $v=0$ since the integrand possesses a pole there. This small deformation provides us the prescription on how to regularize the singularity (it is PV-like prescription). In the following, we neglect this issue and regularize this singularity using an analytic regularization, which is more straightforward and produces the same result.

Next we use the following representation of the Legendre polynomials $\mathcal{P}_n(x)$

$$\mathcal{P}_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x-1)^n \left(\frac{x+1}{x-1}\right)^k \tag{3.59}$$

For $x = \coth\frac{v}{2}$ this representation simplifies

$$\mathcal{P}_{2\lambda}\left(\coth\frac{v}{2}\right) = \left(\frac{1}{e^v-1}\right)^{2\lambda}\sum_{k=0}^{2\lambda}\binom{2\lambda}{k}^2 e^{kv} \tag{3.60}$$

and our integral becomes

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = \frac{16\pi^2(-1)^{2\lambda}\lambda\kappa^{2\lambda}\Gamma(2\lambda)\kappa\tau}{2^{2\lambda+4}\pi^{\frac{5}{2}}\kappa^{\frac{3}{2}}\tau^{\frac{3}{2}}}\sum_{k=0}^{2\lambda}\tbinom{2\lambda}{k}^2\int_{-\infty}^{\infty} dv\; e^{-\frac{v^2}{4\kappa\tau}}\left(\frac{1}{e^v-1}\right)^{4\lambda} e^{v(k+\lambda)}. \tag{3.61}$$

We can now expand part of the integrand in terms of generalized Bernoulli polynomials

$$\left(\frac{1}{e^v-1}\right)^{4\lambda} e^{v(k+\lambda)} = \sum_{n=0}^{\infty}\frac{\mathrm{B}_n^{(4\lambda)}(k+\lambda)}{n!}\, v^{n-4\lambda}. \tag{3.62}$$

This expansion explicitly exhibits the aforementioned poles present in the integrand. Equation (3.62) is a Laurent series which contains negative powers up to -4λ. Then we have to compute

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = \frac{16\pi^2(-1)^{2\lambda}\lambda\kappa^{2\lambda}\Gamma(2\lambda)\kappa\tau}{2^{2\lambda+4}\pi^{\frac{5}{2}}\kappa^{\frac{3}{2}}\tau^{\frac{3}{2}}}\sum_{k=0}^{2\lambda}\tbinom{2\lambda}{k}^2\sum_{n=0}^{\infty}\frac{\mathrm{B}_n^{(4\lambda)}(k+\lambda)}{n!}\int_{-\infty}^{\infty} \mathrm{d}v\; v^{n-4\lambda}e^{-\frac{v^2}{4\kappa\tau}}. \tag{3.63}$$

Since 4λ is even, the integral is different from zero only for even n. If we set $n = 2p$ with $p \in \mathbb{N}$, the gaussian integral can be now easily performed and we always get

$$\int_{-\infty}^{\infty} \mathrm{d}v\; v^{2(p-2\lambda)}e^{-\frac{v^2}{4\kappa\tau}} = (2\sqrt{\kappa\tau})^{2(p-2\lambda)+1}\Gamma\left[\frac{2p-4\lambda+1}{2}\right], \tag{3.64}$$

where we have defined the integral for negative powers of v by analytic continuation. By substituting it in (3.63) we find

$$\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = \frac{(2\lambda)!(-1)^{2\lambda}}{2^{2\lambda}\pi^{\frac{1}{2}}}\frac{1}{\tau^{2\lambda}}\sum_{p=0}^{\infty}\frac{(4\kappa\tau)^p}{(2p)!}\sum_{k=0}^{2\lambda}\tbinom{2\lambda}{k}^2\mathrm{B}_{2p}^{(4\lambda)}(k+\lambda)\Gamma\left[\frac{2p-4\lambda+1}{2}\right]. \tag{3.65}$$

To better understand the structure of this perturbative expansion it is convenient to separate positive and negative powers of τ. Recalling the value of the Gamma function for seminteger values of the argument, we immediately find

$$\begin{aligned}\langle \mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta\to\infty} = \frac{(2\lambda)!}{\tau^{2\lambda}}\Bigg\{&\sum_{p=0}^{2\lambda-1}\frac{(\kappa\tau)^p}{(2p)!}(-1)^p\,\tfrac{(2\lambda-p)!}{(4\lambda-2p)!}\sum_{k=0}^{2\lambda}\tbinom{2\lambda}{k}^2\mathrm{B}_{2p}^{(4\lambda)}(k+\lambda)+\\ &+\tfrac{(-1)^{2\lambda}}{2^{2\lambda}}\sum_{r=2\lambda}^{\infty}\frac{(2\kappa\tau)^r}{(2r)!}(2r-4\lambda-1)!!\sum_{k=0}^{2\lambda}\tbinom{2\lambda}{k}^2\mathrm{B}_{2r}^{(4\lambda)}(k+\lambda)\Bigg\}\end{aligned} \tag{3.66}$$

One can easily check that for $\lambda = \frac{1}{2}$ and $\lambda = 1$ this result exactly reproduces the expressions given by [90], where the first perturbative orders are presented.

3.5 Mordell integrals

As anticipated in the introduction, in this section our goal is to show that the bi-local correlator can be in general expressed as a combination of Mordell integrals. Concretely, let us go back to the case $2\lambda \in \mathbb{N}$ and to be more specific we focus on $\lambda = 1/2$ for the disk. The expression (3.32) can be rewritten as follows

$$\langle \mathcal{O}^{(\frac{1}{2})}(\tau)\rangle_\beta^{\text{disk}} = -\frac{2\pi^{\frac{3}{2}} i\kappa^{-\frac{1}{2}}\beta^{\frac{3}{2}} e^{\frac{i\pi\tau}{\beta}}}{\tau^{3/2}(\beta-\tau)^{3/2}} \int_{-\infty}^{+\infty} \mathrm{d}v \left(\underset{(a)}{v^2} + \underset{(b)}{\frac{i(\beta-2\tau)}{\beta} v} + \underset{(c)}{\frac{\tau(\beta-\tau)}{\beta^2}} \right) \frac{e^{-\frac{\pi^2\beta v^2}{\kappa\tau(\beta-\tau)}+\pi v}}{e^{2\pi v} - e^{\frac{2\pi i\tau}{\beta}}}, \tag{3.67}$$

where we have also scaled the integration variable by 2π. We recognize three different contributions: the third one can be immediately identified with the so-called Mordell integral [101], which appears in number theory and in the theory of Mock-theta functions [136]. The general form of the Mordell integral is

$$\mathcal{M}(x,\theta,\omega) = \int_{-\infty}^{\infty} dt\, \frac{e^{\pi i\omega t^2 - 2\pi x t}}{e^{2\pi t} - e^{2\pi i\theta}} = e^{-\pi i(\theta^2\omega + 2\theta x + 2\theta)} \frac{F[(x+\theta)/\omega, -1/\omega] + i\omega F[x+\theta\omega, \omega]}{\omega\theta_{11}(x+\theta\omega,\omega)}. \tag{3.68}$$

The function $F(x,\omega)$ admits a $q-$expansion of the form

$$F[x,\omega] = -i \sum_{m\in\mathbb{Z}} \frac{(-1)^m q^{(m+1/2)^2} e^{2\pi i(m+1/2)x}}{1+q^{2m+1}}, \tag{3.69}$$

where $q = e^{\pi i\omega}$. The denominator is one of the usual Jacobi theta function and its $q-$expansion is

$$\theta_{11}(x,\omega) = -i \sum_{m\in\mathbb{Z}} (-1)^m q^{(m+1/2)^2} e^{2\pi i(m+1/2)x}. \tag{3.70}$$

In our case, we have $x = -\frac{1}{2}$, $\theta = \frac{\tau}{\beta}$ and $\omega = \frac{\pi i\beta}{\kappa\tau(\beta-\tau)}$. The other two contributions, (a) and (b), are proportional to the second and first derivatives of the Mordell integral with respect to x. Then the complete result for the bi-local correlator at $\lambda = 1/2$:

$$\begin{aligned}\langle \mathcal{O}^{(\frac{1}{2})}(\tau)\rangle_\beta^{\text{disk}} = -\frac{2\pi^{\frac{3}{2}} i\kappa^{-\frac{1}{2}}\beta^{\frac{3}{2}} e^{\frac{i\pi\tau}{\beta}}}{\tau^{3/2}(\beta-\tau)^{3/2}} \left(\frac{1}{4\pi^2}\partial_x^2\mathcal{M}\left(-\frac{1}{2},\frac{\tau}{\beta},\frac{\pi i\beta}{\kappa\tau(\beta-\tau)}\right) - \right.\\ \left. -\frac{i(\beta-2\tau)}{2\pi\beta}\partial_x\mathcal{M}\left(-\frac{1}{2},\frac{\tau}{\beta},\frac{\pi i\beta}{\kappa\tau(\beta-\tau)}\right) + \frac{\tau(\beta-\tau)}{\beta^2}\mathcal{M}\left(-\frac{1}{2},\frac{\tau}{\beta},\frac{\pi i\beta}{\kappa\tau(\beta-\tau)}\right)\right)\end{aligned} \tag{3.71}$$

The structure of the bi-local correlator for 2λ generic integer is not so different. In fact, by carefully inspecting (A.86) we can easily verify that it is given by a sum of integrals of the following form

$$\int_{-\infty}^{\infty} dt\ t^n \frac{e^{\pi i\omega t^2 - 2\pi x t}}{(e^{2\pi t} - e^{2\pi i\theta})^m}, \tag{3.72}$$

where m and n are integers. However any integral of this kind can be evaluated in terms of the original Mordell integral:

$$\int_{-\infty}^{\infty} dt\ t^n \frac{e^{\pi i\omega t^2 - 2\pi x t}}{(e^{2\pi t} - e^{2\pi i\theta})^m} = \frac{(-1)^n}{(m-1)!}\left(\frac{1}{2\pi}\partial_x\right)^n \left(\frac{e^{-2\pi i\theta}}{2\pi i}\partial_\theta\right)^{m-1} \mathcal{M}(x,\theta,\omega) \tag{3.73}$$

In other words, the correlators are completely controlled by this kind of functions.

Resurgence in JT gravity at finite cutoff 4

A promising and challenging attempt to obtain more "local" observables in quantum gravity certainly consists in putting the gravitational degrees of freedom in a box, imposing suitable boundary conditions on the metric at some finite spatial extent. A natural question in this context is how to correctly interpret the definition of gravity on bounded regions of spacetime in the spirit of AdS/CFT and holograhic correspondence.

We have argued in 2.3.3 that a renewed interest in a class of solvable irrelevant deformations of two-dimensional CFTs, the $T\bar{T}$ deformations [135, 118, 19], has suggested an innovative strategy to address the above issue, at least in low dimensions. It was proposed in [89] that $T\bar{T}$-deformed CFTs could realize the holographic dual of AdS_3 gravities on a finite patch, as we have sketched in 2.3.3. [1] This correspondence has been checked in various ways [70, 47, 68], but the consistency of the proposal is still under scrutiny. An unsatisfying feature of this duality is that for large enough energies, the spectrum of the deformed boundary theory becomes complex, implying a potential breakdown of unitarity for the bulk theory.

While there have been attempts to generalize the conjecture to higher dimensions [123], a simpler context, where we can accurately study the status of the proposal, is to consider the finite cutoff version of Jackiw–Teitelboim (JT) gravity [76, 125]. In its mordern formulation [86], the JT path integral itself is defined as a limit procedure from a cut-off theory: after imposing Dirichlet boundary conditions at some finite distance, the proper boundary length and the boundary value for the dilaton are scaled appropriately in the large area limit to preserve the boundary degrees of freedom . We have shown in 2.1.2 that, in so doing, JT gravity reduces to a solvable one-dimensional theory, the Schwarzian quantum mechanics [6, 119, 95]. The natural expectation is that at finite cutoff the relevant dual formulation is provided by a $T\bar{T}$-deformed version of such a theory [63, 62].

The partition function of JT gravity restricted on a finite AdS_2 subregion has been computed in [73], using two different approaches based on either canonical or path-integral quantization[2]. The results of both methods are mutually consistent and are directly related to the $T\bar{T}$ deformation of the Schwarzian theory. An important issue addressed in [73] concerns the spectrum of the deformed theory, which complexifies above a certain energy threshold. This fact can be easily verified looking at equation (2.241), which expresses the spectrum of $T\bar{T}$ deformed Schwarzian theory. As a consequence, the naïve integration prescription does not generate a well-defined partition function. A consistent partition function can be obtained instead by adding contributions originating from a nonperturbative branch, but the related spectral density becomes not positive definite, calling for a physical interpretation. Moreover, in the analysis of

[1]The crucial element in favor of this conjectured duality is that the conformal Ward identity of the relevant CFT translates, in the presence of the deformation, into a second-order functional differential equation that closely resembles the Wheeler–DeWitt equation of AdS_3 gravity.

[2]See also the interesting alternative investigation [121], relying on a completely different method

[73], the nonperturbative completion seems to be accompanied by certain ambiguities that the authors cannot wholly fix in their approach. Last but not least, the construction of partition functions for arbitrary topologies configurations, relevant for the nonperturbative definition of JT gravity itself [113, 120], is left unexplored.

In this chapter, we present the results obtained in our paper [56] about JT gravity at finite cutoff, enlightening the miracolous role played by nonperturbative effects in providing a consistent definition of the theory. Specifically, we reexamine the theory starting from its definition in terms of the $T\bar{T}$-deformed Schwarzian quantum mechanics. We begin by studying the $T\bar{T}$ flow purely at a perturbative level and compute the entire perturbative series associated with the deformation parameter, both for the disk and the trumpet partition functions. We find that the resulting series has a vanishing radius of convergence and, as such, requires an appropriate nonperturbative completion. We then exploit the standard resurgence technique [32, 3], using the properties of the lateral Borel resummation, to take into account nonperturbative contributions. This procedure unambiguously brings into the game the nonperturbative configurations associated with the new energy branch and prescribes the correct integration contour. For the disk topology, we obtain the partition function in terms of a modified Bessel function of the first type, an expression already considered in [73]. The energy spectrum naturally spans a finite interval; however, the associated spectral density is not positive definite. The trumpet partition function experiences an even more dramatic modification: the nonperturbative corrections completely smooth out naïve singularity associated with the fact that the cutoff boundary could overlap with the geodesic boundary.

Relying on this observation, in the second part of the chapter, we explore the construction of the deformed version of the partition functions for arbitrary topologies, using the same gluing procedure derived for the undeformed theory [113] and exposed in the review 2.1.4. We remark that without the nonperturbative corrections, the relevant gluing integral would be ill-defined. The gluing procedure results in a consistent deformation of the standard Eynard–Orantin recursion relations [41] associated with the original theory (2.1.5): the deformed spectral curve and the higher-genus correlation functions are fully compatible with the flow equation of the $T\bar{T}$ deformation, and we find a precise mapping that encodes the flow. We stress that the non-positivity of the input spectral density does not spoil the consistency of the recursion relations, although its actual physical interpretation is still missing in our case. An essential step in our construction is the explicit evaluation of the cylinder partition function: it is closely related to the kernel necessary to engineer the Eynard–Orantin topological recursion formula [41] and is responsible for the "ramp" growth in the spectral form factor [112, 113, 111]. We derive in this last perspective its late-time behavior and observe the transition between the slope and the ramp phase at finite cutoff. Quite interestingly, the change of regime does not seem to depend on the value of the finite cutoff.

The chapter is structured as follows. We start in Section 4.1 where, after recalling basic results of JT gravity and its $T\bar{T}$ deformation, we compute the perturbative series arising from the deformation and its completion using the theory of resurgence, both for the disk and the trumpet. Section 4.2 is devoted to the spectral properties of the deformed theory: the relevant partition functions are seen arising from a compact spectrum and computed with a suitable integration contour. The spectral density is derived and found to be not positive defined. In Section 4.3 we construct the partition functions for arbitrary topologies, exploiting the gluing prescription of the undeformed theory. The consistency of this construction with the $T\bar{T}$ flow equation is discussed in Section 4.4. The extension of the Eynard–Orantin recursion relations in the deformed case is presented in Section 4.5, opening the possibility to interpret holographically the theory at finite cutoff. Finally, in Section 4.6, we study the deformed spectral form factor, deriving the behavior of the “slope” and “ramp” regimes.

4.1 Disk and trumpet

This section is devoted to studying both the disk and the trumpet partition functions for the theory at finite cutoff.

Our starting point is the partition function for the disk, which we computed in 2.1.2 by quantizing the dual Schwarzian theory on the boundary. As we have seen, the final result (2.42) is one-loop exact and can written as a Boltzman integral over a continuos spectral density (2.84):

$$Z^{\text{disk}}_{\text{Schw}} = \int_0^\infty \mathrm{d}E \, \frac{\phi_r \sinh\left(2\pi\sqrt{2\phi_r E}\right)}{2\pi^2} \, e^{-\beta E} \, . \tag{4.1}$$

where we have restored the dependence on the renormalized dilaton value ϕ_r.

For the trumpet, we know from (2.203) the corresponding Boltzmann integral is characterized by a modified density of states,

$$Z^{\text{trumpet}}_{\text{Schw}} = \int_0^\infty \mathrm{d}E \, \frac{\phi_r \cos\left(b\sqrt{2\phi_r E}\right)}{\pi\sqrt{2\phi_r E}} \, e^{-\beta E} \, . \tag{4.2}$$

We then saw that in [63] a certain integral deformation of the Schwarzian theory was considered, which is the one-dimensional analogue of the $T\bar{T}$ deformation 2.3.3. It introduces a shift of the energy levels of the theory which is exactly solvable in terms of a parameter t. As explicitly shown in equation (2.241), the shift is controlled by the following differential equation for the Hamiltonian H,

$$2\partial_t H = \frac{\phi_r H^2}{1-\phi_r t H} \, , \tag{4.3}$$

which governs the flow of the theory under the deformation.[3] The solutions form two branches

$$H_\pm(t) = \frac{1}{\phi_r t}\left(1 \mp \sqrt{1-2\phi_r t E}\right) \, , \tag{4.4}$$

however, only $H_+(t)$ reproduces the expected undeformed limit for $t \to 0$.

The deformed partition function[4] is defined by introducing the level shift in (4.1) and (4.2),

$$Z^{\text{disk}}_{T\bar{T}} = \int_0^\infty \mathrm{d}E \, \frac{\phi_r \sinh\left(2\pi\sqrt{2\phi_r E}\right)}{2\pi^2} \, e^{-(\beta/\phi_r t)(1-\sqrt{1-2\phi_r t E})} \, , \tag{4.5}$$

$$Z^{\text{trumpet}}_{T\bar{T}} = \int_0^\infty \mathrm{d}E \, \frac{\phi_r \cos\left(b\sqrt{2\phi_r E}\right)}{\pi\sqrt{2\phi_r E}} \, e^{-(\beta/\phi_r t)(1-\sqrt{1-2\phi_r t E})} \, . \tag{4.6}$$

[3]The $T\bar{T}$ deformation parameter t is defined is such a way to match the expansion parameter ϕ_b^{-2} that we will use for the bulk theory. To match the conventions of [63, 62, 73], one should set $t = 4\lambda/\phi_r$.

[4]The integrals (4.5) and (4.6) are actually ill-defined if the integration region spans the entire positive line. In [73], it was suggested to Wick-rotate the bare parameters of the theory to make sense of these integrals. In the following, we shall choose a different approach to address this issue.

It was then argued [73] that the deformed Schwarzian theory is the holographic dual of JT gravity at finite cutoff, establishing a correspondence between the $T\bar{T}$ deformation parameter t and the bulk gravity cutoff ϵ, that we introduced in 2.1.2 as a regulator for the radial position of the asymptotic AdS boundary in Poincaré coordinates (the boundary of AdS_2 should sit in fact at $z = 0$). For finite values of the boundary length, as the boundary of Σ is pushed away from the asymptotic boundary of AdS_2 and into the bulk, the boundary theory flows accordingly.

At infinite cutoff, the ϵ parameter is introduced to take the double-scaling limit where both the boundary length and the value of the dilaton on the boundary diverge, while their ratio $u = \beta/\phi_r$ remains constant. In principle, when considering the theory at finite cutoff, ϵ becomes redundant since the theory should only depend on its bare parameters. However, as mentioned in Section 2.1.2, ϵ plays an important role, as it parametrizes the deviation from the infinite cutoff limit and is the analog of the deformation parameter $t = \epsilon^2/\phi_r^2$ in the $T\bar{T}$-deformed Schwarzian theory. As proven in [63] for the first orders in the bulk ϵ-expansion, flowing the Schwarzian under $T\bar{T}$ is equivalent to considering more and more subleading terms in the expansion (2.24) around the $\epsilon \to 0$ pure Schwarzian theory.

The starting point of our analysis is to study the theory from the point of view of its perturbative expansion in t. For this reason, throughout most of the paper, we will find it convenient to express the results in terms u and t.

With a simple change of variables, we can recast the disk and the trumpet partition functions in (4.5) and (4.6) as

$$Z^{\text{disk}}(u,t) = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}s\, s\sinh(2\pi s)\, e^{-I(u,t;s)}\,, \tag{4.7}$$

$$Z^{\text{trumpet}}(u,b,t) = \frac{1}{\pi}\int_0^\infty \mathrm{d}s\, \cos(bs)\, e^{-I(u,t;s)}\,, \tag{4.8}$$

where the t-deformed action reads

$$I(u,t;s) = \frac{u}{t}\left(1-\sqrt{1-ts^2}\right)\,. \tag{4.9}$$

For any $t > 0$, the part of the action (4.9) depending on s becomes imaginary in the region $s \in (1/\sqrt{t}, +\infty)$ and the integral diverges. Moreover, the expression above is ambiguous as it is not clear a priori which of the two branches of the square root one should take when crossing the branch point at $s = 1/\sqrt{t}$. As we will see in the following, these aspects ultimately signal the presence of non-analytic (namely instanton-like) contributions in the parameter t, and care should be taken in choosing the correct prescription to take these into account and make sense of the integrals above.

4.1.1 Perturbative expansion

Despite the possible ambiguities in defining the integrals (4.7) and (4.8), they can be used to yield well-defined asymptotic series in t for the partition functions. This is achieved by first expanding the exponential term as

$$e^{-I(u,t;s)} = e^{-us^2/2}\left(1+\sum_{n=1}^{\infty} A_n(s,u)\, t^n\right)\,. \tag{4.10}$$

The coefficients A_n can be expressed in terms of generalized Laguerre polynomials,

$$A_n(s,u) = -\frac{u\,s^{2n+2}}{2^{2n+1}n}\,L_{n-1}^{n+1}\left(\frac{us^2}{2}\right)\,. \tag{4.11}$$

In Appendix A.12, we show in detail how the expression above is derived. By integrating each term in the series, we rewrite the partition functions as

$$Z^{\text{disk}}(u,t) = \sum_{n=0}^{\infty} Z_n^{\text{disk}}(u)\,t^n\,, \tag{4.12}$$

$$Z^{\text{trumpet}}(u,b,t) = \sum_{n=0}^{\infty} Z_n^{\text{trumpet}}(u,b)\,t^n\,. \tag{4.13}$$

The t^0 terms come from taking the integral against the undeformed action factorized in (4.10) and, as such, correctly reproduce the known results computed in the $t \to 0$ limit [113],

$$Z_0^{\text{disk}}(u) = \frac{u^{-3/2}}{\sqrt{2\pi}}\,e^{2\pi^2/u}\,, \tag{4.14}$$

$$Z_0^{\text{trumpet}}(u,b) = \frac{u^{-1/2}}{\sqrt{2\pi}}\,e^{-b^2/2u}\,. \tag{4.15}$$

For $n > 1$ we have the following expressions:

$$Z_n^{\text{disk}}(u) = \frac{1}{2\pi^2}\int_0^{\infty} \mathrm{d}s\; s\sinh(2\pi s)\,e^{-us^2/2}\,A_n(s,u)\,, \tag{4.16}$$

$$Z_n^{\text{trumpet}}(u,b) = \frac{1}{\pi}\int_0^{\infty} \mathrm{d}s\;\cos(bs)\,e^{-us^2/2}\,A_n(s,u)\,. \tag{4.17}$$

For any $n \in \mathbb{N}$, the above integrals are real and convergent. By expanding in t, we have apparently cured the ambiguity arising from the integration over s. We will see in a moment where the subtlety is now hiding. We first need to perform the integration. In order to deal with both integrals at once, we compute

$$\begin{aligned} a_j &= \int_0^{\infty} \mathrm{d}s\; s^{2j}\,e^{-us^2/2}\,A_n(s,u) \\ &= -\frac{1}{n!}\,2^{j-n-1/2}u^{-j-n-1/2}\left(\frac{1}{2}-j\right)_{n-1}\Gamma\left(j+n+\frac{3}{2}\right)\,, \end{aligned} \tag{4.18}$$

were $(x)_n$ denotes the Pochhammer symbol. Then, we simply use the Taylor expansion of $s\sinh(2\pi s)$ and $\cos(bs)$ to obtain

$$\begin{aligned} Z_n^{\text{disk}}(u) &= \frac{1}{2\pi^2}\sum_{j=0}^{\infty}\frac{(2\pi)^{2j+1}\,a_{j+1}}{(2j+1)!} \\ &= Z_0^{\text{disk}}(u)\,\frac{(2n)!}{n!}\,(-2u)^{-n}\,L_{2n}^{3/2-n}\left(-\frac{2\pi^2}{u}\right) \\ &= \frac{(2u)^{-n}}{n!\sqrt{2\pi^3u^3}}\,\Gamma\left(n-\frac{3}{2}\right)\Gamma\left(n+\frac{5}{2}\right){}_1F_1\left(n+\frac{5}{2};\frac{5}{2}-n;\frac{2\pi^2}{u}\right)\,, \end{aligned} \tag{4.19}$$

and

$$
\begin{aligned}
Z_n^{\text{trumpet}}(u,b) &= \frac{1}{\pi}\sum_{j=0}^{\infty}\frac{(-b^2)^j\,a_j}{(2j)!} \\
&= Z_0^{\text{trumpet}}(u)\,\frac{(2n)!}{n!}\,(-2u)^{-n}\,L_{2n}^{1/2-n}\left(\frac{b^2}{2u}\right) \\
&= -\frac{(2u)^{-n}}{n!\sqrt{2\pi^3 u}}\,\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right){}_1F_1\left(n+\frac{3}{2};\frac{3}{2}-n;-\frac{b^2}{2u}\right). \qquad (4.20)
\end{aligned}
$$

Conveniently, the two expressions above capture also the $n=0$ cases in (4.14) and (4.15). In both cases, the perturbative coefficients are the undeformed partition functions times a polynomial in $1/u$ of degree $3n$. Finally, we must remark that these two series expansions can also be directly obtained by solving the flow equation perturbatively in t (see Section 4.4 for some details) without making any reference to the integral representations (4.7) and (4.8).

4.1.2 Resurgence

The coefficients in (4.19) and (4.20) grow asymptotically as $n!$. This means that the perturbative expansions in (4.12) and (4.13) should be understood as formal power series, since both have vanishing radius of convergence. It is possible to associate a finite result to these series by performing a *Borel resummation.*

The Borel sum of a series

$$\Phi(z) = \sum_n \omega_n\, z^n \qquad (4.21)$$

is defined as follows. First, one should take the *Borel transform* of Φ,

$$\mathcal{B}[\Phi](\zeta) = \sum_n \omega_n\,\frac{\zeta^n}{n!}\,. \qquad (4.22)$$

If the coefficients ω_n grow as $n!$, $\mathcal{B}[\Phi]$ has finite radius of convergence, thus defining a germ of an analytic function at $\zeta=0$. Then, the directional Borel resummation of Φ along a chosen direction θ on the complex ζ-plane is defined as

$$\mathcal{S}_\theta\Phi(z) = \frac{1}{z}\int_0^{e^{i\theta}\infty} d\zeta\, e^{-\zeta/z}\,\mathcal{B}[\Phi](\zeta)\,, \qquad (4.23)$$

where the integral, taken along the ray with $\arg\zeta=\theta$, is also known as a *directional Laplace transform.* The directional resummation $\mathcal{S}_\theta\Phi(z)$ defines an analytic function in the wedge $\mathrm{Re}(e^{-i\theta}z)>0$ that, upon expansion in z, reproduces (4.21).

In our case, rather than directly computing the Borel transform of (4.12) and (4.13), it is convenient to use the power series representation of the Kummer confluent hypergeometric function to first recast (4.19) and (4.20) as

$$Z_n^{\text{disk}}(u) = \sum_{m=0}^{\infty}(-2u)^{-n}\,\frac{u^{-3/2}}{\sqrt{2\pi}}\left(\frac{2\pi^2}{u}\right)^m\frac{\Gamma\left(m+n+\frac{5}{2}\right)}{n!m!\,\Gamma\left(m-n+\frac{5}{2}\right)}\,, \qquad (4.24)$$

$$Z_n^{\text{trumpet}}(u,b) = \sum_{m=0}^{\infty}(-2u)^{-n}\,\frac{u^{-1/2}}{\sqrt{2\pi}}\left(-\frac{b^2}{2u}\right)^m\frac{\Gamma\left(m+n+\frac{3}{2}\right)}{n!m!\,\Gamma\left(m-n+\frac{3}{2}\right)}\,, \qquad (4.25)$$

and then perform the Borel transform on each term in the sum over m[5]. When summing over n, each series has finite radius of convergence,

$$\mathcal{B}[Z^{\text{disk}}](u,\zeta) = \frac{u^{-3/2}}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{2\pi^2}{u}\right)^m {}_2F_1\left(-m-\frac{3}{2}, m+\frac{5}{2}; 1; \frac{\zeta}{2u}\right), \tag{4.26}$$

$$\mathcal{B}[Z^{\text{trumpet}}](u,b,\zeta) = \frac{u^{-1/2}}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{b^2}{2u}\right)^m {}_2F_1\left(-m-\frac{1}{2}, m+\frac{3}{2}; 1; \frac{\zeta}{2u}\right). \tag{4.27}$$

Now, in order to complete the Borel summation and obtain an analytic expression for the formal series in (4.12) and (4.13) one should proceed as in (4.23). However, the hypergeometric functions appearing in the Borel transforms (4.26) and (4.27) have branch cuts located on the positive real axis in the range $\zeta \in (2u, +\infty)$. The branch cut identifies a *Stokes line* at $\arg\zeta = 0$, i.e. a singular direction in the ζ plane. Namely, when taking a directional Laplace transform at $\theta = 0$, one runs into an ambiguity since the results obtained by approaching the Stokes line from above and below differ.

In the theory of resurgence, Stokes lines are associated with nonperturbative contributions, encoded by the discontinuity $(\mathcal{S}_{\theta_\star^+} - \mathcal{S}_{\theta_\star^-})\,\Phi(z)$ in the directional Borel resummation as the ray of angle θ crosses the Stokes line at $\theta_\star$. The directional Borel resummations approaching the Stokes lines from both sides are usually referred to as *lateral Borel resummations*. Because of the nonperturbative nature of the discontinuity, both $\mathcal{S}_{\theta_\star^+}\Phi(z)$ and $\mathcal{S}_{\theta_\star^-}\Phi(z)$ share the same expansion in z, but crucially differ by instantonic contributions. In general, the correct nonperturbative completion of $\Phi(z)$ is obtained by choosing some combination of the two. If Φ is real and the Stokes line lies at $\theta_\star = 0$, under some general assumptions the correct real completion of $\Phi(z)$ is given by the *median resummation*

$$\mathcal{S}_{\text{med}}\Phi(z) = \frac{1}{2}\left(\mathcal{S}_{0^+} + \mathcal{S}_{0^-}\right)\Phi(z). \tag{4.28}$$

Let us apply this to the case at hand. In Appendix A.13, we provide details on how to compute Laplace transforms of Gauss hypergeometric functions above and below the cut. These, in turn, give us the lateral Borel resummations of the disk and trumpet partition functions starting from the expressions for their Borel transforms in (4.26) and (4.27),

$$\begin{aligned} \mathcal{S}_{0^\pm} Z^{\text{disk}}(u,t) &= \frac{e^{-\frac{u}{t}}}{\pi u\sqrt{t}} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{2\pi^2}{u}\right)^m \left[\pi\, I_{m+2}\left(\frac{u}{t}\right) \pm (-1)^m \mathrm{i}\, K_{m+2}\left(\frac{u}{t}\right)\right], \\ \mathcal{S}_{0^\pm} Z^{\text{trumpet}}(u,b,t) &= \frac{e^{-\frac{u}{t}}}{\pi\sqrt{t}} \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{b^2}{2u}\right)^m \left[\pi\, I_{m+1}\left(\frac{u}{t}\right) \pm (-1)^m \mathrm{i}\, K_{m+1}\left(\frac{u}{t}\right)\right], \end{aligned} \tag{4.29}$$

as depicted in Figure 5.3. We see that the median resummation mentioned above, indeed, cancels the imaginary terms in the two lateral Borel resummations and gives the real disk and trumpet partition functions

$$Z^{\text{disk}}(u,t) = \frac{e^{-\frac{u}{t}}}{u\sqrt{t}} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{2\pi^2}{u}\right)^m I_{m+2}\left(\frac{u}{t}\right), \tag{4.30}$$

$$Z^{\text{trumpet}}(u,b,t) = \frac{e^{-\frac{u}{t}}}{\sqrt{t}} \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{b^2}{2u}\right)^m I_{m+1}\left(\frac{u}{t}\right). \tag{4.31}$$

[5] In fact, for fixed m, the modulus of the coefficient of the series in n behaves as $\frac{(n-1)!}{\pi m!}$ when n approaches infinity.

Both sums can be performed through the Bessel multiplication theorem

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{z(\lambda^2-1)}{2} \right)^k I_{n+k}(z) = \lambda^{-n} I_n(\lambda z) \,, \tag{4.32}$$

to obtain

$$Z^{\text{disk}}(u,t) = \frac{u}{\sqrt{t}} \frac{e^{-u/t}}{u^2+4\pi^2 t} I_2\!\left(\frac{1}{t}\sqrt{u^2+4\pi^2 t} \right) , \tag{4.33}$$

$$Z^{\text{trumpet}}(u,b,t) = \frac{u}{\sqrt{t}} \frac{e^{-u/t}}{\sqrt{u^2-b^2 t}} I_1\!\left(\frac{1}{t}\sqrt{u^2-b^2 t} \right) . \tag{4.34}$$

Through resurgence, we have been able to unambiguously fix the nonperturbative completions of both the disk and the trumpet partition functions with just their perturbative expansions at $t=0$ as inputs. These corrections naturally carry the information of the nonperturbative branch H_- of the $T\bar{T}$ deformation in (4.4).

4.2 The spectrum

The results for the disk and the trumpet partition functions obtained in (4.33) and (4.34) through resurgence can be reproduced by changing the prescription for the integration contour in (4.7) and (4.8). At the level of the boundary theory, this prescription induces a cutoff on the spectrum for any finite value of t and gives rise to instantonic contributions associated with a region in the spectrum where the density of states becomes negative. The present section is dedicated to discussing these aspects.

4.2.1 Integration contour

The action (4.9) has two branch points, located at $s=-1/\sqrt{t}$ and $s=+1/\sqrt{t}$. We can extend the definition of $I(t,u;s)$ to the complex s-plane by placing a branch cut in the interval $s \in (-1/\sqrt{t},+1/\sqrt{t})$. Then, we can replace the original contour, running along the real axis, with an integration contour $\mathfrak{S}$ surrounding the branch cut.

Let us consider a generic integral

$$W = \int_{\mathfrak{S}} \mathrm{d}s\, f(s)\, e^{-I(t,u;s)} \,, \tag{4.35}$$

where f is some entire function. The integral is easily computed in terms of the discontinuity of the action across the branch cut,

$$\begin{aligned} W &= 2e^{-u/t} \int_{-1/\sqrt{t}}^{+1/\sqrt{t}} \mathrm{d}s\, f(s) \sinh\!\left(\frac{u}{t}\sqrt{1-ts^2} \right) \\ &= \frac{2e^{-u/t}}{\sqrt{t}} \int_0^{\pi} \mathrm{d}\theta\, \sin\theta\, f\!\left(\frac{\cos\theta}{\sqrt{t}} \right) \sinh\!\left(\frac{u}{t}\sin\theta \right) , \end{aligned} \tag{4.36}$$

where in the last step we introduced the change of variable $\cos\theta = \sqrt{t}s$. By replacing the hyperbolic sine with its Taylor expansion, we find

$$W = \sum_{j=0}^{\infty} \left(\frac{u}{t} \right)^{2j+1} \frac{2e^{-u/t}}{(2j+1)!\sqrt{t}} \int_0^{\pi} \mathrm{d}\theta\, (\sin\theta)^{2j+2} f\!\left(\frac{\cos\theta}{\sqrt{t}} \right) . \tag{4.37}$$

Disk. The disk partition function is obtained by setting $f(s) = s\sinh(2\pi s)/(4\pi^2)$. Actually, because the original integral is even in s and the integration range is symmetric about the origin, one can equivalently use $f(s) = s\exp(2\pi s)/(4\pi^2)$, which gives

$$Z^{\text{disk}}(u,t) = \sum_{j=0}^{\infty} \left(\frac{u}{t}\right)^{2j+1} \frac{e^{-u/t}}{2\pi^2(2j+1)!t} \int_0^{\pi} \mathrm{d}\theta\, (\sin\theta)^{2j+2} \cos\theta \exp\left(\frac{2\pi\cos\theta}{\sqrt{t}}\right) . \tag{4.38}$$

The term

$$(\sin\theta)^{2j+2}\cos\theta = \frac{1}{2j+3}\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)^{2j+3} \tag{4.39}$$

can be used to integrate by parts and obtain

$$Z^{\text{disk}}(u,t) = \sum_{j=0}^{\infty} \left(\frac{u}{t}\right)^{2j+1} \frac{e^{-u/t}}{\pi(2j+1)!(2j+3)t^{3/2}} \int_0^{\pi} \mathrm{d}\theta\, (\sin\theta)^{2j+4} \exp\left(\frac{2\pi\cos\theta}{\sqrt{t}}\right) . \tag{4.40}$$

We perform the integration by using the integral representation of the modified Bessel function, that, for $j \in \mathbb{Z}$, reads

$$I_j(z) = \frac{z^j 2^j j!}{(2j)!\pi} \int_0^{\pi} \mathrm{d}\theta\, (\sin\theta)^{2j} \exp(z\cos\theta) . \tag{4.41}$$

This gives

$$Z^{\text{disk}}(u,t) = \frac{ue^{-\frac{u}{t}}}{4\pi^2 t^{3/2}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{u^2}{4\pi t^{3/2}}\right)^j I_{j+2}\left(\frac{2\pi}{\sqrt{t}}\right) , \tag{4.42}$$

which, upon summation with (4.32), reproduces the result computed in (4.33).

Trumpet. Likewise, the trumpet partition function is recovered from W by setting $f(s) = \cos(bs)/(2\pi)$,

$$Z^{\text{trumpet}}(u,b,t) = \sum_{j=0}^{\infty} \left(\frac{u}{t}\right)^{2j+1} \frac{e^{-u/t}}{\pi(2j+1)!\sqrt{t}} \int_0^{\pi} \mathrm{d}\theta\, (\sin\theta)^{2j+2} \cos\left(\frac{b\cos\theta}{\sqrt{t}}\right) . \tag{4.43}$$

We use the integral representation

$$J_j(z) = \frac{z^j 2^j j!}{(2j)!\pi} \int_0^{\pi} \mathrm{d}\theta\, (\sin\theta)^{2j} \cos(z\cos\theta) , \tag{4.44}$$

to find

$$Z^{\text{trumpet}}(u,b,t) = \frac{ue^{-\frac{u}{t}}}{bt} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{u^2}{2bt^{3/2}}\right)^j J_{j+1}\left(\frac{b}{\sqrt{t}}\right) . \tag{4.45}$$

Using the multiplication theorem

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{z(1-\lambda^2)}{2}\right)^k J_{n+k}(z) = \lambda^{-n} J_n(\lambda z) , \tag{4.46}$$

we finally get

$$Z^{\text{trumpet}}(u,b,t) = \frac{u}{\sqrt{t}} \frac{e^{-u/t}}{\sqrt{b^2 t - u^2}} J_1\left(\frac{\sqrt{b^2 t - u^2}}{t}\right) , \tag{4.47}$$

which agrees with (4.34), since $I_n(x) = \mathrm{i}^{-n} J_n(\mathrm{i}x)$.

4.2.2 The density of states

The prescription on the integration contour $\mathfrak{S}$ translates into a prescription for the integration over the spectrum of the boundary theory. Changing the integration variable to $\mathcal{E} = s^2$ brings the disk partition function into the form

$$Z^{\text{disk}}(u,t) = \int_0^{1/t} \mathrm{d}\mathcal{E}\, \frac{\sinh\!\left(2\pi\sqrt{\mathcal{E}}\right)}{4\pi^2} \left(e^{-u(1-\sqrt{1-t\mathcal{E}})/t} - e^{-u(1+\sqrt{1-t\mathcal{E}})/t}\right). \tag{4.48}$$

The above differs from the naïve integral (4.5) associated with the $T\bar{T}$-deformed Schwarzian theory in two ways: the integration range is now capped at $\mathcal{E} = 1/t$, and there is an additional term of instantonic origin. The spectrum is always real within the integration range, and in the $t \to 0$ limit, one recovers the undeformed Schwarzian partition function.

The deformed density of states $\rho(E;t)$ can be obtained as follows. We split the exponential terms into two separate integrals and apply on both an appropriate change of variables to obtain integrals of the type

$$\int \mathrm{d}E\, \rho(E;t)\, e^{-\phi_r u E}\,, \tag{4.49}$$

where the density of states is weighted by a Boltzmann factor with inverse temperature $\beta = \phi_r u$. This amounts to set

$$E = -\frac{1}{\phi_r t}\left(\pm\sqrt{1-t\mathcal{E}} - 1\right), \qquad \mathcal{E} = \frac{1}{t}\left(1 - (1-\phi_r t E)^2\right). \tag{4.50}$$

The the two integrals combine nicely as

$$\begin{aligned} Z^{\text{disk}}(u,t) &= \int_0^{1/(\phi_r t)} \mathrm{d}E\, \rho(E;t)\, e^{-\phi_r u E} - \int_{2/(\phi_r t)}^{1/(\phi_r t)} \mathrm{d}E\, \rho(E;t)\, e^{-\phi_r u E} \\ &= \int_0^{2/(\phi_r t)} \mathrm{d}E\, \rho(E;t)\, e^{-\phi_r u E}\,, \end{aligned} \tag{4.51}$$

where the t-deformed density of states is given by

$$\begin{aligned} \rho(E;t) &= \frac{1}{4\pi^2} \sinh\!\left(2\pi\sqrt{\mathcal{E}(E)}\right) \frac{\mathrm{d}\mathcal{E}(E)}{\mathrm{d}E} \\ &= \frac{\phi_r(1-t\phi_r E)}{2\pi^2} \sinh\!\left(2\pi\sqrt{\phi_r E(2-t\phi_r E)}\right). \end{aligned} \tag{4.52}$$

Here and in the following, whenever we Laplace-transform from u to E, we adopt the widely-used convention of setting $\phi_r = 1/2$. With this choice, the density of states reads

$$\rho(E;t) = \frac{1-tE/2}{4\pi^2} \sinh\!\left(2\pi\sqrt{E(1-tE/4)}\right). \tag{4.53}$$

To interpret the integral as a conventional Laplace transform we simply extend the integration range to the entire real positive E axis and define $\rho(E;t)$ to have support on the interval $E \in (0, 4/t)$. At $t = 0$, the above reproduces the familiar Schwarzian density $\rho \propto \sinh\!\left(2\pi\sqrt{E}\right)$ growing exponentially in $\sqrt{E}$. At finite t, the result is qualitatively rather different. In the "perturbative range" $0 < E < 2/t$ the density is positive, but after an initial growth it decreases and reaches a zero at $E = 2/t$. In the "nonperturbative range" $2/t < E < 4/t$ the density becomes negative; the two ranges are related by the symmetry property $\rho(4/t - E, t) = -\rho(E;t)$ and thus the integral of $\rho(E;t)$ over the entire spectrum vanishes.

4.3 Other topologies

The disk partition function $Z^{\text{disk}}(u,t)$ computed in the previous sections is the partition function associated with a manifold of genus zero whose boundary has a single connected component of (rescaled) length u,

$$Z_{0,1}(u;t) = Z^{\text{disk}}(u,t)\,. \tag{4.54}$$

In general, one can compute partition functions on orientable manifolds with arbitrary topology. These are classified by the number n of connected components of the boundary, and by the genus g. The resulting partition function, $Z_{g,n}$ will depend on the lengths $u_1,\ldots,u_n$ of the connected boundaries.

As already explained in 2.1.4, in a theory of quantum gravity, the path integral receives contributions from different spacetime topologies. We briefly recall here the key aspects of what is known as “third quantization”.

For any given choice of n, the full partition function should really be a sum over the $Z_{g,n}$ obtained for any value of the genus g. Each term is weighted by the topological (Einstein–Hilbert) action term that gives a factor of $(e^{S_0})^{\chi}$, where $\chi = 2-2g-n$ is the Euler characteristic. At fixed n the full partition function reads

$$\mathbf{Z}_n(u_1,\ldots,u_n;t) = e^{(2-n)S_0}\sum_{g=0}^{\infty} e^{-2gS_0}\, Z_{g,n}(u_1,\ldots,u_n;t)\,. \tag{4.55}$$

In [113], it was shown that the partition function $Z_{g,n}$ for a generic choice of n and g can be obtained in terms of a certain topological decomposition. Each boundary component of length u_i is associated with a trumpet $Z^{\text{trumpet}}(u_i,b_i,t)$ that is glued to a bordered Riemann surface of genus g through a common geodesic boundary of length b_i. In Figure 4.2, we show the simple case of $Z_{1,1}$. The gluing is performed by taking an integral over the length b_i of each geodesic boundary,

$$\begin{aligned} Z_{g,n}(u_1,\ldots,u_n;t) = \int_0^{\infty}\mathrm{d}b_1\, b_1 \ldots \int_0^{\infty}\mathrm{d}b_n\, b_n\, V_{g,n}(b_1,\ldots,b_n) \\ \times Z^{\text{trumpet}}(u_1,b_1,t)\ldots Z^{\text{trumpet}}(u_n,b_n,t)\,. \end{aligned} \tag{4.56}$$

The formula is written in terms of $V_{g,n}(b_1,\ldots,b_n)$, the Weil–Petersson volume of the moduli space of hyperbolic Riemann surfaces of genus g with n geodesic boundaries of lengths $b_1,\ldots,b_n$. We have seen in (2.74) how the Weil–Petersson volumes naturally emerge in the gravitational path integral, where one inserts n geodesic boundaries which have no action (their extrinsic curvature is $\kappa = 0$).

Besides the disk, the only other topology that represents an exception to the formula above is given by the cylinder

$$Z_{0,2}(u_1,u_2;t) = \int_0^{\infty}\mathrm{d}b\, b\, Z^{\text{trumpet}}(u_1,b,t)\, Z^{\text{trumpet}}(u_2,b,t)\,, \tag{4.57}$$

which is obtained by directly gluing together two trumpets along their geodesic boundary, as shown in Figure 4.1.

In principle, it is not obvious why the gluing prescription should still be valid at finite t. In fact, for any $t > 0$, there is a portion of the integration range where the length b of the geodesic boundary exceeds the length $u/\sqrt{t}$ of the Dirichlet boundary of the same trumpet. However, as already noticed in [73], the inclusion of nonperturbative terms had the effect of making the trumpet partition function in (4.34) regular at $b = u/\sqrt{t}$ and real across the entire integration range, thus making the integral well-defined. Although we do not have an *ab initio* derivation of (4.56) and (4.57) for finite t, we take these formulas as prescriptions for the computation of partition functions for any topology. We will show later that the results they generate have remarkable properties.

4.3.1 Cylinder

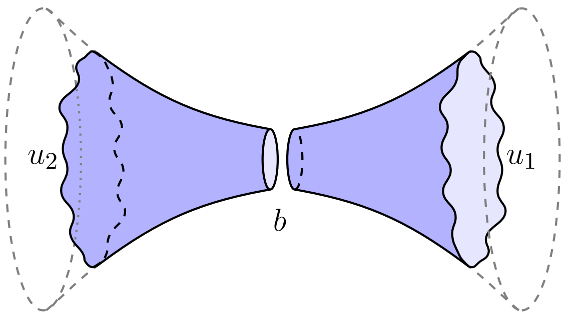


Figure 4.1: The topological decomposition of the cylinder in terms of two trumpets glued along their geodesic boundary.

We start our analysis of by considering the cylinder $Z_{0,2}(u_1,u_2;t)$. The series representation in (4.45) turns out to be particularly useful when dealing with integrals over b. We plug that expression in (4.57) and use

$$\int_0^\infty \frac{\mathrm{d}b}{b^{j+k+1}}\, J_{j+1}\left(\frac{b}{\sqrt{t}}\right) J_{k+1}\left(\frac{b}{\sqrt{t}}\right) = \frac{1}{2j!k!(j+k+1)}\left(\frac{1}{2\sqrt{t}}\right)^{j+k} \tag{4.58}$$

to write

$$Z_{0,2}(u_1,u_2;t) = 2u_1u_2\, e^{-(u_1+u_2)/t} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{u_1^{2j}u_2^{2k}}{(j!)^2(k!)^2}\left(\frac{1}{4t^2}\right)^{j+k+1} \frac{1}{(j+k+1)}\,. \tag{4.59}$$

To split the sums, we can use the trivial identity

$$\left(\frac{1}{4t^2}\right)^{j+k+1} \frac{1}{(j+k+1)} = \int_0^{1/(2t)^2} \mathrm{d}x\; x^{j+k} \tag{4.60}$$

and perform the summations

$$\sum_{j=0}^{\infty} \frac{(xu^2)^j}{(j!)^2} = I_0(2\sqrt{x}u) \tag{4.61}$$

to obtain

$$Z_{0,2}(u_1,u_2;t) = 2u_1u_2\, e^{-(u_1+u_2)/t} \int_0^{1/(2t)^2} \mathrm{d}x\; I_0(2\sqrt{x}u_1)\, I_0(2\sqrt{x}u_2)\,. \tag{4.62}$$

The last integral is easily evaluated and gives

$$Z_{0,2}(u_1,u_2;t) = \frac{u_1 u_2\, e^{-(u_1+u_2)/t}}{t(u_1^2-u_2^2)} \left[u_1\, I_0\!\left(\frac{u_2}{t}\right) I_1\!\left(\frac{u_1}{t}\right) - u_2\, I_0\!\left(\frac{u_1}{t}\right) I_1\!\left(\frac{u_2}{t}\right)\right] . \tag{4.63}$$

It is not immediate to see how this extends the result at infinite cutoff, but performing an expansion at the first few orders in t, one finds

$$Z_{0,2}(u_1,u_2;t) = \frac{1}{2\pi}\frac{\sqrt{u_1}\sqrt{u_2}}{(u_1+u_2)} + \frac{1}{26\pi}\frac{1}{\sqrt{u_1}\sqrt{u_2}}\, t + \frac{9}{256\pi}\frac{u_1+u_2}{(\sqrt{u_1}\sqrt{u_2})^3}\, t^2 + O(t^3)\,, \tag{4.64}$$

which exactly matches the $t \to 0$ limit previously computed in [113] and presented in (2.83). However, a power expansion in t necessarily misses nonperturbative contributions that, as we will discuss later, constitute a crucial feature of the theory at finite cutoff.

Furthermore, in the context of intepreting JT gravity as an average ensemble of random matrices 2.1.5, we have seen a key observable in the matrix model is represented by the connected correlators of resolvents. Specifically, from the cylinder partition function one can extract the resolvent by taking a Laplace transform over both u_1 and u_2,[6]

$$R_{0,2}(E_1,E_2;t) = \frac{1}{4}\int_0^\infty \mathrm{d}u_1 \int_0^\infty \mathrm{d}u_2\; Z_{0,2}(u_1,u_2;t)\, e^{u_1 E_1/2 + u_2 E_2/2}\,. \tag{4.66}$$

We assume $E_1, E_2 < 0$ and apply the identity

$$\int_0^\infty \mathrm{d}u\; u\, e^{-\alpha u}\, I_0(\beta u) = \frac{\alpha}{(\alpha^2-\beta^2)^{3/2}}\,, \tag{4.67}$$

which holds for $\mathrm{Re}\,\alpha > |\mathrm{Re}\,\beta|$, to (4.62). This gives

$$\begin{aligned} R_{0,2}(E_1,E_2;t) &= \frac{1}{2}\int_0^{1/(2t)^2} \mathrm{d}x\; \frac{1/t - E_1/2}{[(1/t-E_1/2)^2-4x]^{3/2}}\, \frac{1/t - E_2/2}{[(1/t-E_2/2)^2-4x]^{3/2}} \\ &= \frac{t^2(1-tE_1/2)(1-tE_2/2)(tE_1^2/4 + tE_2^2/4 - E_1 - E_2)}{4[(1-tE_1/2)^2-(1-tE_2/2)^2]^2\sqrt{-E_1(1-tE_1/4)}\sqrt{-E_2(1-tE_2/4)}} \\ &\quad - \frac{t^2[(1-tE_1/2)^2+(1-tE_2/2)^2]}{4[(1-tE_1/2)^2-(1-tE_2/2)^2]^2}\,. \end{aligned} \tag{4.68}$$

When continuing the resolvent to arbitrary complex values of E_1 and E_2, the square roots in (4.68) induce branch cuts at $E_1 \in (0, 4/t)$ and $E_2 \in (0, 4/t)$. The double-discontinuity of the resolvent across the real E_1 and E_2 lines, appropriately normalized by a $-1/(4\pi^2)$ factor, gives the two-point correlator of the density $\rho(E;t)$,

$$\rho_{0,2}(E_1,E_2;t) = \frac{t^2(1-tE_1/2)(1-tE_2/2)(tE_1^2/4+tE_2^2/4-E_1-E_2)}{4\pi^2[(1-tE_1/2)^2-(1-tE_2/2)^2]^2\sqrt{E_1E_2(1-tE_1/4)(1-tE_2/4)}}\,. \tag{4.69}$$

Interestingly, its support coincides with the one computed for the one-point function of $\rho(E;t)$ in (4.53), obtained from the disk partition function. In fact, the above expression is valid within the ranges where the branch cuts extend, i.e. for $E_1 \in (0,4/t)$ or $E_1 \in (0,4/t)$. Outside those ranges, $\rho_{0,2}$ vanishes, since the double discontinuity of the resolvent $R_{0,2}$ is zero.

[6]Here, we make use of the trivial change of variables

$$-\int_0^\infty \mathrm{d}\beta\, e^{-\beta E} f(\beta) = -\phi_r \int_0^\infty \mathrm{d}u\, e^{-\phi_r u E} f(\phi_r u)\,, \tag{4.65}$$

together with the convention, stated at the end of Section 4.2.2, according to which $\phi_r = 1/2$.

4.3.2 The general case

The Weil–Petersson volume $V_{g,n}$ is a polynomial of degree $3g-3+n$ in the squared lengths $b_1^2,\dots,b_n^2$ of the geodesic boundaries. By linearity, (4.56) can be computed by splitting the contribution of each monomial as a product of integrals where a single $Z^{\text{trumpet}}(u,b,t)$ is integrated against some even power of b. It is sufficient to use

$$\begin{aligned}\tilde{Z}_m(u,t) &= \int_0^\infty \mathrm{d}b\, b\, Z^{\text{trumpet}}(u,b,t)\, b^{2m} \\ &= \frac{m!}{\sqrt{t}}\, 2^m u^{m+1} e^{-u/t}\, I_m\left(\frac{u}{t}\right)\end{aligned} \tag{4.70}$$

to read off $Z_{g,n}$ from the coefficients in the polynomial $V_{g,n}$. Moreover, we can use the representation of $V_{g,n}$ in (2.76) as an integral over $\overline{\mathcal{M}}_{g,n}$ together with (4.70) to recast (4.56) as[7]

$$Z_{g,n}(u_1,\dots,u_n;t) = \frac{1}{t^{n/2}} \int_{\overline{\mathcal{M}}_{g,n}} e^{\omega} \prod_{i=1}^{n} u_i\, e^{-u_i/t} \sum_{j=0}^{\infty} (\psi_i\, u_i)^j\, I_j\left(\frac{u_i}{t}\right) . \tag{4.72}$$

A similar approach can be used to compute generic resolvents $R_{g,n}(E_1,\dots,E_n;t)$. Instead of taking an integral transform of the result $Z_{g,n}(u_1,\dots,u_n;t)$, it is possible to first apply the transformation to a single trumpet. This generates a term, for $E<0$,

$$\begin{aligned}T(E,b,t) &= -\frac{1}{2}\int_0^\infty \mathrm{d}u\; Z^{\text{trumpet}}(u,b,t)\, e^{uE/2} \\ &= -\frac{\sqrt{t}}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\Gamma(k+1/2)}{(1-tE/2)^{2k}} \left(\frac{2\sqrt{t}}{b}\right)^k J_k\left(\frac{b}{\sqrt{t}}\right)\end{aligned} \tag{4.73}$$

which can be glued to different topologies as in (4.56) to compute directly the resolvent for any g and n,

$$R_{g,n}(E_1,\dots,E_n;t) = \int_0^\infty \mathrm{d}b_1\, b_1 \dots \int_0^\infty \mathrm{d}b_n\, b_n\, V_{g,n}(b_1,\dots,b_n)\, T(E_1,b_1,t)\dots T(E_n,b_n,t) . \tag{4.74}$$

Again, because of linearity, one simply needs to use

$$\begin{aligned}\tilde{R}_m(E,t) &= \int_0^\infty \mathrm{d}b\, b\, T(E,b,t)\, b^{2m} \\ &= -\frac{(2m+1)!\,(1-tE/2)}{2(-E(1-tE/4))^{m+1}\sqrt{-E(1-tE/4)}} ,\end{aligned} \tag{4.75}$$

to immediately obtain the result for $R_{g,n}$ for any given $V_{g,n}$. To compute arbitrary correlators of $\rho(E;t)$ one can take the discontinuity

$$\begin{aligned}\tilde{\rho}_m(E,t) &= \frac{\tilde{R}_m(E-\mathrm{i}0,t)-\tilde{R}_m(E+\mathrm{i}0,t)}{2\pi\mathrm{i}} \\ &= (-1)^{m+1}\frac{(2m+1)!\,(1-tE/2)}{2\pi(E(1-tE/4))^{m+1}\sqrt{E(1-tE/4)}}\,\theta(E)\,\theta(4-tE) .\end{aligned} \tag{4.76}$$

[7]The infinite sum can be rewritten in terms of Lommel functions of two variables as

$$\sum_{j=0}^{\infty} (\psi_i\, u_i)^j\, I_j\left(\frac{u_i}{t}\right) = V_0\left(\frac{\mathrm{i}}{t\psi_i}, \frac{\mathrm{i}u_i}{t}\right) - \mathrm{i}V_1\left(\frac{\mathrm{i}}{t\psi_i}, \frac{\mathrm{i}u_i}{t}\right) . \tag{4.71}$$

Examples. As an example of the above setup, we compute the partition function for the disk at genus one (see Figure 4.2). The relevant Weil–Petersson volume is given by

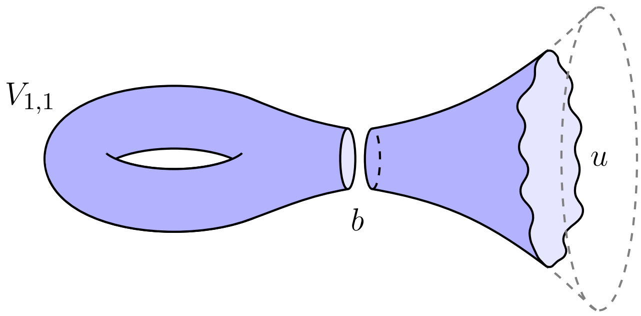


Figure 4.2: The topological decomposition of a disk at genus one. A trumpet is glued to genus-one Riemann surface along a common geodesic boundary.

$$V_{1,1}(b) = \frac{1}{48}(4\pi^2 + b^2) \; . \tag{4.77}$$

By replacing each power of b with the appropriate term in (4.70), we find

$$\begin{aligned} Z_{1,1}(u;t) &= \frac{1}{48}(4\pi^2 \tilde{Z}_0(u,t) + \tilde{Z}_2(u,t)) \\ &= \frac{u\, e^{-u/t}}{24\sqrt{t}} \left[2\pi^2 I_0\left(\frac{u}{t}\right) + u\, I_1\left(\frac{u}{t}\right)\right] \; . \end{aligned} \tag{4.78}$$

When computing results at higher g and n, which involve higher powers of b, one can recursively apply the identity

$$I_{n+1}(z) = I_{n-1}(z) - \frac{2n}{z} I_n(z) \; , \tag{4.79}$$

to rewrite the result solely in terms of modified Bessel functions of order zero and one. For instance, from

$$V_{2,1}(b) = \frac{1}{2211840}(4\pi^2 + b^2)(12\pi^2 + b^2)(6960\pi^4 + 384\pi^2 b^2 + 5b^4) \; , \tag{4.80}$$

$$V_{1,2}(b_1, b_2) = \frac{1}{192}(4\pi^2 + b_1^2 + b_2^2)(12\pi^2 + b_1^2 + b_2^2) \; , \tag{4.81}$$

$$V_{0,3}(b_1, b_2, b_3) = 1 \; , \tag{4.82}$$

we can compute the disk at genus two,

$$\begin{aligned} Z_{2,1}(u;t) = &\frac{u\, e^{-u/t}}{5760\sqrt{t}}(870\pi^8 + 278\pi^4 u^2 - 232\pi^2 t u^2 + 120 t^2 u^2 + 5u^4)\, I_0\left(\frac{u}{t}\right) \\ &+ \frac{u\, e^{-u/t}}{2880\sqrt{t}}(338\pi^6 - 278\pi^4 t + 232\pi^2 t^2 - 120 t^3 + 29\pi^2 u^2 - 20 t u^2)\, I_1\left(\frac{u}{t}\right) \; , \end{aligned} \tag{4.83}$$

the cylinder at genus one,

$$Z_{1,2}(u_1,u_2;t) = \frac{u_1 u_2}{24t}\, e^{-(u_1+u_2)/t} \left[u_1 \left(2\pi^2 - t\right) I_1\left(\frac{u_1}{t}\right) I_0\left(\frac{u_2}{t}\right) + (u_1 \leftrightarrow u_2) \right.$$
$$\left. + \left(u_1^2 + u_2^2 + 6\pi^4\right) I_0\left(\frac{u_1}{t}\right) I_0\left(\frac{u_2}{t}\right) + u_1 u_2\, I_1\left(\frac{u_1}{t}\right) I_1\left(\frac{u_2}{t}\right) \right] , \tag{4.84}$$

and the topology with three boundaries at genus zero

$$Z_{0,3}(u_1,u_2,u_3;t) = \frac{u_1 u_2 u_3}{t^{3/2}}\, e^{-(u_1+u_2+u_3)/t}\, I_0\left(\frac{u_1}{t}\right) I_0\left(\frac{u_2}{t}\right) I_0\left(\frac{u_3}{t}\right) . \tag{4.85}$$

4.4 Flow equation

The shift in the spectrum of the $T\bar{T}$-deformed theory as a function of the deformation parameter is controlled by (4.3). The equation imposes a constraint at the level of the thermal partition function, in the form of a partial differential equation in both the deformation parameter t and the (rescaled) temperature u. In fact, the differential operator

$$\mathbf{F}(u,t) = u\, \frac{\partial^2}{\partial u^2} + 2t\, \frac{\partial^2}{\partial u\, \partial t} - 2\left(\frac{t}{u} - 1\right) \frac{\partial}{\partial t} \tag{4.86}$$

has the property that, for any density of states $\varrho(\mathcal{E})$,

$$\mathbf{F}(u,t) \int_0^\infty \mathrm{d}\mathcal{E}\; \varrho(\mathcal{E})\; e^{-u(1-\sqrt{1-t\mathcal{E}})/t} = 0 . \tag{4.87}$$

This induces a recursion relation

$$2u(n+1)\, Z_{n+1}(u) = 2n\, Z_n(u) - 2nu\, Z_n'(u) - u^2 Z_n''(u) \tag{4.88}$$

for both the disk and the trumpet coefficients in the t-expansion introduced in Section 4.1.1. A simple check on the explicit forms derived in (4.19) and (4.20) shows that, indeed, the above holds true.

With the introduction of nonperturbative corrections, however, the integral of the type in (4.87) should be modified according to the prescriptions discussed in Section 4.2.2. Interestingly, the modified integral is still a solution of the flow equation,

$$\mathbf{F}(u,t) \int_0^{1/t} \mathrm{d}\mathcal{E}\; \varrho(\mathcal{E}) \left(e^{-u(1-\sqrt{1-t\mathcal{E}})/t} - e^{-u(1+\sqrt{1-t\mathcal{E}})/t} \right) = 0 . \tag{4.89}$$

As a consequence, both (4.33) and (4.34) are solutions of the flow equation, as one can explicitly check. The nonperturbative contributions to both the disk and the trumpet partition function correct the perturbative series with the addition of a trans-series term of the form

$$Z_{\text{inst.}}(u,t) = e^{-2u/t} \sum_{n=0}^{\infty} Z_n^*(u)\, t^n . \tag{4.90}$$

The presence of the exponential associated with the instantonic saddle has the effect of modifying the action of the flow equation operator by flipping the sign of u,

$$\mathbf{F}(u,t)\, Z_{\text{inst.}}(u,t) = -e^{-2u/t}\, \mathbf{F}(-u,t) \sum_{n=0}^{\infty} Z_n^*(u) . \tag{4.91}$$

As a consequence, the coefficients $Z_n^*(u)$ of the expansion about the instantonic saddle obey the modified equation

$$2u(n+1)\,Z_{n+1}^*(u) = -2n\,Z_n^*(u) + 2nu\,Z_n^{*\prime}(u) + u^2 Z_n^{*\prime\prime}(u)\,. \tag{4.92}$$

The fact that the trumpet partition function is annihilated by $\mathbf{F}(u,t)$ has important implications at $g>0$. In fact, let us consider the gluing formula (4.56) for $n=1$. Since the dependence on u and t comes exclusively from the single $Z^{\text{trumpet}}(u,b,t)$ inside the integral, we can immediately conclude that the disk partition function is a solution of the flow equation, not just at genus zero, but at any genus g:

$$\mathbf{F}(u,t)\,Z_{g,1}(u;t) = 0\,. \tag{4.93}$$

A similar conclusion can be drawn for other topologies, i.e. when $n>1$, but it requires a modification in the way we assign boundary conditions. So far, we considered a specific way of assigning Dirichlet boundary conditions. Specifically, we imposed on each boundary the same value $\phi_b = 1/\sqrt{t}$ for the dilaton field. In principle, nothing prevents us from computing higher topologies by gluing trumpets associated with different values of t. The generalization for the gluing formulas presented in Section 4.3 is actually straightforward, and so is the generalization of the result written in terms of the building blocks (4.70). This is effectively a refinement of the results considered so far, since for any given topology, the generalized partition function $Z_{g,n}$ is now a function of n different deformation parameters $t_1,\ldots,t_n$. At the level of the flow equation, to each boundary is associated a differential operator $\mathbf{F}(u_i,t_i)$ which annihilates the partition function:

$$\mathbf{F}(u_i,t_i)\,Z_{g,n}(u_1,\ldots,u_n;t_1,\ldots,t_n) = 0\,. \tag{4.94}$$

For the purpose of the present paper, we will not consider further this refinement, and we will only deal with the case where the dilaton field takes the same value on each disconnected component of the boundary.

4.5 Topological recursion

In Section 4.3, we have given a prescription to obtain the resolvents $R_{g,n}$ for any topology.

At $t=0$, we have explained in 2.1.5 that these functions have a natural interpretation in terms of correlators computed in a certain dual double-scaled Hermitian matrix integral [113]. The correlators enjoy Schwinger–Dyson-like identities, known as *loop equations*, that allow to recursively compute results at all orders in the large-N expansion [39]. This procedure goes under the name of *topological recursion*, and the set of data initiating it can be captured by a single mathematical object: the *spectral curve* [40].

Thanks to the simplicity of the undeformed trumpet partition function (4.15), at $t=0$ the resolvents $R_{g,n}$ are, essentially, the Laplace transforms of the Weil–Petersson volumes $V_{g,n}$. In fact, the undeformed topological recursion [113] is given precisely by the recursion formula of Eynard and Orantin [41], which is the Laplace-transformed version of the recursion formula derived by Mirzakhani [99].

Remarkably, we find that the deformation induced by t represents a consistent deformation of the spectral curve (2.97) for JT gravity at infinite cutoff. By this, we mean that the resolvents $R_{g,n}$ computed through the Weil–Petersson gluing, as described in Section 4.3, can be reproduced by the topological recursion associated with a t-deformation of the Eynard–Orantin spectral curve capturing the $t=0$ case.

To define the spectral curve we introduce the map

$$E(z) = -z^2\,, \tag{4.95}$$

which, in turn, determines the functions

$$\begin{aligned} W_{0,1}(z_1;t) &= \mathrm{i}\pi\rho(E(z_1),t)\,E'(z_1) \\ &= \frac{z_1(2+tz_1^2)}{4\pi}\sin\!\left(\pi z_1\sqrt{4+tz_1^2}\right), \end{aligned} \tag{4.96}$$

and

$$\begin{aligned} W_{0,2}(z_1,z_2;t) &= \left(R_{0,2}(E(z_1),E(z_2);t) - \frac{1}{(E(z_1)-E(z_2))^2}\right)E'(z_1)\,E'(z_2) \\ &= \frac{4(2+tz_1^2)\,(2+tz_2^2)}{(z_1^2-z_2^2)^2\,[4+t(z_1^2+z_2^2)]^2}\left(2z_1z_2+\frac{4(z_1^2+z_2^2)+t(z_1^4+z_2^4)}{\sqrt{4+tz_1^2}\sqrt{2+tz_2^2}}\right). \end{aligned} \tag{4.97}$$

Notice how both functions are meromorphic in some neighborhood of the origin.

For any choice of g and n other than the two cases above, the topological recursion computes the $W_{g,n}$ functions through the recursion formula[8]

$$\begin{aligned} W_{g,n}(z_1,\dots,z_n;t) = \operatorname*{Res}_{z\to 0} K(z_1,z;t)\Bigg[&W_{g-1,n+1}(z,-z,z_2,\dots,z_n;t) \\ &+ \sum_{\substack{h_1+h_2=g\\ I_1\cup I_2=J}}^{*} W_{h_1,1+|I_1|}(z,I_1;t)\,W_{h_2,1+|I_2|}(-z,I_2;t)\Bigg]\,, \end{aligned} \tag{4.98}$$

where $J=\{z_2,\dots,z_n\}$, and the symbol $*$ over the sum indicates that one should exclude terms where $(h_1,I_1)=(g,J)$ or $(h_2,I_2)=(g,J)$. The recursion kernel K that appears in (4.98) is defined as

$$\begin{aligned} K(z_1,z;t) &= \frac{1}{2[W_{0,1}(z;t)+W_{0,1}(-z;t)]}\int_{-z}^{z}\mathrm{d}z_2\,W_{0,2}(z_1,z_2;t) \\ &= \frac{(2+tz_1^2)\sqrt{4+tz^2}}{(2+tz^2)\sqrt{4+tz_1^2}}\,\frac{4\pi\csc\!\left(\pi z\sqrt{4+tz^2}\right)}{(z_1^2-z^2)\,[4+t(z_1^2+z^2)]}\,. \end{aligned} \tag{4.99}$$

The functions $W_{g,n}$ computed by the recursion are closely related to the resolvents $R_{g,n}$. Specifically, one can obtain the former by simply acting on the latter with the change of variables induced by $E(z)$,

$$W_{g,n}(z_1,\dots,z_n;t) = R_{g,n}(E(z_1),\dots,E(z_n);t)\;E'(z_1)\dots E'(z_n)\,. \tag{4.100}$$

[8] There are more general formulations of the topological recursion. The formulas in (4.98) and (4.99) are valid for a map E such that $\mathrm{d}E$ vanishes at $z=0$, where its local Galois involution is $\sigma: z\mapsto -z$.

As such, $W_{g,n}$ can be computed, from a bulk perspective, through the gluing (4.56). In the spirit of Section 4.3, we define

$$\begin{aligned}\tilde{W}_m(z,t) &= \tilde{R}_m(E,t)\,E'(z)\\ &= (2m+1)!\,\frac{2+tz^2}{(4+tz^2)^{m+3/2}}\left(\frac{2}{z}\right)^{2m+2}\,,\end{aligned} \tag{4.101}$$

which provides the contribution associated with a single trumpet integrated against b^{2m}.

To prove that the topological recursion (4.98) indeed matches the results obtained in Section 4.3, we will show that the $W_{g,n}(z_1,\ldots,z_n;t)$ are connected to the undeformed ones, $W_{g,n}(z_1,\ldots,z_n;0)$, through a change of variables.[9] A simple way to show this is to consider an alternative choice for the map E. In particular, we consider a map with an explicit dependence on t,

$$\hat{E}(\zeta) = -\frac{2}{t}\left(\sqrt{1+t\zeta^2}-1\right)\,, \tag{4.103}$$

This choice has two important properties. The first is that it correctly reproduces the undeformed map when t vanishes, i.e. $\lim_{t\to 0}\hat{E}(\zeta) = E(\zeta)$. The second is that it eliminates any dependence from t in $W_{0,1}$ and $W_{0,2}$, which then necessarily agree with the undeformed ones. Specifically,

$$\begin{aligned}\hat{W}_{0,1}(\zeta_1) &= \frac{\zeta_1\sin(2\pi\zeta_1)}{2\pi} & \hat{W}_{0,2}(\zeta_1,\zeta_2) &= \frac{1}{(\zeta_1-\zeta_2)^2}\\ &= W_{0,1}(\zeta_1;0)\,, & &= W_{0,2}(\zeta_1,\zeta_2;0)\,.\end{aligned} \tag{4.104}$$

The same, then, holds true for the recursion kernel

$$\begin{aligned}\hat{K}(\zeta_1,\zeta) &= \frac{\pi\csc(2\pi\zeta)}{\zeta_1^2-\zeta^2}\\ &= K(\zeta_1,\zeta;0)\,.\end{aligned} \tag{4.105}$$

As anticipated at the beginning of this section, the undeformed recursion induced by (4.104) and (4.105) is the Eynard–Orantin topological recursion, while the undeformed $W_{g,n}$ are connected to the Weil–Petersson volumes $V_{g,n}$ by a simple integral transform, as it can be easily seen from the expression of the integrated trumpet

$$\begin{aligned}\hat{\tilde{W}}_m(z) &= (2m+1)!\,z^{-2m-2}\\ &= \tilde{W}_m(z,0)\,.\end{aligned} \tag{4.106}$$

We can therefore argue as follows. We start from the undeformed case, where the recursion formula is known to hold, and we notice that all quantities can be lifted to the case of finite t through the change of variables induced by $E\circ\hat{E}^{-1}$. Because the recursion formula

[9] As it is clear from the definitions in (4.96), (4.97) and (4.100), the functions $W_{g,n}$ transform as a differential n-forms. In fact, the spectral curve and the topological recursion are naturally formulated in terms of differential forms

$$\omega_{g,n}(z_1,\ldots,z_n) = W_{g,n}(z_1,\ldots,z_n)\,\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_n\,. \tag{4.102}$$

The recursion kernel K, on the other hand, defines a tensor of type $(1,1)$.

(4.98) is covariant under change of variables induced by maps that are bi-holomorphic in some neighborhood of $z=0$, the topological recursion is guaranteed to hold at any finite t.

As an example, we notice that both the recursion and the standard gluing procedure give, for the disk at genus one,

$$W_{1,1}(z_1;t) = \frac{(2+tz_1^2)\,[6+\pi^2 z_1^2\,(4+tz_1^2)]}{3z_1^4\,(4+tz_1^2)^{5/2}}\,. \tag{4.107}$$

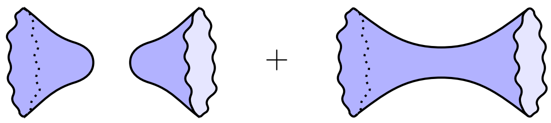
4.6 The spectral form factor

At $t=0$ [113], JT gravity was observed to reproduce the characteristic shape of a spectral form factor associated with an ensemble of Hamiltonians with random-matrix statistics. From a bulk perspective, the spectral form factor can be interpreted as a transition amplitude in the Hilbert space of two copies of JT gravity [111]. It is computed by the analytic continuation of two boundaries, $u_1 \mapsto u+\mathrm{i}\tau$, $u_2 \mapsto u-\mathrm{i}\tau$, which introduces a timescale τ. The quantity includes terms coming from different topologies, each weighted by the usual topological factor,

$$F(u,\tau;t) = e^{2S_0}\, Z_{0,1}(u+\mathrm{i}\tau;t)\, Z_{0,1}(u-\mathrm{i}\tau;t) + Z_{0,2}(u+\mathrm{i}\tau, u-\mathrm{i}\tau;t) + \dots\,, \tag{4.108}$$

where the dots correspond to subleading terms associated with higher-genus topologies. We can rewrite the definition in a graphical way as

$F = e^{2S_0}$



$+ \quad \dots\,.$ (4.109)

Interestingly, different features of the spectral form factor are associated with contributions coming from different topologies. The initial "slope" region comes from considering two disjoint disks (4.30). The characteristic shape of the slope can be observed by looking at its large-τ regime,

$$Z_{0,1}(u+\mathrm{i}\tau;t)\, Z_{0,1}(u-\mathrm{i}\tau;t) \sim \frac{1}{2\pi\tau^3}\left(1-e^{-4u/t}\right) + \frac{e^{-2u/t}}{\pi\tau^3}\sin\left(\frac{2\tau}{t}\right). \tag{4.110}$$

The first term gives a cubic decay that reproduces the known $t\to 0$ limit, while the second term is an oscillation of period πt whose amplitude is exponentially suppressed in $1/t$.

Eventually, the slope phase will end, and other topologies will dominate the form factor. The characteristic "ramp" region comes, in fact, from the connected topology, i.e., from the cylinder (4.63), which represents a Euclidean wormhole connecting the two boundaries. The large-τ behavior is again dominated by two terms,

$$Z_{0,2}(u+\mathrm{i}\tau, u-\mathrm{i}\tau;t) \sim \frac{\tau}{4\pi u}\left(1-e^{-4u/t}\right) - \frac{e^{-2u/t}}{2\pi}\cos\left(\frac{2\tau}{t}\right). \tag{4.111}$$

In Figure 4.3 and Figure 4.4, the slope and the ramp are plotted as functions of τ for various values of t.

In [111] the spectral form factor has been discussed using the long-time behavior of the Hartle–Hawking wavefunction associated with JT gravity. The physical origin of the slope,

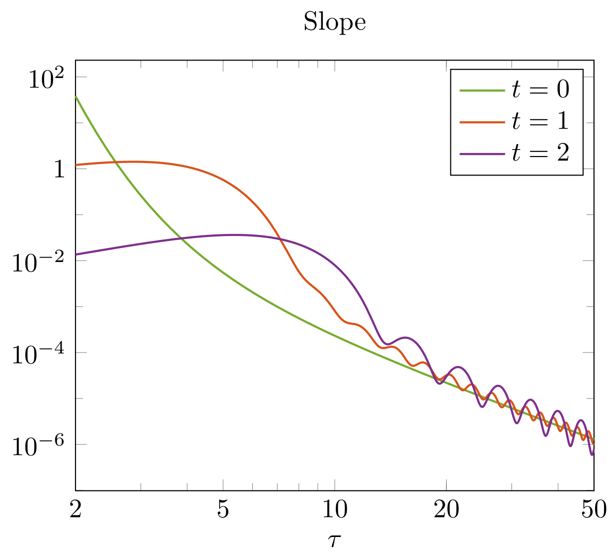


Figure 4.3: A log–log plot of the slope $Z_{0,1}(1+\mathrm{i}\tau;t)\,Z_{0,1}(1-\mathrm{i}\tau;t)$ for various values of t.

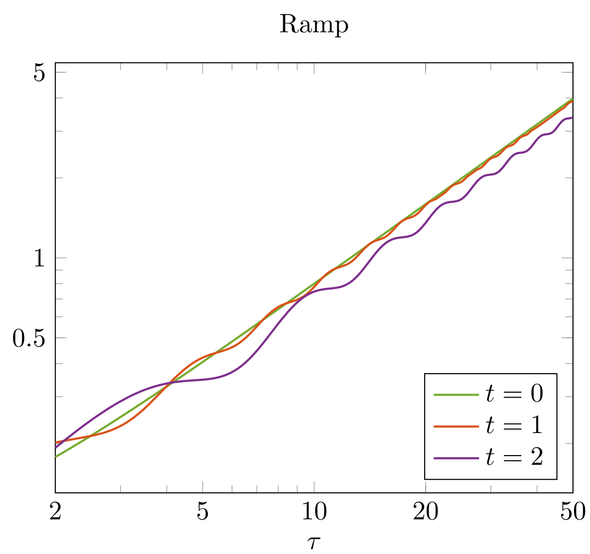


Figure 4.4: A log–log plot of the ramp $Z_{0,2}(1+\mathrm{i}\tau,1-\mathrm{i}\tau;t)$ for various values of t.

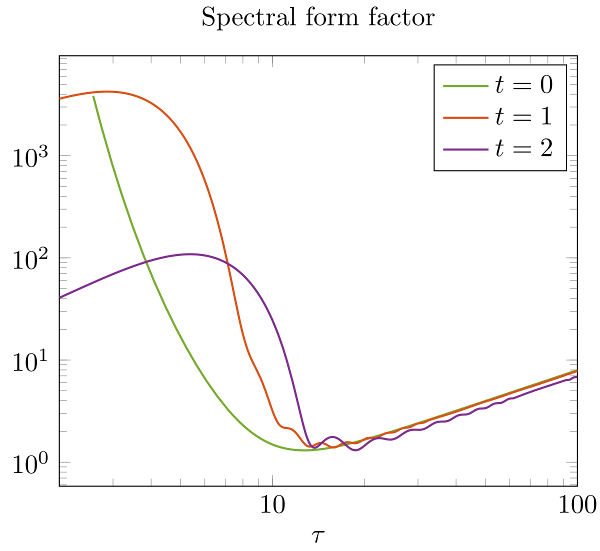


Figure 4.5: A log–log plot of the spectral form factor $F(1,\tau,t)$ with $S_0 = 4$ for various values of t. It includes contributions from the slope and the ramp; other topologies are discarded.

and in particular, of its decaying character, has been interpreted probabilistically. As the time τ increases, the amplitude for the time evolved Hartle–Hawking state to have a small Einstein–Rosen bridge decreases, being mostly supported at large values of the bridge length. The relevant initial state is localized at small Einstein–Rosen bridge lengths, and thus the transition amplitude decreases with τ. The exchange of a baby universe instead explains the ramp in the spectral form factor: at late times τ, Euclidean wormholes allow transitions from the initial Hartle–Hawking state to a state with a short Einstein–Rosen bridge and a large baby universe, with a size of order τ. The amplitude for this process, while exponentially small in the entropy, does not decay as τ increases, and the linear growth comes from the τ different ways in which the baby universe can be rotated before being absorbed [111].

At finite cutoff, we see from (4.110) and (4.111) that with respect to the undeformed case, the leading τ-behaviors are diminished by exponential finite-size effects while novel periodic fluctuations appear. We observe that both the ramp term (4.111) and the slope (4.110) share the same damping factor in the non-oscillating part, which indeed suggests we are capturing a universal effect of gravity at finite volume and confirms our result for the cylinder partition function. As a matter of fact, the intersection of these two regimes is independent of the cutoff parameter t, up to exponentially-suppressed terms, leading to a transition time of order[10]

$$\tau \sim (2u)^{1/4} e^{S_0/2} \,. \tag{4.112}$$

In Figure 4.5, the spectral form factor is plotted as a function of τ for various values of t.

A more physical understanding of the late-time behavior at finite cutoff should come from repeating the analysis of [111] using the real Hartle–Hawking wave function obtained in [73]. It considers both an expanding and a contracting branch, and it is nonperturbative in the parameter t: we expect, in particular, an interpretation of the oscillating terms as coming from some interference effects.

[10]The numerical factors are due to the choice of our conventions but can be simply reabsorbed into a redefinition of the entropy S_0.

More generally, the appearance of nonperturbative contributions in the disk partition functions seems to suggest the presence of some kind of tunneling process. At $t = 0$, the boundary of the AdS_2 patch extends to asymptotic infinity, which corresponds to having an infinite potential wall. However, as we move the boundary towards the interior of AdS_2 by increasing t, the potential barrier turns out to be not infinite anymore, and a hard wall replaces it through Dirichlet conditions. Equivalently, according to the holographic RG picture presented in [89], one would expect that integrating out the geometry between the asymptotic boundary and a finite radial distance would result in the $T\bar{T}$ deformation of the original Schwarzian. It is not unreasonable that the effective dynamics could support nontrivial tunneling amplitudes among different vacua in both cases.

Part III

Exact $T\bar{T}$ deformation of 2d Yang-Mills theory

$T\bar{T}$-deformed $U(N)$ Yang-Mills on the sphere 5

The present chapter is mainly based on our papers [54] and [55] and deals with the deformation of gauge theories. Pure Yang–Mills theory in two dimensions is quite different from its higher-dimensional counterparts in that it does not allow for propagating degrees of freedom. As we reviewed in 2.2.1, the theory is invariant under a large group of spacetime symmetries that make the dependence on the geometry almost trivial and render the theory solvable [110, 129].

In the context of the $T\bar{T}$ deformation, Yang–Mills theory was studied in [25, 75, 115, 105] and its large-N limit was explored in [116, 48].

In the present work, we shed new light on the subject by taking a different root, i.e. we focus on the $\mathrm{U}(N)$ gauge theory at genus zero and find its exact $T\bar{T}$ deformed version [55], although we belive much of our results can be generalized to arbitrary groups and topologies. As a warm-up, we present the abelian $U(1)$ case, the content of [54], which is less difficult from the computational point of view but still retains all key aspects.

Motivations In 2.3.4 we derived the flow equation satisfied by the Yang-Mills partition function, once deformed by the $T\bar{T}$ operator. We can now specialize to $G \simeq \mathrm{U}(N)$ and use the effective coupling α introduced above. Moreover, for later convenience we introduce the rescaled deformation parameter $\tau = \mu N^3 g_{\mathrm{YM}}^2$. In terms of these variables, the flow equation (2.253) for the partition function is rewritten as

$$\mathbf{F}_{\alpha,\tau}\, Z(\alpha,\tau) = 0\,, \tag{5.1}$$

where we have introduced the differential operator

$$\mathbf{F}_{\alpha,\tau} = \frac{\partial}{\partial\tau} + \frac{2\alpha}{N^2}\frac{\partial^2}{\partial\alpha^2}\,. \tag{5.2}$$

In 2.3.4 we also derived a flow equation for the Hamiltonian, or deformed Yang-Mills potential in the first order formalism. It is easy to obtain a closed expression for the deformed Hamiltonian by solving the relevant Burgers differential equation (2.251) for $\mathscr{U}$:

$$H = \frac{g_{\mathrm{YM}}^2 C_2(R)/2}{1-\tau C_2(R)/N^3}\,. \tag{5.3}$$

However, for any $\tau \neq 0$ there is always an infinite number of representations whose energy is arbitrary close to the limit value $-1/(2\mu)$. Consequently, the partition function defined through such a deformed Hamiltonian would necessarily diverge for $\mathbf{g} < 2$.

For $\tau > 0$, the Hamiltonian, intended as a function of C_2, is pathological at $C_2 = N^3/\tau$. Namely, $H \to \mp\infty$ as $C_2 \to (N^3/\tau)^{\pm}$. One would suspect that (5.3) should really only hold for representations for which $C_2 < N^3/\tau$. Extending (5.3) beyond said range appears devoid of any physical meaning. However, if one hopes to determine the partition function as a sum over representations, one must necessarily understand how the deformation acts on the entire spectrum, not just a portion of it. From a physical standpoint, one is led to postulate that whenever a given representation falls out of the allowed range for the solution (5.3), it should be removed from the physical spectrum. Yet, this requirement leads to nontrivial analytic properties for the deformed partition function $Z(\alpha, \tau)$ and makes the study of such a quantity as a solution of the flow equation (5.1) less obvious.

For $\tau < 0$, the situation is more subtle. The partition function is still naively divergent, but the deformed spectrum (5.3) appears well-defined in any range of values. To obtain a well-defined partition function, one is led to study solutions of (5.1) that involve instanton-like corrections in the deformation parameter.[1] However, the question of how to unambiguously determine such nonperturbative corrections is nontrivial and will be addressed in detail in later sections.

As a conclusion, we find the naive $T\bar{T}$ deformation of the Migdal heat-kernel representation (2.102) when one replaces the quadratic Casimir with its deformed version, according to (5.3), shows some deep problematic features, for both signs of τ.

Summary of results In order to find a dynamical explanation for both aspects, we construct the deformed partition function for each flux sector $\mathcal{Z}_{\mathfrak{m}}$. These are sectors of the theory associated with stationary points of the classical action, labeled by the quantized magnetic flux vector $\mathfrak{m} \in \mathbb{Z}^N$. We reviewed the instanton expansion for Yang-Mills theory in detail in 2.2.2. To determine the correct solutions of the differential equation describing the $T\bar{T}$ flow,

$$\frac{\partial \mathcal{Z}_{\mathfrak{m}}}{\partial \mu} + 2a \frac{\partial^2 \mathcal{Z}_{\mathfrak{m}}}{\partial a^2} = 0 , \tag{5.4}$$

(a denotes the total area) we must impose suitable boundary conditions. These correspond to the two important physical regimes we have access to. The first is the undeformed theory in its fully-quantum regime. The second is the deformed theory in its semiclassical limit.

This approach is essential for two reasons. For $\mu > 0$, the postulated truncation of the spectrum must be associated with nonanalyticities of the partition function. In the abelian theory [54], we will show these can be interpreted in terms of an infinite number of infinite-order quantum phase transitions. This peculiar behavior only emerges when taking the sum over $\mathfrak{m}$: each $\mathcal{Z}_{\mathfrak{m}}$ is, in fact, analytic when $\mu > 0$. On the other hand, for $\mu < 0$, we need to fix nonperturbative terms to which the undeformed limit $\mu \to 0$ is insensitive. Crucially, these are relevant for the semiclassical regime of the theory that can be probed by taking a specific double scaling limit. Only by having the result for the deformed $\mathcal{Z}_{\mathfrak{m}}$ can we match its semiclassical limit against the deformed action evaluated on the associated classical saddle.

In analogy with the undeformed case, the partition function can be expressed through a sum over inequivalent irreducible representations of the gauge group. Schematically, we find that the full deformed partition function, written in terms of the deformed Hamiltonian

$$H = \frac{g_{\mathrm{YM}}^2 C_2(R)/2}{1 - \mu g_{\mathrm{YM}}^2 C_2(R)} , \tag{5.5}$$

[1] The presence of nonperturbative ambiguities in the context of $T\bar{T}$-deformed theories has been studied in [1, 73, 56].

reads

$$Z = \sum_{R|H>0} (\dim R)^2 \, e^{-aH} \qquad \text{for } \mu > 0, \tag{5.6}$$

$$Z = \sum_{R} (\dim R)^2 \left(e^{-aH} - \sum_{k=0}^{k_{\max}} \lambda_k \right) + \mathcal{R} \qquad \text{for } \mu < 0. \tag{5.7}$$

For $\mu > 0$, we have thus proven dynamically that the sum extends over the finite number of representations R for which $H > 0$. For $\mu < 0$, the sum is unrestricted, though in order for it to converge a finite number of terms λ_k is subtracted. These are the first few terms in the μ-expansion of e^{-aH}, and the upper bound $k_{\max}$ is the minimum value for which the sum over R converges. Each of the λ_k carries a factor of $e^{a/(2\mu)}$, making each term nonperturbative in μ. The same factor appears in the residual term $\mathcal{R}$, which is itself a solution of the flow equation (5.4).

The chapter is organized as follows. In Section 5.1, we expose the full exact $T\bar{T}$ deformation of Maxwell theory. In Section 5.2, we turn to Yang-Mills and in particular we construct the deformed zero-flux sector by Borel resumming the associated power expansion in the deformation parameter. The analytic properties of the associated Borel transform signal the presence of nonperturbative contributions at $\mu < 0$. We determine the form of such terms with resurgence theory. In Section 5.3, we compute the partition function for arbitrary flux sectors by solving the relevant flow equation. To reproduce the correct undeformed limit, we project the initial condition on a complete set of solutions using the Ramanujan master theorem. In Section 5.4, we sum over all flux sectors to obtain the explicit form of the full deformed partition function. It involves the use of the multidimensional Poisson summation formula and certain generalizations thereof. In Section 5.5, we show how the deformed flux sectors obtained in Section 5.3 reproduce the correct semiclassical limit, thus confirming our choice of nonperturbative corrections. We argue that the truncation of the spectrum is due to destructive interference between deformed flux sectors.

5.1 Preliminary case: Maxwell theory

In this section, we examine the simplest non-trivial case of $T\bar{T}$-deformed gauge theories: Maxwell theory on the torus. Although placing the theory on the torus topology leads to a clear interpretation in terms of a finite-volume thermal system, the abelian theory is insensitive to the underlying topology. As such, our results, which are taken by [54], apply to any genus. We will derive the exact expression of the partition function Z for both signs of the deformation parameter, solving nonperturbatively the flow equation for the U(1) gauge model

$$\frac{\partial Z}{\partial \mu} + 2A \frac{\partial^2 Z}{\partial A^2} = 0, \tag{5.8}$$

where A is the torus area [2]. As already mentioned, a certain number of nontrivial features are displayed by Maxwell theory that generalize to the nonabelian case.

[2] It's just (5.1) with $N = 1$

Maxwell theory and its deformation We consider Maxwell theory defined on the Euclidean torus $S^1 \times S^1$ with lengths R and β. The spectrum of the theory reads $E_n = g_{\mathrm{ym}}^2 R n^2/2$, where g_{ym} is the gauge coupling. Accordingly, the partition function takes the simple form

$$Z = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-g_{\mathrm{ym}}^2 A n^2/2} = \vartheta_3(\mathrm{e}^{-g_{\mathrm{ym}}^2 A/2}) \,, \tag{5.9}$$

where $A = R\beta$. A dual representation of the partition function is obtained by performing Poisson summation on (5.9),

$$Z = \sqrt{\frac{2\pi}{g_{\mathrm{ym}}^2 A}} \sum_{\mathfrak{m}\in\mathbb{Z}} \mathrm{e}^{-\frac{2\pi^2}{g_{\mathrm{ym}}^2 A}\mathfrak{m}^2} \,, \tag{5.10}$$

where the expression in the exponent represents the classical instanton action for configurations of (quantized) magnetic flux $\mathfrak{m}$, while the factor in front is due to quantum fluctuations. The $T\overline{T}$-deformed spectrum is readily obtained from the undeformed one by solving the relevant Burgers equation:

$$E_n = \frac{g_{\mathrm{ym}}^2 R n^2/2}{1-\mu e^2 n^2} \,. \tag{5.11}$$

Despite its simplicity, this immediately exhibits pathologies akin to the case of conformal theories. For $\mu > 0$, an infinite number of energy levels become negative, namely those with $n^2 > \frac{1}{\mu e^2}$, thus signaling an instability: one would like to truncate the spectrum and to explain the absence of the associated states dynamically. On the other hand, for $\mu < 0$ the spectrum remains positive, but it saturates on a maximum energy $E_{\mathrm{c}} = \frac{R}{2\mu}$. One can quickly compute the density of states in this limit and observe that it diverges as $(E_{\mathrm{c}} - E)^{-3/2}$. Moreover, in both cases the naive partition function is clearly ill-defined as it is given in terms of a divergent sum.

Solving the flow equation in each flux sector Since (5.8) is linear, it should separately apply to each term of the sum in (5.10). In fact, it is reasonable to expect that the deformation should lead to a well-defined result for each instanton, with the total partition function still expressible as a sum over "deformed" instantons and their fluctuations. Our strategy will be precisely to construct the full result as a sum over $T\overline{T}$-deformed flux sectors. As a bonus, deriving the contribution associated with each $\mathfrak{m}$, we can check that the classical deformed action for U(1) obtained in [26] dominates the semiclassical expansion.

The flow equation can be easily solved by separation of variables. The generic solution,

$$\frac{1}{\tau^s}\left[c_1 U\left(s,0,\frac{\alpha}{2\tau}\right) + c_2 \frac{\alpha}{2\tau}\, {}_1F_1\left(s+1;2;\frac{\alpha}{2\tau}\right)\right] \,, \tag{5.12}$$

is labeled by a real parameter s. Here, c_1 and c_2 are arbitrary constants, while U and ${}_1F_1$ respectively denote the Tricomi and Kummer confluent hypergeometric functions. We have also introduced the two adimensional quantities $\tau = \mu e^2$ and $\alpha = g_{\mathrm{ym}}^2 A$ that naturally appear in the generic solution. Denoting with $z_{\mathfrak{m}}(\alpha,\tau)$ the deformed partition function for a given flux $\mathfrak{m}$, we obtain its general form by considering a linear combination of the fundamental solutions in (5.12). The coefficients of such a combination are constrained to reproduce the undeformed result at $\tau = 0$ and to lead to a convergent expression upon summation over $\mathfrak{m}$. The behavior of (5.12) around $\tau = 0$ is sensitive to the sign of the deformation parameter. We consider the two choices separately.

The undeformed result, as it appears in (5.10), can be written as a convergent power series in $1/\alpha$ with

$$z_{\mathfrak{m}}(\alpha,0) = \sqrt{2\pi}\sum_{k=0}^{\infty}\frac{(-2\pi^2\mathfrak{m}^2)^k}{k!}\frac{1}{\alpha^{k+1/2}}. \tag{5.13}$$

Next, we consider the expansion of (5.12) as $\tau\to 0^+$. In this limit, the Kummer function blows up as $e^{\frac{\alpha}{2\tau}}$ for generic values of s. On the other hand, $U(s,0,x)\sim x^{-s}$ for $x\to\infty$. Thus, the obvious choice to match the expansion (5.13) is

$$z_{\mathfrak{m}}(\alpha,\tau) = \sqrt{\frac{\pi}{\tau}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{\pi^2\mathfrak{m}^2}{\tau}\right)^k U\left(k+\frac{1}{2},0,\frac{\alpha}{2\tau}\right). \tag{5.14}$$

This solution is precisely the one obtained by Borel-resumming the asymptotic series obtained through a power expansion in τ of the generic deformed flux sector $z_{\mathfrak{m}}(\alpha,\tau)$.[3]

By replacing the Tricomi function with its integral representation, we can perform the sum over k and rewrite (5.14) as the Fourier transform

$$z_{\mathfrak{m}}(\alpha,\tau) = \int_{-\infty}^{\infty}\mathrm{d}y\, e^{2\pi i\mathfrak{m}y}\,\phi(y) \tag{5.16}$$

of the smooth function with compact support

$$\phi(y) = e^{-\frac{\alpha y^2}{2(1-y^2\tau)}}\,\Theta(1-y^2\tau)\,, \tag{5.17}$$

where Θ is the step function. Now, the sum of $e^{2\pi i\mathfrak{m}x}$ over $\mathfrak{m}$ simply yields the Dirac comb of period 1. This allows us to trivially evaluate the integral in y and to obtain the full deformed partition function

$$Z(\alpha,\tau) = \sum_{n=-\lfloor\frac{1}{\sqrt{\tau}}\rfloor}^{\lfloor\frac{1}{\sqrt{\tau}}\rfloor} e^{-\frac{\alpha n^2}{2(1-n^2\tau)}}\,, \tag{5.18}$$

where the symbol $\lfloor x\rfloor$ stands for the integer part of x. In the above, we still sum over the deformed spectrum, but now all negative energies are excluded. Focusing on a specific level n, we see that its energy grows as τ increases and it blows up at $\tau = n^{-2}$. Above this threshold, the level drops out of the spectrum. As a consequence, only a finite number of energy levels survives when $\tau>0$. For $\tau>1$, the deformed spectrum contains only the ground state and the partition function becomes trivial: $Z(\alpha,\tau)=1$.

[3]Specifically, the associated Borel transform

$$\mathcal{B}z_{\mathfrak{m}}(\alpha,\zeta) = \sqrt{\frac{2\pi}{\alpha}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{2\pi^2\mathfrak{m}^2}{\alpha}\right)^k \times {}_2F_1\left(k+\frac{1}{2},k+\frac{3}{2};1;-\frac{2\zeta}{\alpha}\right) \tag{5.15}$$

produces the desired result when taking a directional Laplace transform along the positive real ζ-axis.

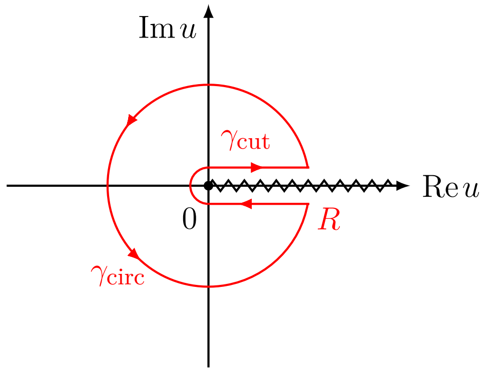


Figure 5.1: The contour γ for the integrals in (5.20) and (5.21) is the union of a Hankel-like contour γ_{cut} and a circle γ_{circle} of radius R. In $u = 1/\tau$ both integrands have an essential singularity.

When $\tau < 0$, the Tricomi function in (5.12) develops an imaginary part. However, we can easily engineer a new ansatz for real solutions of the flow equation exploiting the second family of hypergeometrics in (5.12),

$$z_{\mathfrak{m}}(\alpha,\tau) = \frac{\pi\alpha}{2}\sum_{k\in K}\frac{(4\pi^2\mathfrak{m}^2)^k}{(2k)!\,(-\tau)^{k+3/2}}\,{}_1F_1\left(k+\frac{3}{2};2;\frac{\alpha}{2\tau}\right)\,, \tag{5.19}$$

where $K = \{0,1,\ldots\}$. This expression is manifestly real and reproduces the $1/\alpha$-expansion in (5.13) for $\tau \to 0^-$. In fact, ${}_1F_1(s+1,2,-x) \sim x^{-s-1}/\Gamma(1-s) + \mathrm{e}^{-x}\ldots$ for $x \to \infty$.

The presence of exponentially-suppressed terms in the expansion of the Kummer function indicates that the solution is nonanalytic at $\tau = 0$. However, these are unavoidable if one wants to preserve the reality of the partition function. Again, the origin and necessity of nonanalytic terms also emerge if we carefully examine the perturbative solution of the flow equation through the tools provided by resurgence.[4]

Exploiting an integral representation of the Kummer function, we can perform the sum over k in (5.19) and obtain

$$z_{\mathfrak{m}}(\alpha,\tau) = -\oint_\gamma \mathrm{d}u\,\frac{\mathrm{i}\alpha\sinh\left(2\pi\mathfrak{m}\sqrt{-u}\right)\mathrm{e}^{-\frac{\alpha u}{2-2\tau u}}}{4\pi\mathfrak{m}(\tau u-1)^2}\,, \tag{5.20}$$

where the contour γ is depicted in FIG. 5.1. Suppose we now shrink γ around the essential singularity in $1/\tau$ and pick up the dominant contribution at large $|\mathfrak{m}|$. We can check that (5.19) grows exponentially in this limit. Thus, the sum over the fluxes does not converge, and (5.19) does not define a sensible partition function for $\tau < 0$.

We must remark that the existence of nonanalytic contributions solving the flow equation suggests that there is a degree of arbitrariness in writing down the ansatz (5.19): we are free to modify it by adding any combination of solutions exponentially vanishing at $\tau = 0$. The minimal modification of (5.19) that cancels the unsatisfactory behavior for large $|\mathfrak{m}|$ is given by extending the sum to half-integers with $K = \{0,1/2,1,\ldots\}$. We will see later that this

[4]The branch cut of (5.15) for $\zeta \in (-\infty,-\alpha/2)$ signals the need for instanton-like corrections in τ, when $\tau < 0$.

nonperturbative completion has remarkable properties. Each new term solves the flow equation and vanishes exponentially as $\tau \to 0^-$. Now

$$z_{\mathfrak{m}}(\alpha,\tau) = -\int_\gamma \mathrm{d}u\, \frac{\mathrm{i}\alpha\, \mathrm{e}^{-2\pi|\mathfrak{m}|\sqrt{-u}}\, \mathrm{e}^{-\frac{\alpha u}{2-2\tau u}}}{4\pi|\mathfrak{m}|(\tau u-1)^2}\,. \tag{5.21}$$

By shrinking the contour again around the essential singularity, we can verify that $z_{\mathfrak{m}}(\alpha,\tau)$ decays exponentially for large $|\mathfrak{m}|$, yielding a convergent sum over the fluxes. This behavior is exactly the one suggested by the semiclassical analysis where $z_{\mathfrak{m}}(\alpha,\tau)$ is expected to decay as the exponential of the deformed action [25].

The easiest way to compute the integral (5.21) is to consider γ as the sum of two contours $\gamma_{\text{cut}} \cup \gamma_{\text{circle}}$ (see FIG. 5.1). The integration over γ_{circle} vanishes in the limit of large R. Instead, the contribution of γ_{cut} is evaluated by taking the discontinuity of the integrand across the cut of the square root. Upon integrating by parts and setting $u = y^2$, we find

$$z_{\mathfrak{m}}(\alpha,\tau) = 2\int_0^\infty \mathrm{d}y\, \left(\mathrm{e}^{-\frac{\alpha}{2}\frac{y^2}{1-\tau y^2}} - \mathrm{e}^{\frac{\alpha}{2\tau}}\right)\cos(2\pi\mathfrak{m}y)\,. \tag{5.22}$$

Once again, we observe the appearance of the Dirac comb when summing over $\mathfrak{m}$, leading to

$$Z(\alpha,\tau) = \sum_{n=-\infty}^{\infty}\left(\mathrm{e}^{-\frac{\alpha}{2}\frac{n^2}{1-\tau n^2}} - \mathrm{e}^{\frac{\alpha}{2\tau}}\right). \tag{5.23}$$

We notice that the entire deformed spectrum survives, but we have an additional subtraction in the partition function, nonperturbative in τ, that ensures the convergence of the sum. The subtraction matches the asymptotic value of the first term for large n. Again, (5.23) solves the flow equation and reproduces the correct undeformed limit for $\tau \to 0^-$.

One would be tempted to conclude that an infinite number of nonperturbative states of energy E_{c} is present in the spectrum, having negative norms and regularizing the thermal trace. This feature is reminiscent of other instances of $T\overline{T}$-deformed theories which are associated with nonpositive-definite densities of states [56].

The deformed instanton action. In a suitable semiclassical limit, we expect that $z_{\mathfrak{m}}(\alpha,\tau)$ should be dominated by the exponential of the classical deformed action evaluated on the corresponding U(1) instanton configuration [19].

First, we check this property in the case $\tau \geq 0$. Performing the change of variable $y = \tau^{-1/2}\tanh(x)$ in (5.16), we can rearrange the integral as

$$z_{\mathfrak{m}}(\alpha,\tau) = \frac{1}{\sqrt{\tau}}\int_{-\infty}^{\infty}\mathrm{d}x\, \frac{\mathrm{e}^{-4\pi^2\mathfrak{m}^2\chi(x)/\alpha}}{\cosh^2(x)}\,, \tag{5.24}$$

where

$$\chi(x) = \frac{\sinh^2(x) - 2\mathrm{i}|\mathfrak{m}|\sqrt{\sigma}\tanh(x)}{2\mathfrak{m}^2\sigma} \tag{5.25}$$

and $\sigma = 4\pi^2\tau/\alpha^2$. This representation suggests considering a double scaling in which α and τ are taken small with σ fixed. In this limit, we expect the integral to be dominated by the saddles of χ. Posing $x = \mathrm{i}\arctan(\sqrt{\sigma}w|\mathfrak{m}|)$, the equation $\chi'(x) = 0$ translates into

$$\mathfrak{m}^4\sigma^2 w^4 + 2\mathfrak{m}^2\sigma w^2 - w + 1 = 0. \tag{5.26}$$

For $\sigma < \frac{27}{256m^2}$ we have a real saddle, which is smoothly connected to the one of the undeformed theory

$$w_\star = {}_3F_2\left(\frac{1}{2},\frac{3}{4},\frac{5}{4};\frac{4}{3},\frac{5}{3};\frac{256}{27}\,\mathfrak{m}^2\sigma\right)\,. \tag{5.27}$$

The integral around this saddle is easily evaluated using standard steepest descent approximation leading to

$$z_{\mathfrak{m}}(\alpha,\tau) \sim \sqrt{\frac{2\pi}{\alpha\eta}}\,\mathrm{e}^{-\frac{3\pi^2}{2\alpha\sigma}\left[{}_3F_2\left(-\frac{1}{2},-\frac{1}{4},\frac{1}{4};\frac{1}{3},\frac{2}{3};\frac{256}{27}\mathfrak{m}^2\sigma\right)-1\right]}\,, \tag{5.28}$$

where $\eta = 2w_\star^{-3/2} - w_\star^{-1} - 2\mathfrak{m}^2\sigma w_\star^{1/2}$. The dominant term is exactly proportional to the exponential of the classical deformed action obtained in [19].

While this analysis is certainly consistent at fixed $\mathfrak{m}$, the necessity to sum over the U(1) fluxes poses a problem with the branch cut of the hypergeometric function, the classical action becoming complex for $\mathfrak{m}^2 > 27\sigma/256$. A closer inspection of the interval $0 < \mathfrak{m}^2\sigma < 27/256$ unveils a second real solution of (5.26), which provides a subdominant contribution to our expansion. The two real solutions become closer and closer, and collide exactly when $\mathfrak{m}^2\sigma = 27/256$, emerging further as two complex conjugate solutions. We expect them to both contribute when $\mathfrak{m}^2\sigma > 27/256$, combining into a real expression for the full partition function. This dramatic change in the nature of the instanton expansion is presumably related to the truncation of the spectrum observed for $\tau > 0$.

We can readily repeat the same analysis in the case of $\tau < 0$. At variance with the previous situation, the hypergeometric function stays real for any value of $\mathfrak{m}$ (no branch cut is present when σ is negative). The saddle point connected to the undeformed case always dominates $z_{\mathfrak{m}}(\alpha,\tau)$ in the double-scaling limit. One can estimate the behavior of the instanton series at large $|\mathfrak{m}|$ as

$$z_{\mathfrak{m}}(\alpha,\tau) \sim \mathrm{e}^{-\frac{2\pi|\mathfrak{m}|}{\sqrt{-\tau}}}\,, \tag{5.29}$$

confirming the convergence of the sum over the U(1) fluxes.

Wilson loops and quantum phase transitions. The partition function (5.18) is nonanalytic whenever $\tau^{-1/2}$ is integer ($\tau = 0$ is a limit point for such a set of values). Nonetheless, it is always smooth in τ. Such nonanaliticities are the signs of phase transitions of infinite order [98]. We will show how Wilson-loop correlators act as order parameters for such transitions.

In accordance with our previous discussion, we introduce the partition function for an arbitrary topology with b boundaries

$$Z_b(\alpha,\tau,\theta_1,\ldots,\theta_b) = \sum_{n=-\lfloor\frac{1}{\sqrt{\tau}}\rfloor}^{\lfloor\frac{1}{\sqrt{\tau}}\rfloor} \mathrm{e}^{-\frac{\alpha n^2}{2(1-n^2\tau)}+\mathrm{i}(\theta_1 n_1+\ldots+\theta_1 n_1)}\,, \tag{5.30}$$

where the θ's parametrize the boundary holonomies. A correlator of two homological Wilson loops is given by

$$\langle W_{q_1}W_{q_2}\rangle = \int_0^{2\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_2}{2\pi}\,\mathrm{e}^{\mathrm{i}(\theta_1 q_1+\theta_2 q_2)} \times \frac{Z_2(\alpha_1,\tau,\theta_1,\theta_2)\,\overline{Z_2(\alpha_2,\tau,\theta_1,\theta_2)}}{Z(\alpha_1+\alpha_2,\tau)}\,, \tag{5.31}$$

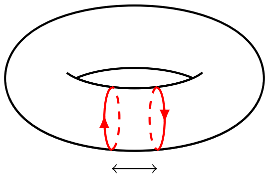


Figure 5.2: Two parallel Wilson loops, set at a distance L, wrapping the fundamental cycle of radius β. In the decompactification limit, the area of the torus diverges, while L and β are kept fixed.

where $q_1, q_2 \in \mathbb{Z}$ label the U(1) representations of the Wilson loops. The above is nonvanishing for $q_1 = -q_2$ and its computation is straightforward. In the decompactification limit, where $\alpha = \alpha_1 + \alpha_2 \to \infty$ while $\alpha_2 = e^2 L\beta$ is kept fixed, we find

$$\langle W_q W_{-q} \rangle \sim \mathrm{e}^{-e^2 L\beta \frac{q^2}{2(1-\tau q^2)}}\, \Theta(1-\tau q^2)\,. \tag{5.32}$$

One can think of the two Wilson-loop insertions as the wordlines of a particle-antiparticle test pair of charge qe set at a distance L and wrapping around the thermal circle. See FIG. 5.2.

For $\tau < q^{-2}$ the pair experiences an attractive potential that grows linearly with L, typical of a confined phase. However, as τ increases, the interaction gets stronger with an effective charge $e_{\text{eff}} = eq/\sqrt{1-\tau q^2}$. For $\tau > q^{-2}$, the potential diverges with the particles of charge eq seemingly decoupling from the theory.

To conclude this section, we have derived the exact partition function for the $T\overline{T}$-deformed U(1) gauge theory on the torus. Depending on the sign of the deformation, we have found radically different behaviors.

For $\mu > 0$, the spectrum of the theory undergoes a drastic reduction with only a finite number of states (namely, the ones with positive energy) surviving as indicated in Eq. (5.18). The truncation of the spectrum comes with an infinite number of quantum phase transitions, each associated with the vanishing of a certain correlator of Polyakov loops.

For $\mu < 0$, the appearance of a tower of nonperturbative contributions cures the naive divergence of the partition function. Conservatively, the origin of this tower can be traced back to the existence of a state of energy E_{c} with the negative norm for each U(1) flux. The spectral properties of the theory in this regime are encoded in the resolvent

$$\begin{aligned} R(E,\tau) &= \int_0^\infty \mathrm{d}\alpha\, \mathrm{e}^{\alpha E}\, Z(\alpha,\tau) \\ &= -\frac{\sqrt{2}\pi}{\sqrt{E}(2E\tau+1)^{3/2}} \cot\left(\frac{\sqrt{2}\pi\sqrt{E}}{\sqrt{2E\tau+1}}\right)\,. \end{aligned} \tag{5.33}$$

As already observed in the case of JT gravity [56], for negative values of the deformation parameter, the partition function can be reproduced by acting on the undeformed resolvent with the change of variables induced by the flow equation for the spectrum. In the case at hand, we observe that $R(E,\tau)$ is obtained from $R(E,0)$ with

$$E \longmapsto \frac{E}{1+2\tau E}\,. \tag{5.34}$$

The same approach cannot be used when $\mu > 0$ since the above map cease to be continuous. Still, it would be interesting to gain a better understading of (5.23), in particular from the perspective of the topological composition rules of the original gauge theory [129]. These, in fact, while fully compatible with the truncation of the spectrum for $\mu > 0$, fail to naively hold for (5.23).

5.2 The zero-flux sector

After having studied the simpler case of Maxwell theory, we now move to $U(N)$ Yang-Mills theory on the sphere, whose general features have been exposed in 2.2.3. Specifically, we start by studying the deformed zero-flux sector, describing quantum fluctuations about the "trivial" vacuum. In the present section, our approach is to regard $\mathcal{z}_{\mathbf{0}}$ as a power series in the deformation parameter τ. To this end, it is convenient to introduce the differential operator

$$\mathbf{D}_{\alpha,\tau} = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \left(-\frac{2\alpha}{N^2} \frac{\partial^2}{\partial \alpha^2} \right)^n . \tag{5.35}$$

Since $\mathbf{F}_{\alpha,\tau} \circ \mathbf{D}_{\alpha,\tau} = 0$, $\mathbf{D}_{\alpha,\tau}$ effectively generates power-series solutions of the flow equation when acting on the corresponding "undeformed" function encoding the initial condition at $\tau = 0$. However, the sum in (5.35) should be regarded as a formal power series in τ, since in general, it could have vanishing radius of convergence.

Before applying the above to $\mathcal{z}_{\mathbf{0}}$, it is convenient to write the undeformed zero-flux partition function (2.140) as a power series in α,[5]

$$\mathcal{z}_{\mathbf{0}}(\alpha, 0) = C_N \sum_{j=0}^{\infty} \frac{\alpha^{j-N^2/2}}{j!} \left(\frac{N^2-1}{24} \right)^j . \tag{5.36}$$

Then,

$$\begin{aligned} \mathcal{z}_{\mathbf{0}}(\alpha, \tau) &= \mathbf{D}_{\alpha,\tau}\, \mathcal{z}_{\mathbf{0}}(\alpha, 0) \\ &= C_N \sum_{j=0}^{\infty} \frac{\alpha^{j-N^2/2}}{j!} \left(\frac{N^2-1}{24} \right)^j \sum_{n=0}^{\infty} \left(\frac{2\tau}{N^2 \alpha} \right)^n \omega_n , \end{aligned} \tag{5.37}$$

where

$$\omega_n = \frac{(-1)^n \, \Gamma(j - N^2/2)\, \Gamma(1 + j - N^2/2)}{n!\, \Gamma(j - n - N^2/2)\, \Gamma(1 + j - n - N^2/2)} . \tag{5.38}$$

Let us now consider the sum over n: as anticipated, the series

$$\Phi(t) = \sum_{n=0}^{\infty} \omega_n \, t^{-n} \tag{5.39}$$

[5] To enforce convergence for small α, one can regard N as a complex number and choose an appropriate region in the complex N-plane.

is asymptotic, having $\omega_n \sim n!$ for large n. In order to apply the standard machinery of Borel resummation, we first consider its Borel transform,

$$\begin{aligned}\mathcal{B}\Phi(\zeta) &= \sum_{n=0}^{\infty} \zeta^n \frac{\omega_n}{n!} \\ &= {}_2F_1(N^2/2 - j, N^2/2 - j + 1; 1; -\zeta)\,.\end{aligned} \tag{5.40}$$

We observe two different behaviors depending on sign of τ.

5.2.1 $\tau > 0$

For positive values of τ, and hence of

$$t = \frac{N^2\alpha}{2\tau}\,, \tag{5.41}$$

we can simply Borel-resum the above by taking a Laplace transform along the positive real axis in the complex ζ-plane

$$\begin{aligned}\mathcal{S}_0\Phi(t) &= t\int_0^{\infty} \mathrm{d}\zeta\; \Phi(\zeta)\, e^{-t\zeta} \\ &= t^{N^2/2-j}\, U(N^2/2 - j, 0, t)\,.\end{aligned} \tag{5.42}$$

Here, U denotes the Tricomi confluent hypergeometric function; we refer the reader to Appendix A.14 for a brief survey on its properties. Plugging the resummed series back into (5.37) gives

$$\mathcal{Z}_{\mathbf{0}}(\alpha,\tau) = C_N \left(\frac{N^2}{2\tau}\right)^{N^2/2} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{\tau(N^2-1)}{12N^2}\right)^{j} U\left(\frac{N^2}{2} - j, 0, \frac{N^2\alpha}{2\tau}\right). \tag{5.43}$$

The sum is easily performed through the multiplication theorem for the Tricomi confluent hypergeometric function. This leads to the final expression

$$\mathcal{Z}_{\mathbf{0}}(\alpha,\tau) = C_N\, e^{X}\, Y^{N^2/2}\, U(N^2/2, 0, W)\,, \tag{5.44}$$

where we defined

$$X = \frac{N^2(N^2-1)\alpha}{2(N^2(12+\tau)-\tau)}\,, \tag{5.45}$$

$$Y = \frac{N^2(12+\tau)-\tau}{24\tau}\,, \tag{5.46}$$

$$W = \frac{6N^4\alpha}{\tau(N^2(12+\tau)-\tau)}\,. \tag{5.47}$$

It is immediate to check that, indeed, (5.44) is a solution of the flow equation (6.1) and reproduces the correct undeformed limit (2.140) for $\tau \to 0^+$.

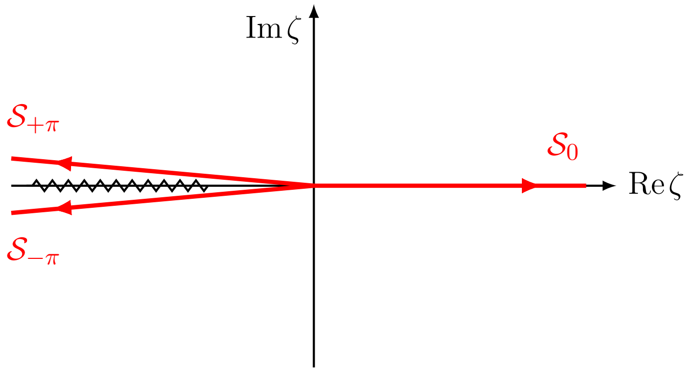


Figure 5.3: The contours where the directional Laplace transformations are taken. For $\tau > 0$, $\mathcal{S}_0$ gives the full result of the Borel summation, free from nonperturbative ambiguities. The Stokes line on the negative ζ axis induces two different lateral Laplace transformations, $\mathcal{S}_{+\pi}$ and $\mathcal{S}_{-\pi}$, that are relevant for the $\tau < 0$ regime.

5.2.2 $\tau < 0$

When $\tau < 0$, which means $t < 0$, the series should be resummed by taking a directional Laplace transform along the negative real ζ axis. However, as shown in Figure 5.3, the Borel transform (5.40) has a Stokes line at $\arg \zeta = \pi$ due to a cut that extends over $\zeta \in (-\infty, -1)$. By approaching the cut from above and below with lateral Laplace transforms one finds

$$\begin{aligned} \mathcal{S}_{\pm\pi}\Phi(t) &= t\int_0^{e^{\pm i\pi}\infty} \mathrm{d}\zeta\, e^{-t\zeta}\,{}_2F_1(N^2/2-j, N^2/2-j+1;1;-\zeta) \\ &= -t\int_0^{\infty} \mathrm{d}x\, e^{tx}\,{}_2F_1(N^2/2-j, N^2/2-j+1;1;-e^{\pm i\pi}x) \\ &= (t \mp i0)^{N^2/2-j}\, U(N^2/2-j, 0, t\mp i0)\,. \end{aligned} \tag{5.48}$$

We can use the analytic properties of the Tricomi confluent hypergeometric function to rewrite the above as

$$\begin{aligned} \mathcal{S}_{+\pi}\Phi(t) &= (t-i0)^{N^2/2-j}\left[U(N^2/2-j,0,t) - \frac{2i\pi t}{\Gamma(N^2/2-j)}\,{}_1F_1(N^2/2-j+1;2;t)\right], \\ \mathcal{S}_{-\pi}\Phi(t) &= (t+i0)^{N^2/2-j}\, U(N^2/2-j,0,t)\,. \end{aligned} \tag{5.49}$$

Since we are interested in a real partition function for real values of the deformation parameter τ and the effective 't Hooft coupling α, we can remove the nonperturbative ambiguities associated with the presence of the Stokes line by employing a prescription known as *median resummation*,

$$\mathcal{S}_{\text{med}}\Phi(t) = \frac{1}{2}(\mathcal{S}_{+\pi}\Phi(t) + \mathcal{S}_{-\pi}\Phi(t))\,. \tag{5.50}$$

For odd N, this prescription leads to

$$\begin{aligned} z_{\mathbf{0}}(\alpha,\tau) &= \frac{\pi C_N}{\Gamma(N^2/2)}\,\alpha\left(-\frac{N^2}{2\tau}\right)^{N^2/2+1} \\ &\quad\times\sum_{j=0}^{\infty}\frac{(1-N^2/2)_j}{j!}\left(-\frac{N^2-1}{N^2}\frac{\tau}{12}\right)^j {}_1F_1\left(\frac{N^2}{2}-j+1;2;\frac{N^2\alpha}{2\tau}\right) \\ &= -\frac{\pi C_N}{\Gamma(N^2/2)}We^X(-Y)^{N^2/2}{}_1F_1(N^2/2+1;2;W)\,. \end{aligned} \tag{5.51}$$

For N even, we find instead

$$\begin{aligned} z_{\mathbf{0}}(\alpha,\tau) &= C_N\left(-\frac{N^2}{2\tau}\right)^{N^2/2}\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{N^2-1}{N^2}\frac{\tau}{12}\right)^j \\ &\quad\times\left[U\left(\frac{N^2}{2}-j,0,\frac{N^2\alpha}{2\tau}\right)-\frac{\mathrm{i}\pi}{\Gamma(N^2/2-j)}\frac{N^2\alpha}{2\tau}{}_1F_1\left(\frac{N^2}{2}-j+1;2;\frac{N^2\alpha}{2\tau}\right)\right] \\ &= C_N\,e^X\,Y^{N^2/2}\left[U(N^2/2,0,W)-\frac{\mathrm{i}\pi W}{\Gamma(N^2/2)}{}_1F_1(N^2/2+1;2;W)\right]\,. \end{aligned} \tag{5.52}$$

Because the second term is purely imaginary, it necessarily cancels the imaginary part of the first one. In fact, we can rewrite the two expressions as a single formula that holds for any N as

$$z_{\mathbf{0}}(\alpha,\tau) = \mathrm{Re}\Big(C_N\,e^X\,Y^{N^2/2}\,U(N^2/2,0,W)\Big)\,. \tag{5.53}$$

In both cases, we made use of multiplication theorems (A.137) and (A.138) for confluent hypergeometric functions. For the theorems to hold, we need $\tau > \tau_{\min}$, where

$$\tau_{\min} = -\frac{12N^2}{N^2-1}\,. \tag{5.54}$$

In fact, when approaching $\tau_{\min}$ from the left, $W\to+\infty$. Consequently, the partition function diverges in this limit since the instanton-like terms blow up. We are therefore forced to regard $\tau > \tau_{\min}$ as a constraint on the validity of (5.51) and (5.52). When this condition is obeyed, the two expressions above are real, satisfy the flow equation, and reproduce the undeformed limit for $\tau\to 0$. One could, in principle, extend the range of validity by studying the relevant nonperturbative contributions for $\tau < \tau_{\min}$. However, upon summing over $\mathfrak{m}$, we will find an expression for the total partition function that does not exhibit any pathological behavior at $\tau = \tau_{\min}$ and that can be taken to hold for any value of the deformation parameter.

5.3 Any flux sector

The results of the previous section suggest an ansatz for the structure of the full solution of the flow equation (5.1) when written in terms of the variables Y and W,[6]

$$z_{\mathfrak{m}}(\alpha,\tau) = C_N\, e^X\, Y^{N^2/2} f(Y,W)\,. \tag{5.55}$$

On the above, the flow equation becomes

$$W\partial_W f(Y,W) - W\partial_W^2 f(Y,W) + Y\partial_Y f(Y,W) + \frac{N^2}{2} f(Y,W) = 0\,. \tag{5.56}$$

This equation can be easily solved by separation of variables. Specifically, if we choose $f(Y,W) = \Upsilon(Y)\,\Omega(W)$, we obtain two ordinary equations

$$Y\Upsilon'(Y) = s\Upsilon(Y)\,, \tag{5.57}$$

$$W\Omega''(W) - W\Omega'(W) - (N^2/2+s)\,\Omega(W) = 0\,, \tag{5.58}$$

with solutions

$$\Upsilon(Y) = cY^s\,, \tag{5.59}$$

$$\Omega(W) = u\,U(N^2/2+s,0,W) + v\,W\,{}_1F_1(N^2/2+s+1;2;W)\,. \tag{5.60}$$

Here, s is just an integration constant labeling different solutions.

We rewrite the ansatz as a generic linear combination of the above, where the coefficients are chosen in order to reproduce the boundary value at $\tau = 0$, which is fixed by the undeformed theory. From (2.136) and (2.137), it is easy to see that a generic undeformed flux sector can be expressed as a convergent expansion in $1/\alpha$ with structure

$$z_{\mathfrak{m}}(\alpha,0) = \sum_{k=0}^{\infty} \frac{a_{\mathfrak{m},k}}{\alpha^{k+N^2/2}}\,. \tag{5.61}$$

The behavior of $z_{\mathfrak{m}}(\alpha,\tau)$ for small τ is sensitive to the sign of the deformation. We separately study the two choices.

5.3.1 $\tau > 0$

When $\tau \to 0^+$, $W \to +\infty$ and ${}_1F_1\big(N^2/2+s+1,2,W\big)$ becomes exponentially divergent. This is not surprising as one expects the Kummer confluent hypergeometric function to bring non-perturbative contributions in τ, which should be absent for positive values of the deformation parameter. Therefore, when $\tau > 0$, we consider a general solution of the flow equation written as the linear combination

$$z_{\mathfrak{m}}(\alpha,\tau) = C_N\, e^X\, Y^{N^2/2} \sum_{s=0}^{\infty} \frac{p_{\mathfrak{m},s}}{s!} (-Y)^s\, U(N^2/2+s,0,W)\,. \tag{5.62}$$

[6] Here, X should be regarded as the shorthand

$$X = \frac{(N^2-1)W}{1+24Y-N^2}\,.$$

The undeformed limit is given by

$$\lim_{\tau\to 0^+} z_{\mathfrak{m}}(\alpha,\tau) = z_{\mathbf{0}}(\alpha,0)\sum_{s=0}^{\infty}\frac{p_{\mathfrak{m},s}}{s!}(-\alpha)^{-s}\,. \tag{5.63}$$

For the zero-flux sector, one can trivially determine the coefficients in the sum by comparing with (2.140). This gives $p_{\mathbf{0},s}=\delta_{s,0}$, which reproduces the result (5.44) derived in the previous section. For a generic flux sector, we can determine the coefficients $p_{\mathfrak{m},s}$ by exploiting the so-called *Ramanujan's Master Theorem.*[7] We find

$$p_{\mathfrak{m},s} = \frac{1}{\Gamma(-s)}\int_0^\infty \mathrm{d}\alpha\,\alpha^{s-1}\,\frac{z_{\mathfrak{m}}(\alpha,0)}{z_{\mathbf{0}}(\alpha,0)}\,. \tag{5.64}$$

The formula is understood for some region in the complex s-plane where the above is well-defined. The result is then analytically continued to positive integer values of s.

To determine the coefficient $p_{\mathfrak{m},s}$ for a generic flux $\mathfrak{m}$ we start with the definition (2.134). Performing the integral is nontrivial due to the presence of the square of the Vandermonde determinant in (2.133). However, within the Fourier integral, we can trade it for the differential operator

$$\mathbf{V} = \frac{\Delta^2(\partial_{\mathfrak{m}_1},\ldots,\partial_{\mathfrak{m}_N})}{(-4\pi^2)^{N(N-1)/2}}\,, \tag{5.65}$$

leading to

$$\begin{aligned} z_{\mathfrak{m}}(\alpha,0) &= z_{\mathbf{0}}(\alpha,0)\int_{\mathbb{R}^N}\mathrm{d}\ell_1\ldots\mathrm{d}\ell_N\, e^{-2\pi\mathrm{i}\mathfrak{m}\cdot\boldsymbol{\ell}}\,\frac{\Delta^2(\ell_1,\ldots,\ell_N)\,e^{-\frac{\alpha}{2N}\langle\boldsymbol{\ell}\rangle}}{(2\pi)^{N/2}N!\,G(N+1)\,(N/\alpha)^{N^2/2}} \\ &= z_{\mathbf{0}}(\alpha,0)\,\frac{(\alpha/N)^{\nu}}{N!\,G(N+1)}\,(-1)^m\,\mathbf{V}\,e^{-2\pi^2N|\mathfrak{m}|^2/\alpha}\,. \end{aligned} \tag{5.66}$$

We have used the fact that $\mathbf{V}(e^{-\pi\mathrm{i}m}f(\mathfrak{m})) = e^{-\pi\mathrm{i}m}\,\mathbf{V}f(\mathfrak{m})$. Now we can plug the above in (5.64). The integration can be performed by assuming $\mathrm{Re}\,s<0$. This gives the coefficient

$$p_{\mathfrak{m},s} = \frac{(-1)^{m+\nu}N^s}{N!\,G(N+1)}\,\frac{\Gamma(s+1)}{\Gamma(s+\nu+1)}\,\mathbf{V}\left(2\pi^2|\mathfrak{m}|^2\right)^{s+\nu}\,, \tag{5.67}$$

written in terms of the operator in (5.65) and the shorthand $\nu = N(N-1)/2$.

[7]According to the theorem, if a complex function f has an expansion of the form

$$f(x) = \sum_{k=0}^{\infty}\frac{\varphi(k)}{k!}(-x)^k\,,$$

then its Mellin transform reads

$$\int_0^\infty \mathrm{d}x\,x^{s-1}f(x) = \Gamma(s)\,\varphi(-s)\,.$$

5.3.2 $\tau < 0$

Let us now consider the case where $\tau < 0$. From the results of the previous section and the abelian case [54], we know that the deformed flux sector should receive nonperturbative corrections in τ. Indeed, the presence of the Kummer confluent hypergeometric function, necessary to construct real solutions of the flow equation (5.1) at $\tau < 0$, brings instanton-like contributions for $\tau \to 0^-$ (see Eq. (A.134)). As noticed in the previous section, the results will differ according to the parity of N. To avoid repeating the analysis for both choices, the rest of the present work will only focus on the case where N is odd, whenever $\tau < 0$. Upon taking the sum over all flux sectors, we will find an expression for the full deformed partition function that applies to any N.

We choose the ansatz

$$\mathcal{z}_{\mathfrak{m}}(\alpha,\tau) = -\pi C_N W e^X (-Y)^{N^2/2} \sum_{s\in K} \frac{(-1)^{2s} p_{\mathfrak{m},s}}{s!\,\Gamma(s+N^2/2)} (-Y)^s \,{}_1F_1(N^2/2+s+1;2;W)\,. \tag{5.68}$$

Let us for the moment set $K = \{0,1,2,\ldots\}$. The ansatz is carefully defined in such a way to reproduce the same limit as above, but now taken from negative values of τ, i.e. with this choice, it follows from (A.134)

$$\lim_{\tau\to 0^-} \mathcal{z}_{\mathfrak{m}}(\alpha,\tau) = \mathcal{z}_{\mathbf{0}}(\alpha,0) \sum_{s=0}^{\infty} \frac{p_{\mathfrak{m},s}}{s!} (-\alpha)^{-s}\,, \tag{5.69}$$

and one can still use Eq. (5.64) to find the coefficients of the sum. This is consistent with the result for the zero-flux sector derived in (5.51).

However, it is possible to add to the sum in (5.68) any half-integer s with $s > 1 - N^2/2$ without modifying the undeformed limit (5.69). This is due to the fact that for such values of s, the Kummer confluent hypergeometric function acts as a purely nonperturbative contribution to the result. As we will see later, the inclusion of these additional terms is crucial to ensure the convergence of the sum over $\mathfrak{m}$ producing the full partition function $Z(\alpha,\tau)$ and to generate the expected semiclassical limit of each flux sector. We will address both of these important points in the following sections.

For the moment, we simply choose $K = K^+ \cup K^-$, where

$$K^+ = \left\{0,\frac{1}{2},1,\frac{3}{2},2,\ldots\right\}, \tag{5.70}$$

$$K^- = \left\{1-\frac{N^2}{2},2-\frac{N^2}{2},\ldots,-\frac{1}{2}\right\}. \tag{5.71}$$

As observed at the end of the last section, the ansatz in (5.68) holds for $\tau > \tau_{\min}$, since the expression diverges for $\tau \to \tau_{\min}^-$. As we will see in the following, this is a constraint that applies only to the individual flux sectors since the expression for the full deformed partition function will hold for any value of τ.

5.4 The full partition function

In this section, we will compute the full deformed partition function by summing over all the deformed flux sectors. For these, we rely on the results of Section 5.2 and Section 5.3.

5.4.1 $\tau > 0$

For positive values of the deformation parameter, we can use (5.62) to compute a generic deformed flux sector, starting from (5.67). The sum over s is performed by first replacing the Tricomi confluent hypergeometric function with its integral representation (A.135) and then by using the identity

$$\sum_{s=0}^{\infty} \frac{x^s}{\Gamma(s+a)\Gamma(s+b)} = \frac{{}_1F_2(1;a,b;x)}{\Gamma(a)\Gamma(b)} = {}_1\tilde{F}_2(1;a,b;x)\,. \tag{5.72}$$

This readily gives

$$\begin{aligned} z_{\mathfrak{m}}(\alpha,\tau) = {} & \frac{(-1)^{m+\nu} C_N\, e^X\, Y^{N^2/2}}{N!\,G(N+1)} \int_0^\infty \mathrm{d}t\, \frac{e^{-tW}}{t(1+t)} \left(\frac{t}{1+t}\right)^{N^2/2} \\ & \times \mathbf{V}\left[\left(2\pi^2|\mathfrak{m}|^2\right)^{\nu} {}_1\tilde{F}_2\left(1;\tfrac{N^2}{2},\nu+1;-2\pi^2 NY\tfrac{t}{1+t}|\mathfrak{m}|^2\right)\right]. \end{aligned} \tag{5.73}$$

The above expression can be simplified by changing integration variable with

$$t = \frac{r^2}{2NY - r^2} \tag{5.74}$$

and by taking advantage of the following property of the hypergeometric function:

$${}_1\tilde{F}_2\left(1;\tfrac{N^2}{2},\nu+1;-z^2\right) = (-1)^{\nu} z^{1+N/2-N^2} J_{N/2-1}(2z) - \sum_{s=0}^{\nu-1} \frac{(-z^2)^{s-\nu}}{s!\,\Gamma(s+N/2)}\,. \tag{5.75}$$

We observe that, when inserted into to our integral, the finite sum appearing above combines with $|\mathfrak{m}|^{2\nu}$ to produce a polynomial of degree $2(\nu-1)$ in the $\mathfrak{m}$'s, and it vanishes under the action of $\mathbf{V}$, which is a differential operator of order 2ν. We are then left with

$$z_{\mathfrak{m}}(\alpha,\tau) = \frac{(-1)^m e^X}{N!\,G^2(N+1)} \mathbf{V}\left[2\pi \int_0^{\sqrt{2NY}} \mathrm{d}r\, r^{N/2} e^{-\frac{r^2 W}{2NY-r^2}} |\mathfrak{m}|^{1-N/2} J_{N/2-1}(2\pi r|\mathfrak{m}|)\right]. \tag{5.76}$$

In the above, we recognize the N-dimensional Fourier transform[8]

$$\begin{aligned} z_{\mathfrak{m}}(\alpha,\tau) &= \frac{(-1)^m e^X}{N!\,G^2(N+1)} \mathbf{V}\left[\int \mathrm{d}\ell_1 \ldots \mathrm{d}\ell_N\, e^{-2\pi i \mathfrak{m}\cdot\boldsymbol{\ell}}\, e^{-\frac{W|\boldsymbol{\ell}|^2}{2NY-|\boldsymbol{\ell}|^2}}\, \Theta(2NY-|\boldsymbol{\ell}|^2)\right] \\ &= \int \mathrm{d}\ell_1 \ldots \mathrm{d}\ell_N\, e^{-2\pi i \mathfrak{m}\cdot\boldsymbol{\ell}}\, \hat{z}_{\boldsymbol{\ell}}(\alpha,\tau)\,, \end{aligned} \tag{5.77}$$

[8]Let $f(\mathbf{x})$ be a spherically-symmetric function on $\mathbb{R}^N$. We denote $f(\mathbf{x}) = F(|\mathbf{x}|)$. Then

$$\int_{\mathbb{R}^N} \mathrm{d}\mathbf{x}\, e^{-2\pi i \mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) = 2\pi|\mathbf{k}|^{1-N/2} \int_0^\infty \mathrm{d}r\, J_{N/2-1}(2\pi|\mathbf{k}|r)\, r^{N/2} F(r)\,.$$

of the spherically-symmetric smooth function with compact support,

$$\hat{z}_{\boldsymbol{\ell}}(\alpha,\tau) = \frac{\Theta(2NY - \langle \boldsymbol{\ell} \rangle)}{N!\,G^2(N+1)}\,\Delta^2(\ell_1,\ldots,\ell_N)\,e^{X - \frac{W\langle \boldsymbol{\ell} \rangle}{2NY - \langle \boldsymbol{\ell} \rangle}}\,. \tag{5.78}$$

Here, Θ denotes the Heaviside step function. The full partition function comes from taking the sum over $\mathbf{m}$, which can be traded for a sum over $\boldsymbol{\ell}$ through the Poisson summation formula,

$$\begin{aligned} Z(\alpha,\tau) &= \sum_{\mathbf{m}\in\mathbb{Z}^N} z_{\mathbf{m}}(\alpha,\tau) \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^N} \hat{z}_{\boldsymbol{\ell}}(\alpha,\tau)\,. \end{aligned} \tag{5.79}$$

Analogously to the abelian case [54], the full partition function can be expressed in terms of the deformed Hamiltonian (5.3),

$$Z(\alpha,\tau) = \sum_{R\in\mathcal{R}_{N,\tau}} (\dim R)^2\, e^{-\frac{\alpha}{2N}\frac{C_2(R)}{1-\tau C_2(R)/N^3}}\,. \tag{5.80}$$

Crucially, the range of the sum extends over $\mathcal{R}_{N,\tau}$, the set of inequivalent irreducible representations of $U(N)$ with $\tau C_2 < N^3$. As a consequence, for any $\tau > 0$ the deformed partition function is a sum over a finite set, which is necessarily convergent. Moreover, the number of terms in the sum in (5.80) varies with τ. Specifically, as τ increases, a given representation R drops out of the sum when τ reaches the critical value $\tau_R = N^3/C_2(R)$. This prevents the partition function from being analytic in τ at $\tau = \tau_R$. However, as $\tau \to \tau_R^-$, the term in the sum associated with R vanishes together with its derivatives of any order, thus making the partition function smooth. These critical values of τ have been studied in [54] for the abelian theory. They were observed to be associated with quantum phase transition of infinite order.

For $\tau > N^2$, only the trivial representation contributes to the sum and the partition function becomes itself trivial with $Z(\alpha,\tau) = 1$.

5.4.2 $\tau < 0$

Let us now turn to the case where the deformation parameter is negative and N is odd. We start from (5.68) and write $z_{\mathbf{m}} = z^+_{\mathbf{m}} + z^-_{\mathbf{m}}$ by splitting the range of the sum over K^+ and K^- respectively, as defined in (5.70) and (5.71).

In Appendix A.15 we show that

$$\begin{aligned} z^+_{\mathbf{m}}(\alpha,\tau) &= \frac{e^X}{N!\,G^2(N+1)}\,\mathbf{V}\Bigg[2\pi\int_0^\infty \mathrm{d}r\, r^{N/2}\,|\mathbf{m}|^{1-N/2}\,J_{N/2-1}(2\pi r|\mathbf{m}|) \\ &\quad\times\left(e^{-\frac{r^2 W}{2NY - r^2}} - e^W - We^W\sum_{k=1}^{k_{\max}}\frac{1}{k}\left(\frac{2NY}{r^2}\right)^k L^1_{k-1}(-W)\right)\Bigg] \\ &= \int \mathrm{d}\ell_1\ldots\mathrm{d}\ell_N\, e^{-2\pi\mathrm{i}\mathbf{m}\cdot\boldsymbol{\ell}}\,\hat{z}^+_{\boldsymbol{\ell}}(\alpha,\tau)\,, \end{aligned} \tag{5.81}$$

where $k_{\max} = (N^2-1)/2$, and

$$\hat{z}^+_{\boldsymbol{\ell}}(\alpha,\tau) = \frac{e^X\,\Delta^2(\ell_1,\ldots,\ell_N)}{N!\,G^2(N+1)}\left(e^{-\frac{W|\boldsymbol{\ell}|^2}{2NY - |\boldsymbol{\ell}|^2}} - e^W - We^W\sum_{k=1}^{k_{\max}}\frac{1}{k}\left(\frac{2NY}{|\boldsymbol{\ell}|^2}\right)^k L^1_{k-1}(-W)\right). \tag{5.82}$$

One would be tempted to apply the standard Poisson summation formula to the above. However, the sum over $\mathfrak{m}$ does not converge.[9] One can check that, as a consequence, $\hat{z}^{+}_{\boldsymbol{\ell}}(\alpha,\tau)$ diverges for $\boldsymbol{\ell}\to\mathbf{0}$. It is possible to subtract the $\boldsymbol{\ell}=\mathbf{0}$ term (see e.g. [38]) with[10]

$$\sum_{\mathfrak{m}\in\mathbb{Z}^N} z^{+}_{\mathfrak{m}}(\alpha,\tau) - \int_{\mathbb{R}^N} \mathrm{d}\mathfrak{m}\; z^{+}_{\mathfrak{m}}(\alpha,\tau) = \sum_{\boldsymbol{\ell}\neq\mathbf{0}} \hat{z}^{+}_{\boldsymbol{\ell}}(\alpha,\tau)\,. \tag{5.83}$$

We use the above to write an expression for the full deformed partition function,

$$Z(\alpha,\tau) = \sum_{\boldsymbol{\ell}\neq\mathbf{0}} \hat{z}^{+}_{\boldsymbol{\ell}}(\alpha,\tau) + \mathcal{R}(\alpha,\tau)\,, \tag{5.84}$$

where the residual term reads[11]

$$\mathcal{R}(\alpha,\tau) = \int_{\mathbb{R}^N} \mathrm{d}\mathfrak{m}\; z^{+}_{\mathfrak{m}}(\alpha,\tau) + \sum_{\mathfrak{m}\neq\mathbf{0}} z^{-}_{\mathfrak{m}}(\alpha,\tau)\,. \tag{5.85}$$

We now have to efficiently express the residual part in terms of U(N)-representation data. To this end, we split the residual term as

$$\mathcal{R}(\alpha,\tau) = \mathcal{R}_{\mathrm{EM}}(\alpha,\tau) + \mathcal{R}_0(\alpha,\tau) \tag{5.86}$$

where

$$\mathcal{R}_{\mathrm{EM}}(\alpha,\tau) = \sum_{\mathfrak{m}\neq\mathbf{0}} z^{-}_{\mathfrak{m}}(\alpha,\tau) - \int_{\mathbb{R}^N} \mathrm{d}\mathfrak{m}\; z^{-}_{\mathfrak{m}}(\alpha,\tau)\,, \tag{5.87}$$

$$\mathcal{R}_0(\alpha,\tau) = \int_{\mathbb{R}^N} \mathrm{d}\mathfrak{m}\; z_{\mathfrak{m}}(\alpha,\tau)\,. \tag{5.88}$$

We remind the reader that convergence in the definition (5.87) should be understood as in Footnote 10.

[9] We prove in (A.152) that the sum of $z_{\mathfrak{m}}$ over all flux sectors is convergent. This can be split as

$$\sum_{\mathfrak{m}\in\mathbb{Z}^N} z^{+}_{\mathfrak{m}}(\alpha,\tau) + \sum_{\mathfrak{m}\in\mathbb{Z}^N} z^{-}_{\mathfrak{m}}(\alpha,\tau)\,.$$

The second sum diverges since $z^{-}_{\mathfrak{m}}$ is a polynomial in $\mathfrak{m}$. As a consequence, the first sum must diverge as well, and it must do so by sharing the same behavior at large $\mathfrak{m}$.

[10] In (5.83) and in other instances throughout the remainder of the present section, we will deal with finite expressions written as the difference between a divergent sum and a divergent integral. In such cases, convergence is ensured by taking the appropriate simultaneous limit of the spherical partial sums. Namely, the combination

$$\sum_{\mathbf{x}} f(\mathbf{x}) - \int \mathrm{d}\mathbf{x}\; g(\mathbf{x})$$

should be understood as the regulated form

$$\lim_{\Lambda\to\infty}\left(\sum_{|\mathbf{x}|\leq\Lambda} f(\mathbf{x}) - \int_{|\mathbf{x}|\leq\Lambda} \mathrm{d}\mathbf{x}\; g(\mathbf{x})\right).$$

[11] We omit the $\mathfrak{m}=\mathbf{0}$ term in the sum, since this vanishes. The result for the zero-flux sector is in (5.51).

From (5.68) and (5.67), and by using

$$ {}_1F_1(k+1;2;W) = \frac{e^W}{k} L^1_{k-1}(-W)\,, \tag{5.89} $$

we find

$$ z^-_{\mathfrak{m}}(\alpha,\tau) = \frac{(-1)^\nu \pi^{1-N/2}\, W e^{X+W}}{N!\,G^2(N+1)} \sum_{k=1}^{k_{\max}} \frac{(-2\pi^2 NY)^k\, L^1_{k-1}(-W)}{\Gamma(k+1)\,\Gamma(k+1-N/2)}\, \mathbf{V}\,|\mathfrak{m}|^{2k-N}\,, \tag{5.90} $$

while its Fourier anti-transform reads

$$ \hat{z}^-_{\boldsymbol{\ell}}(\alpha,\tau) = \frac{W e^{X+W}}{N!\,G^2(N+1)} \sum_{k=1}^{k_{\max}} \frac{(2NY)^k}{k}\, L^1_{k-1}(-W)\, \frac{\Delta^2(\ell_1,\dots,\ell_N)}{|\boldsymbol{\ell}|^{2k}}\,. \tag{5.91} $$

For the last step, we use again a generalized Poisson summation formula [38] to rewrite the residual term (5.87) as

$$ \begin{aligned} \mathcal{R}_{\mathrm{EM}}(\alpha,\tau) &= \sum_{\boldsymbol{\ell}\neq\mathbf{0}} \hat{z}^-_{\boldsymbol{\ell}}(\alpha,\tau) - \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{\ell}\, \hat{z}^-_{\boldsymbol{\ell}}(\alpha,\tau) \\ &= \frac{W e^{X+W}}{N!\,G^2(N+1)} \sum_{k=1}^{k_{\max}} Q^{\mathrm{reg}}_{N,k}\, \frac{(2NY)^k}{k}\, L^1_{k-1}(-W)\,, \end{aligned} \tag{5.92} $$

where the coefficients are given by the generalized Euler–Maclaurin expansion[12]

$$ Q^{\mathrm{reg}}_{N,k} = \sum_{\boldsymbol{\ell}\neq\mathbf{0}} \frac{\Delta^2(\ell_1,\dots,\ell_N)}{|\boldsymbol{\ell}|^{2k}} - \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{\ell}\, \frac{\Delta^2(\ell_1,\dots,\ell_N)}{|\boldsymbol{\ell}|^{2k}}\,. \tag{5.93} $$

We are left with the residual term (5.88) that we rewrite through (A.152) as the residue of an essential singularity,

$$ \begin{aligned} \mathcal{R}_0(\alpha,\tau) &= \frac{e^{X+W}}{N!\,G^2(N+1)}\, P_N\, \mathrm{Res}_{u=2NY}\left(e^{\frac{2NYW}{u-2NY}}\, u^{\nu-1}\right) \\ &= \frac{W e^{X+W}}{N!\,G^2(N+1)}\, P_N\, \frac{(2NY)^\nu}{\nu}\, L^1_{\nu-1}(-W)\,, \end{aligned} \tag{5.94} $$

in terms of the constant of group-theoretic origin

$$ \begin{aligned} P_N &= \pi^2 \int_{\mathbb{R}^N} \mathrm{d}\mathbf{m}\, \mathbf{V}\left[|\mathbf{m}|^{1-\frac{N}{2}}\, \mathrm{i}^{\frac{N}{2}}\, H^{(1)}_{\frac{N}{2}-1}(2\mathrm{i}\pi|\mathbf{m}|)\right] \\ &= \frac{G(N+2)\,\Gamma(N/2)}{(-2)^\nu\,\Gamma(N^2/2)}\,. \end{aligned} \tag{5.95} $$

By taking advantage of certain cancellations, we can write the deformed partition function in a more suggestive way as

$$ \begin{aligned} Z(\alpha,\tau) = \mathcal{R}_0(\alpha,\tau) &+ \sum_{\boldsymbol{\ell}\in\mathbb{Z}^N} \frac{\Delta^2(\ell_1,\dots,\ell_N)}{N!\,G^2(N+1)} \left(e^{X-\frac{W|\boldsymbol{\ell}|^2}{2NY-|\boldsymbol{\ell}|^2}} - e^{X+W}\right) \\ &- e^{X+W} \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{\ell}\, \frac{\Delta^2(\ell_1,\dots,\ell_N)}{N!\,G^2(N+1)}\, W \sum_{k=1}^{k_{\max}} \frac{1}{k} \left(\frac{2NY}{|\boldsymbol{\ell}|^2}\right)^k L^1_{k-1}(-W)\,. \end{aligned} \tag{5.96} $$

[12]See [16] for an introduction to generalized Euler–Maclaurin expansions and their physical applications.

Here, the counterterms have the same form as the ones that appear in (5.84), but are provided by an integral over $\boldsymbol{\ell}$, rather than a sum. In fact, in this expression both the sum and the integral are separately divergent. Once more, convergence should be understood as in Footnote 10.

The sum over $\boldsymbol{\ell}$ is the only term that survives in the abelian theory, and correctly reproduces the result obtained in [54], where a constant is subtracted from the e^{-aH} term associated with the deformed Hamiltonian (5.3). The same constant factor, namely

$$e^{X+W} = e^{\frac{N^2\alpha}{2\tau}} , \tag{5.97}$$

appears in front of the integral and in $\mathcal{R}_0$, and determines the nonperturbative character of every counterterm in (5.96).

So far, we have been able to recast the full partition function from a sum over fluxes into an expression that is the direct deformation of the sum over $\mathrm{U}(N)$ representations in (2.133). We now wish to provide a more compact way to reorganize the result, which should shed more light on the origin of the conterterms. To this end, we observe that in (5.82), instead of subtracting from the exponential the first $k_{\max}+1$ terms of its $1/|\boldsymbol{\ell}|$ expansion, we can equivalently use the rest of the series to write

$$\sum_{\boldsymbol{\ell}\neq\mathbf{0}} \hat{z}^{+}_{\boldsymbol{\ell}}(\alpha,\tau) = \frac{We^{X+W}}{N!\,G^2(N+1)} \sum_{k=k_{\max}+1}^{\infty} Q_{N,k}\,\frac{(2NY)^k}{k}\, L^1_{k-1}(-W)\,, \tag{5.98}$$

in a way that is similar to the residual term in (5.92), if not for the fact that the coefficients

$$Q_{N,k} = \sum_{\boldsymbol{\ell}\neq\mathbf{0}} \frac{\Delta^2(\ell_1,\ldots,\ell_N)}{|\boldsymbol{\ell}|^{2k}} \tag{5.99}$$

are not quite the same as in (5.93). However, as explained in Appendix A.16, $Q^{\mathrm{reg}}_{N,k}$ is the meromorphic continuation of $Q_{N,k}$ in k, and both can be expressed as certain derivatives of the regularized Epstein zeta function. Therefore, we can use the prescription in (A.161) to write the full partition function as

$$Z(\alpha,\tau) = \mathcal{R}_0(\alpha,\tau) + \frac{We^{X+W}}{N!\,G^2(N+1)} \sum_{k=1}^{\infty} \frac{(2NY)^k}{k}\, L^1_{k-1}(-W)\, \mathbf{V}\, \mathrm{Z}^{\mathrm{reg}}\,|\,{}^{\mathbf{0}}_{\mathfrak{m}}|\,(2k)\Big|_{\mathfrak{m}=0} . \tag{5.100}$$

The range of the sum can be safely extended to $k=0$ since the added term vanishes identically. According to this form of the partition function, the counterterms can be seen as originating from the regularization of the Dirichlet-like sums generated by the expansion in inverse powers of $|\boldsymbol{\ell}|$. There is only a finite number of such terms for any N, namely those with $1\leq k\leq k_{\max}$. The expansion necessarily misses the term with $\boldsymbol{\ell}=\mathbf{0}$, which is accounted for by the presence of $\mathcal{R}_0$. This term vanishes in the undeformed theory due to the presence of the Vandermonde determinant, but amounts to a finite nonperturbative contribution in the deformed theory as prescribed by the analysis of the deformed flux sectors.

5.5 The semiclassical limit

In Section 5.3.2, we commented on the fact that in writing (5.68), the partition function of a generic deformed flux sector for $\tau<0$, we were confronted with a choice regarding the nonperturbative part of the expression. At the end of Appendix A.15, we noticed that our choice guarantees the convergence of the sum over the fluxes $\mathfrak{m}$. Although this property is certainly necessary for the validity of our construction, it is not sufficient to fully remove the ambiguity regarding the exact form of the deformed flux sectors. From the point of view of the flow equation (5.1), this arbitrariness comes down to the fact that imposing a boundary condition at $\tau = 0$ is not enough to guarantee the uniqueness of the solution. A second boundary condition can be imposed by taking $\alpha \to 0^+$. However, one should be careful in taking such a limit, since there are a priori various ways in which this can be done. Specifically, keeping τ finite, or alternatively, keeping μ finite, both lead to unphysical regimes.

In pure undeformed Yang–Mills theory, taking the $g_{\rm YM} \to 0$ limit is equivalent to taking the semiclassical limit. The reason for this is that the gauge coupling acts as an overall constant that multiplies the action (2.98), playing a role analogous to that of $\hbar$. As a consequence, when $g_{\rm YM} \to 0$, the path integral localizes on the field configuration that minimize the Euclidean action. However, this feature is not shared by the deformed action, as it can be seen from (2.244) where the two sides of the equation carry different powers of $\mathscr{L}$, and thus of $g_{\rm YM}$. One should rather define a rescaled deformation parameter $\sigma \sim \mu/g_{\rm YM}^2$ so that the deformed Lagrangian density (2.247) depends on the gauge coupling through an overall power.

In terms of the variables at hand, then, the semiclassical limit amounts to taking $\alpha \to 0$ and $\tau \to 0$ simultaneously in such a way that

$$\sigma = \frac{4\pi^2}{N}\frac{\tau}{\alpha^2} \tag{5.101}$$

is kept fixed. Concretely, throughout this section we will replace τ with its expression in terms of α and σ, so that the semiclassical limit of $z_{\mathfrak{m}}$ is simply obtained by studying the regime where $\alpha \to 0$.

By performing the limit at the level of each individual flux sector we expect to find

$$-\log z_{\mathfrak{m}} \sim S_{\rm cl}(\mathfrak{m}, \sigma)\,, \tag{5.102}$$

where

$$S_{\rm cl}(\mathfrak{m}, \sigma) = \frac{3\pi^2 N}{2\alpha\sigma}\left({}_3F_2\left(-\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}; \frac{1}{3}, \frac{2}{3}; \frac{256}{27}|\mathfrak{m}|^2\sigma\right) - 1\right) \tag{5.103}$$

is the deformed action evaluated on the classical instanton configuration, obtained by plugging the classical undeformed action associated to flux sector $\mathfrak{m}$ into the deformed Yang-Mills Lagrangian (2.247). Notice that, since (2.247) is a strictly monotonic function of the undeformed Lagrangian density, the classical instanton configurations are stationary points of the deformed action as well.

The remainder of this section is devoted to the computation of the semiclassical limit of $z_{\mathfrak{m}}$ for both sign choices of τ.

5.5.1 $\tau > 0$

Choosing a positive deformation parameter corresponds to imposing $\sigma > 0$. We start with (5.76) and notice that the integrand is an even function of r. Thus we can rewrite the expression by mirroring the integration range about $r = 0$ and dividing by 2. We then change integration variable with

$$r = \frac{2\pi N|\mathfrak{m}|}{\alpha}\,w\,. \tag{5.104}$$

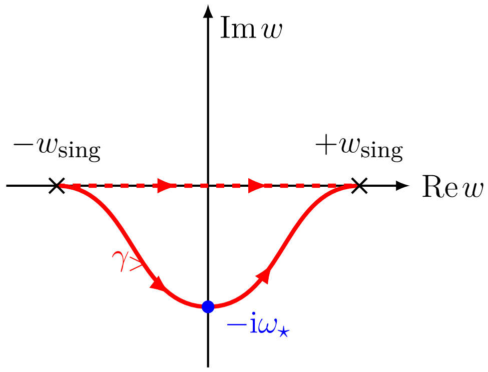


Figure 5.4: The integration contour for (5.107). The original integration contour is represented with a dashed red line and runs along the real w-axis. In the $\alpha \to 0$ limit, we deform the contour according to the steepest-descent approximation which prescribes crossing the saddle point at $-\mathrm{i}\omega_\star$ horizontally. The new contour $\gamma_>$ is represented with a solid red line.

The integral now extends over the range $(-w_{\text{sing}}, w_{\text{sing}})$, with

$$w_{\text{sing}} = \frac{1}{|\mathbf{m}|\sqrt{\sigma}} \,. \tag{5.105}$$

At the endpoints of such a range, the integrand has an essential singularity.

We now consider the $\alpha \to 0$ regime. To this end, we use the asymptotic behavior of the Bessel function, namely

$$J_\nu(2z) \sim \frac{\cos(2z - \nu\pi/2 - \pi/4)}{\sqrt{\pi z}} \qquad \text{for } z \to \infty \,, \tag{5.106}$$

and find

$$\mathcal{Z}_{\mathbf{m}} \sim \frac{(2\pi N/\alpha)^{\frac{N+1}{2}}(-1)^m}{N!\,G^2(N+1)} \int_{\gamma_>} \mathrm{d}w\, (\mathrm{i}w)^{\frac{N-1}{2}}\, \mathbf{V}\left[|\mathbf{m}|\, e^{-\frac{4\pi^2 N|\mathbf{m}|^2}{\alpha}\eta(w)}\right] , \tag{5.107}$$

where

$$\eta(w) = \frac{w^2}{2(1-|\mathbf{m}|^2\sigma w^2)} + \mathrm{i}w \,. \tag{5.108}$$

The trigonometric function in (5.106) was turned into a complex exponential in (5.107) by adding an appropriate odd function of w that gets cancelled under integration. The integration contour $\gamma_>$ is depicted in Figure 5.4.

We employ the steepest-descent approximation method. The solutions of the saddle point equation, that we write as $\eta'(-\mathrm{i}\omega) = 0$, are captured by the quartic

$$|\mathbf{m}|^4\sigma^2\omega^4 + 2|\mathbf{m}|^2\sigma\omega^2 - \omega + 1 = 0 \,. \tag{5.109}$$

There is a saddle,

$$\omega_\star = {}_3F_2\left(\frac{1}{2}, \frac{3}{4}, \frac{5}{4}; \frac{4}{3}, \frac{5}{3}; \frac{256}{27}|\mathbf{m}|^2\sigma\right) , \tag{5.110}$$

which is smoothly connected to the one of the undeformed theory. Due to the presence of a branch cut in the hypergeometric function, $\omega_\star$ is real for $|\mathfrak{m}|^2\sigma < 27/256$, and so is

$$\frac{1}{\eta''(-\mathrm{i}\omega_\star)} = {}_3F_2\Big(\frac{3}{4},\frac{5}{4},\frac{3}{2};\frac{4}{3},\frac{5}{3};\frac{256}{27}|\mathfrak{m}|^2\sigma\Big) > 0\,. \tag{5.111}$$

For this range of parameters, we deform the contour as indicated in Figure 5.4 and find

$$\mathcal{Z}_{\mathfrak{m}} \sim \frac{(2\pi N/\alpha)^{\frac{N}{2}}(-1)^m}{N!\,G^2(N+1)}\,\mathbf{V}\Big[h(|\mathfrak{m}|^2\sigma)\,e^{-S_{\mathrm{cl}}(\mathfrak{m},\sigma)}\Big]\,. \tag{5.112}$$

In the above, we have recognized

$$\frac{4\pi^2N|\mathfrak{m}|^2}{\alpha}\,\eta(-\mathrm{i}\omega_\star) = S_{\mathrm{cl}}(\mathfrak{m},\sigma) \tag{5.113}$$

to be the deformed classical action (5.103), and we have denoted

$$\sqrt{\frac{\omega_\star^{N-1}}{\eta''(-\mathrm{i}\omega_\star)}} = h(|\mathfrak{m}|^2\sigma)\,, \tag{5.114}$$

where

$$\begin{aligned} h(z) &= \Big[{}_3F_2\Big(\frac{1}{2},\frac{3}{4},\frac{5}{4};\frac{4}{3},\frac{5}{3};\frac{256}{27}z\Big)\Big]^{\frac{N-1}{2}}\Big[{}_3F_2\Big(\frac{3}{4},\frac{5}{4},\frac{3}{2};\frac{4}{3},\frac{5}{3};\frac{256}{27}z\Big)\Big]^{\frac{1}{2}} \\ &= 1+(N+2)z+O(z^2)\,. \end{aligned} \tag{5.115}$$

We can observe that the full result for the deformed flux sector in the semiclassical limit maintains the form in (2.136), i.e. it can be decomposed as

$$\mathcal{Z}_{\mathfrak{m}} \sim w_{\mathfrak{m}}(\alpha,\sigma)\,e^{-S_{\mathrm{cl}}(\mathfrak{m},\sigma)}\,. \tag{5.116}$$

This is because the action of the differential operator $\mathbf{V}$ in (5.112) does not spoil the presence of an overall exponential term associated with the deformed classical action. Rather, by acting as in (5.112), $\mathbf{V}$ determines the deformation of the fluctuation term $w_{\mathfrak{m}}(\alpha,\sigma)$.

Notice that for each flux sector there exists a finite neighborhood of $\sigma = 0$ for which $\mathcal{Z}_{\mathfrak{m}}$ is analytic in σ. Conversely, for any $\sigma > 0$ only a finite number flux sectors will have an associated partition function that is analytic at that point. This reflects the fact that, as discussed in Section 5.4.1, the total partition function has peculiar analyticity properties in any neighborhood of $\tau = 0$. The presence of a branch cut can also be understood by noticing that at $|\mathfrak{m}|^2\sigma = 27/256$ the saddle (5.110) collides with another real solution of (5.109). This additional saddle, which is subdominant in the range $0 < |\mathfrak{m}|^2\sigma < 27/256$, should combine with the contribution coming from $\omega_\star$ and modify the asymptotic behavior in (5.112) when $|\mathfrak{m}|^2\sigma > 27/256$.

From the point of view of the full partition function, this feature suggests a semiclassical mechanism for the truncation of the spectrum that can be attributed to a collective behavior of the flux sectors: for any fixed τ, the sum over the fluxes should include contributions coming from both saddles when $|\mathfrak{m}|$ is large enough. Accordingly, an infinite number of oscillatory terms appears in the full sum, since the saddle points are complex conjugates and carry, in general, a nontrivial imaginary part. We argue that a destructive interference occurs among these terms, resulting into the sharp cutoff on the sum over the representations. At a first look, this observation might seem at tension with the fact that the truncation of the spectrum is controlled by the sole τ, given that the location of the branchpoint of the classical action depends on both τ and α. We expect nevertheless that the interference should come from the full tower of fluxes, and as such to be dominated by terms with large $|\mathfrak{m}|$. Nicely, one finds that in said regime, $|S_{\rm cl}|^2 \sim 4\pi^2|\mathfrak{m}|^2 N^3/\tau$, recovering the dependence on the correct cutoff scale, including the expected power of N.

5.5.2 $\tau < 0$

For negative values of the deformation parameter, i.e. for $\sigma < 0$, we start from (A.152) and rescale the integration variable with

$$u = \frac{4\pi^2 N^2 |\mathfrak{m}|^2}{\alpha^2}\, v\ . \tag{5.117}$$

The essential singularity of the integrand sits now at

$$v_{\rm sing} = \frac{1}{|\mathfrak{m}|^2\sigma}\ . \tag{5.118}$$

In taking the $\alpha \to 0$ limit, we make use of (A.150) to write

$$\mathcal{Z}_{\mathfrak{m}}(\alpha,\tau) \sim \frac{(2\pi N/\alpha)^{\frac{N+1}{2}}}{N!\, G^2(N+1)} \frac{\rm i}{2} \oint_{\gamma_<} \frac{{\rm d}v}{v} \left(\sqrt{-v}\right)^{\frac{N+1}{2}} \mathbf{V}\left[|\mathfrak{m}|\, e^{-\frac{4\pi^2 N|\mathfrak{m}|^2}{\alpha}\chi(v)} \right], \tag{5.119}$$

where

$$\chi(v) = \frac{v}{2(1-|\mathfrak{m}|^2\sigma v)} + \sqrt{-v}\ . \tag{5.120}$$

By writing the saddle-point equation as $\chi'(-\omega^2) = 0$, we find that the solutions are again captured by the quartic in (5.109) and consider the solution in (5.110). Since now $\sigma < 0$, the saddle is always real for any range of parameters. Moreover, we find that

$$v_{\rm sing} < -\omega_\star^2 < 0\ . \tag{5.121}$$

In choosing the contour for (5.119) according to the steepest-descent prescription, we notice that $\chi''(-\omega_\star^2) < 0$. Accordingly, we define $\gamma_<$ so that it crosses the saddle point vertically as indicated in Figure 5.5. The final result reads

$$\mathcal{Z}_{\mathfrak{m}} \sim \frac{(2\pi N/\alpha)^{\frac{N}{2}}}{N!\, G^2(N+1)}\, \mathbf{V}\left[h(|\mathfrak{m}|^2\sigma)\, e^{-S_{\rm cl}(\mathfrak{m},\sigma)} \right], \tag{5.122}$$

where, again we find that

$$\frac{4\pi^2 N|\mathfrak{m}|^2}{\alpha}\, \chi(-\omega_\star^2) = S_{\rm cl}(\mathfrak{m},\sigma) \tag{5.123}$$

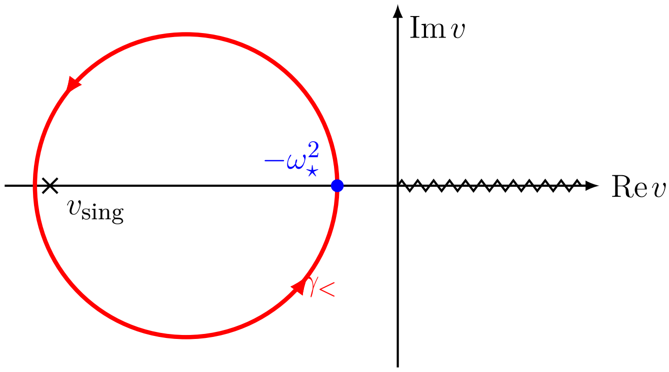


Figure 5.5: The integration contour for the steepest-descent approximation of (5.119) in the $\alpha \to 0$ limit. The contour crosses the saddle point $-\omega_\star^2$ parallel to the imaginary axis. Notice that, when approaching $v_{\rm sing}$ from the left, $\mathrm{Re}\,\chi \to +\infty$.

is the deformed classical action (5.103) evaluated on the classical instanton configuration with total magnetic flux $\mathfrak{m}$, while

$$\sqrt{-\frac{\omega_\star^{N-3}}{4\chi''(-\omega_\star^2)}} = h(|\mathfrak{m}|^2\sigma) \tag{5.124}$$

coincide with the expression obtained in (5.115).

It is remarkable that, for each flux sector, the semiclassical limits obtained for either signs of τ agree.[13] This effectively tells us that the nonperturbative corrections included in the partition function (5.68) are precisely those that guarantee such a match. In fact, each term that appears in the sum generates an instanton-like contribution of the form

$$e^{X+W} = e^{\frac{2\pi^2 N}{\alpha\sigma}} \tag{5.125}$$

that shapes the semiclassical limit.

[13]The only discrepancy between (5.112) and (5.122), namely the presence of the overall sign $(-1)^m$, is merely due to the fact that for $\tau < 0$ this was dropped since it is always trivial for odd N.

6 The phase diagram

Despite its simplicity, we know that Yang-Mills theory retains enough complexity to provide a convenient testing ground for conjectured properties of higher-dimensional models. Specifically, it can be used as a toy model to study various features of the large-N dynamics of gauge theories, such as the analyticity of the strong coupling expansion [34], or the 't Hooft gauge/string duality [122]. In fact, as we anticipated in 2.2.4, two-dimensional Yang–Mills theory has an exact description at large N in terms of a string theory, with $1/N$ playing the role of the string coupling constant. The expression for the $1/N$-expansion of the free energy can be computed in terms of branched covers of the two-dimensional target space, i.e. as string worldsheets of various windings [61, 66, 65].

Further, in the review 2.2.4 we have also explained how the partition function on genus-zero manifolds exhibits a large-N phase transition in the total area a,[1] going from a strongly-coupled string-like phase for large a to a weakly coupled phase for small a. This is a third-order phase transition first observed by Douglas and Kazakov [35]. Its physical origin can be understood from the weak-coupling side in terms of instanton condensation [97, 64] or as a divergence of the string expansion when seen from the strong-coupling region [35, 124].

In the previous chapter, We initiated a systematic study of $T\bar{T}$-deformed gauge theories,[2] deriving exact results for the abelian case [54] and for the nonabelian theory on the sphere [55]. In particular, our results of 5 have revealed a truncation of the spectrum for $\mu > 0$ associated with nonanalyticities in the partition function and the appearance of nonperturbative contribution in the deformation parameter for $\mu < 0$. It is a challenging task, though, to study the large-N limit of the theory from the exact expressions obtained at finite N.

In the present chapter, mainly following our paper [57], we study the deformed theory on the sphere in the limit where N is large. In taking this limit, one obtains a nontrivial dynamics by keeping finite the 't Hooft coupling λ and the dimensionless combination $\tau = \mu\lambda N^2$, which can be regarded as an effective deformation parameter. Having a new coupling τ opens up a new direction in the phase diagram of the theory, which in the undeformed case was simply the half-line $\alpha > 0$ (i.e., $\lambda > 0$). Indeed, studying the full structure of this phase diagram is one of the main goals of this work.

Summary of results. Contrary to previous investigations on the subject [116, 48], we follow an approach based on iteratively solving the system of partial differential equations governing the deformation of the large-N expansion of the free energy. We find exact solutions at all orders in $1/N$. These are obtained by propagating the initial conditions at $\tau = 0$ associated

[1] The 't Hooft coupling λ and the area a form an adimensional coupling $\alpha = \lambda a$, which is the proper coupling of the theory.

[2] The $T\bar{T}$ deformation of gauge fields has also been studied in connection with DBI-like theories [14, 27, 5].

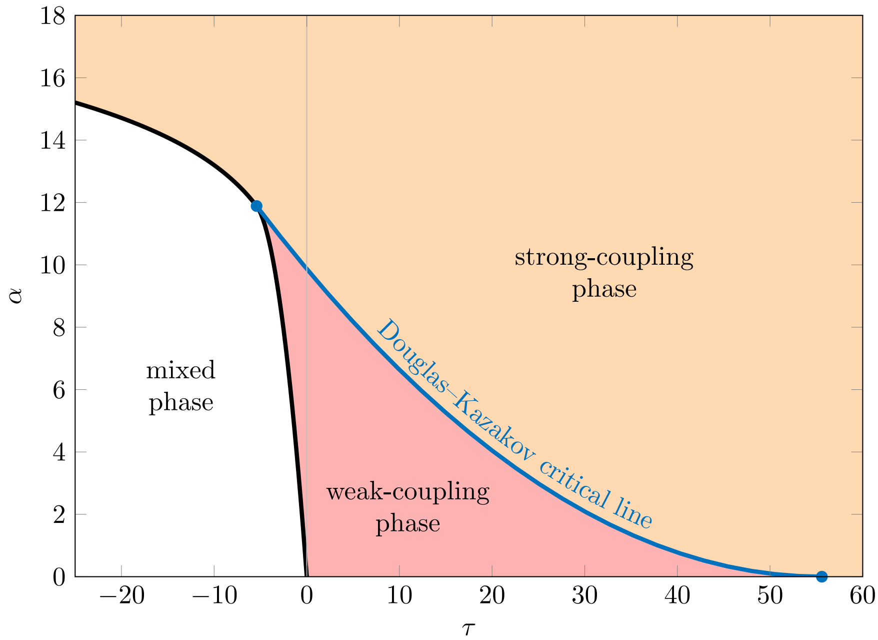


Figure 6.1: The phase diagram of the theory at large N has three phases: weak coupling, strong coupling, and mixed coupling. The blue line is the deformed Douglas–Kazakov critical line, associated with a third-order phase transition. The black line is a critical line associated with a second order-phase transition. The two lines join at a multicritical point represented by a blue dot. The thin gray line at $\tau = 0$ corresponds to the undeformed theory.

with both the weak-coupling and the strong-coupling regime along a system of characteristic curves determined by the leading order F_0 of the free energy. These curves effectively chart the phase diagram of the deformed theory; much of the information on the large-N dynamics can be obtained by studying their properties. The entire phase diagram is shown in Figure 6.

The characteristic curve emanating from the Douglas–Kazakov critical point acts as an interface between the characteristics transporting the weak-coupling and the strong-coupling initial conditions. In other words, the critical point of the undeformed theory is now a critical line with an associated third-order phase transition. The critical line is monotonically decreasing as a function of τ. It reaches the $\alpha = 0$ axis at a value τ_{max} above which the theory exists only in the strong phase. At such a point, the discontinuity of F_0''' diverges.

There is a second endpoint of this critical line where F_0''' again diverges. This happens at $\tau_{\text{mcp}} < 0$. We can interpret this behavior by observing that the Douglas–Kazakov line tangentially joins a novel critical line associated with a second-order phase transition on such a point. This curve is an envelope for the characteristics of both the strong coupling and the weak-coupling phase and effectively acts as a boundary for both phases. From the point of view of the differential equation, this limits the region that can be accessed by propagating the initial condition at $\tau = 0$. This phenomenon has to do with nonperturbative corrections in τ, which cease to be suppressed in the large-N limit upon crossing the envelope. We had already observed in [55] how such corrections introduce ambiguities that can be fixed by imposing a second boundary condition. In the new region, which we refer to as *mixed phase*, the hierarchy between instantons in α, typical of the weak-coupling phase, is also lost.

Figure 6 can then be interpreted as a diagram in which each phase represents a different regime for the instantons. In the weak-coupling phase, all instanton corrections are suppressed. In the strong phase, the instantons in α contribute to the result, while the instantons in τ are suppressed. Finally, in the mixed phase, both types of instantons contribute.

We mentioned earlier that at finite N, the theory exhibits nonanalyticities in the free energy associated with the truncation of the spectrum and with the presence of nonperturbative corrections in the deformation parameter. We can explain the absence of such feature at large N with the way τ scales with N, which makes such a limit well-defined. For $\tau > 0$, the scaling has the effect of restoring an infinite spectrum or, equivalently, of pushing towards $\tau \to +\infty$ the points of nonanalyticity. On the other hand, if $\alpha > 0$, we notice from the phase diagram that there always exists a region for small $\tau < 0$ where the nonperturbative corrections in τ are suppressed,[3] thus ensuring the analyticity of the free energy at $\tau = 0$.

6.1 Preliminary observations

We recall that the flow of the Yang–Mills partition function along the $T\bar{T}$ deformation is controlled by the partial differential equation [25, 75, 115, 55]

$$\frac{1}{\lambda}\frac{\partial Z}{\partial \mu} + 2\alpha \frac{\partial^2 Z}{\partial \alpha^2} = 0 \,, \tag{6.1}$$

where $\lambda = \alpha/a$ is the 't Hooft coupling. For $\mu > 0$, we showed in 5 that the deformed partition function is given by a formula analogous to the heath-kernel expansion (2.102), namely [55]

$$Z = \sum_{C_2(R,\mu)>0} (\dim R)^2 \, e^{-\frac{\alpha}{2N} C_2(R,\mu)} \,, \tag{6.2}$$

where each representation is weighted by the "deformed quadratic Casimir"

$$C_2(R,\mu) = \frac{C_2(R)}{1-\mu\lambda C_2(R)/N} \,, \tag{6.3}$$

and the sum is restricted over the representations for which the above is positive. In other words, whenever the deformation parameter reaches a critical value $\mu_R = N/(\lambda C_2(R))$, the associated representation R is removed from the sum in (6.2). As a consequence, Z is nonanalytic yet smooth for $\mu \in \{\mu_R\}$. Furthermore, for any $\mu > 0$, only a finite number of representations R contribute to the partition function, i.e. such that $\mu_R < \mu$. The only representation always present in the sum is the trivial representation since it has $C_2 = 0$. Next, we find that the two $\mathrm{U}(N)$ representations with the smallest Casimir are the fundamental and the antifundamental representation, namely

$$\mathbf{n}_\mathrm{F} = (+1, 0, \ldots, 0) \,, \tag{6.4}$$

$$\mathbf{n}_\mathrm{A} = (0, \ldots, 0, -1) \,, \tag{6.5}$$

both of which have $C_2 = N$. This means that for every N, the theory becomes completely trivial when $\mu > 1/\lambda$.

[3] These corrections have the form $e^{N^2\alpha/2\tau}$ and are thus suppressed at large N for small τ and large α, consistently with the picture emerging from the phase diagram in Figure 6.

A large-N theory with a finite number of states would necessarily bear no resemblance to the two-phases undeformed theory described in the previous section. To obtain a deformed theory with rich dynamics at large-N, one should find an appropriate double-scaling limit where $\mu \to 0$ when $N \to \infty$, so that the sum over an infinite number of representations is restored. The flow equation (6.1) suggests the correct scaling. If we consider just the leading order in the large-N expansion of the free energy, namely $\log Z \sim N^2 F_0$, the corresponding differential equation reads

$$\frac{1}{\lambda}\frac{\partial F_0}{\partial \mu} + 2N^2\alpha\left(\frac{\partial F_0}{\partial \alpha}\right)^2 + 2\alpha\frac{\partial^2 F_0}{\partial \alpha^2} = 0\,. \tag{6.6}$$

By defining as in [115, 55] the rescaled adimensional deformation parameter $\tau = \mu\lambda N^2$ we provide the right scaling so that the representations contributing to the leading order of the free energy are still present. At the same time, the deformed Casimir remains nontrivial over such a set when N is large. The flow equation for F_0 in terms of τ then reads

$$\frac{\partial F_0}{\partial \tau} + 2\alpha\left(\frac{\partial F_0}{\partial \alpha}\right)^2 = 0\,. \tag{6.7}$$

We have seen in the previous chapter 5 that, for $\mu < 0$, the deformed partition function receives nonperturbative corrections carrying an overall factor of $e^{N^2\alpha/(2\tau)}$, thus making the partition function nonanalytic at $\tau = 0$. However, we notice that at large-N, because of the chosen scaling in N, one expects these instanton-like corrections to be suppressed for small τ, thus making F_0 analytic at $\tau = 0$. In the next section, we will see that this is indeed the case.

Let us now quickly recall some features of the deformed theory at finite N that will be useful to treat the large N limit. In 5, based on [55], the deformed partition function on the sphere was computed by first finding the correct solution of the flow equation associated with each deformed $\mathcal{Z}_{\mathfrak{m}}(\alpha,\tau)$, and by then summing over $\mathfrak{m}$. The partition functions of the various flux sectors are conveniently expressed in terms of the variables

$$X = \frac{N^2(N^2-1)\alpha}{2(N^2(12+\tau)-\tau)}\,, \tag{6.8}$$

$$Y = \frac{N^2(12+\tau)-\tau}{24\tau}\,, \tag{6.9}$$

$$W = \frac{6N^4\alpha}{\tau(N^2(12+\tau)-\tau)}\,, \tag{6.10}$$

and read

$$\mathcal{Z}_{\mathfrak{m}}(\alpha,\tau) = \begin{cases} C_N\, e^X\, Y^{N^2/2} \displaystyle\sum_{s=0}^{\infty} \frac{p_{\mathfrak{m},s}}{s!}(-Y)^s\, U(N^2/2+s,0,W) & \text{for } \tau > 0\,, \\ -\pi C_N\, W e^X (-Y)^{N^2/2} & \\ \times \displaystyle\sum_{s\in K} \frac{(-1)^{2s} p_{\mathfrak{m},s}}{s!\,\Gamma(s+N^2/2)}(-Y)^s\, {}_1F_1(N^2/2+s+1;2;W) & \text{for } \tau < 0,\ N \text{ odd}\,, \end{cases} \tag{6.11}$$

with $K = \{1-\frac{N^2}{2}, 2-\frac{N^2}{2}, \ldots, -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots\}$. With U and ${}_1F_1$ we denote, respectively, the Tricomi and the Kummer confluent hypergeometric functions. For simplicity, we will not deal with the case of even N when $\tau < 0$. The coefficients that appear in the solutions are given by

$$p_{\mathfrak{m},s} = \begin{cases} \delta_{s,0} & \text{for } \mathfrak{m} = \mathbf{0}, \\ \dfrac{(-1)^{m+\nu} N^s}{N! G(N+1)} \dfrac{\Gamma(s+1)}{\Gamma(s+1+\nu)} \mathbf{V} \left(2\pi^2 |\mathfrak{m}|^2\right)^{s+\nu} & \text{for } \mathfrak{m} \neq \mathbf{0}, \end{cases} \tag{6.12}$$

where $\nu = N(N-1)/2$. This ensures that the limit

$$\lim_{\tau \to 0} z_{\mathfrak{m}}(\alpha, \tau) = z_{\mathbf{0}}(\alpha, 0) \sum_{s=0}^{\infty} \frac{p_{\mathfrak{m},s}}{s!} (-\alpha)^{-s} \tag{6.13}$$

matches the correct expression for the undeformed flux sector (2.139).

6.2 The large-N expansion of the free energy

In the last section, we have determined the correct scaling of the effective deformation parameter τ, deriving the flow equation that governs the leading order of the free energy in the large-N limit. The goal now is to study further the flow equation, to obtain exact results for all orders in the $1/N$-expansion, and to identify the main features of the phase diagram of the theory for both positive and negative values of τ.

We first need to write down the flow equation acting on the deformed free energy $F(\alpha, \tau) = \log Z(\alpha, \tau)$. Eq. (6.1) induces a partial differential equation for $F(\alpha, \tau)$ which takes the form

$$N^2 \partial_\tau F + 2\alpha (\partial_\alpha F)^2 + 2\alpha \, \partial_\alpha^2 F = 0 \, . \tag{6.14}$$

Before expanding in powers of N, it is useful to transform (6.14) into an equation with constant coefficients by replacing F with

$$F(\alpha, \tau) = N^2 G(\sqrt{\alpha}, \tau) + \frac{\log \alpha}{4} \, , \tag{6.15}$$

thus obtaining for $G(z, t)$

$$\partial_\tau G + \frac{1}{2} (\partial_z G)^2 + \frac{1}{2N^2} \, \partial_z^2 G = \frac{3}{8N^4 z^2} \, . \tag{6.16}$$

Finally, we assume that F, and thus G, possess an expansion in powers of $1/N^2$, as in the case of the undeformed theory. In particular, we denote

$$G(z, \tau) = \sum_{\ell=0}^{\infty} N^{-2\ell} G_\ell(z, \tau) \, . \tag{6.17}$$

Let us now start by considering the leading order at large N. Instead of directly dealing with the equation for G_0,

$$\partial_\tau G_0 + \frac{1}{2} (\partial_z G_0)^2 = 0 \, , \tag{6.18}$$

it is easier to study the equivalent problem for $\mathcal{E} = \partial_z G_0$, which is described by the well-known inviscid Burgers' equation

$$\partial_\tau \mathcal{E} + \mathcal{E}\,\partial_z \mathcal{E} = 0\,. \tag{6.19}$$

Standard solutions are obtained by studying the characteristics of the differential operator $\mathcal{D} = \partial_\tau + \mathcal{E}\,\partial_z$, i.e. the solutions of the ordinary differential equation $\mathrm{d}z/\mathrm{d}\tau = \mathcal{E}$. According to (6.19), $\mathcal{E}$ is constant along the characteristics, which are then given by

$$z = \xi + \tau\,\varphi(\xi)\,, \tag{6.20}$$

where $\varphi(\xi) = \mathcal{E}(\xi, 0)$ and ξ is some integration constant. The original equation (6.19) is solved by simply inverting (6.20), from which one can write the explicit solution

$$\mathcal{E}(z,\tau) = \varphi(\xi(z,\tau))\,. \tag{6.21}$$

It is not difficult at this point to derive the equations for the subleading terms in the large-N expansions:

$$\begin{aligned}
\mathcal{D}G_1 &= -\frac{1}{2}\,\partial_z^2 G_0\,,\\
\mathcal{D}G_2 &= -\frac{1}{2}\,\partial_z^2 G_1 - \frac{1}{2}(\partial_z G_1)^2 + \frac{3}{8z^2}\,,\\
\mathcal{D}G_\ell &= -\frac{1}{2}\,\partial_z^2 G_{\ell-1} - \frac{1}{2}\sum_{k=1}^{\ell-1}\partial_z G_k\,\partial_z G_{\ell-k}\,, \qquad \text{for } \ell \geq 2.
\end{aligned} \tag{6.22}$$

This recursive system can be conveniently integrated by changing variables with

$$\tilde{G}_\ell(\xi,\tau) = G_\ell(z(\xi,\tau),\tau)\,, \tag{6.23}$$

in terms of which (6.22) becomes

$$\partial_\tau \tilde{G}_\ell = \frac{\tau\,\ddot{\varphi}\,\partial_\xi \tilde{G}_{\ell-1}}{2(1+\tau\,\dot{\varphi})^3} - \frac{\partial_\xi^2 \tilde{G}_{\ell-1} + S_\ell}{2(1+\tau\,\dot{\varphi})^2} + \frac{3\delta_{\ell,2}}{8z^2}\,, \tag{6.24}$$

where $S_1 = 0$, while for $\ell \geq 2$,

$$S_\ell(\xi,\tau) = \sum_{k=1}^{\ell-1} \partial_\xi \tilde{G}_k(\xi,\tau)\,\partial_\xi \tilde{G}_{\ell-k}(\xi,\tau)\,. \tag{6.25}$$

The solutions are now easy to find:

$$\tilde{G}_0(\xi,\tau) = -\frac{1}{2}\int_0^\tau \mathrm{d}t\,\mathcal{E}^2(z(\xi,\tau),t) + F_0(z^2(\xi,\tau),0)\,, \tag{6.26}$$

$$\begin{aligned}
\tilde{G}_\ell(\xi,\tau) = &\int_0^\tau \mathrm{d}t\left(\frac{t\,\ddot{\varphi}(\xi)\,\partial_\xi \tilde{G}_{\ell-1}(\xi,t)}{2(1+t\,\dot{\varphi}(\xi))^3} - \frac{\partial_\xi^2 \tilde{G}_{\ell-1}(\xi,t) + S_\ell(\xi,t)}{2(1+t\,\dot{\varphi}(\xi))^2}\right)\\
&-\frac{\log\xi\,\delta_{\ell,1}}{2} + \frac{3\tau\,\delta_{\ell,2}}{8\xi(\xi+\tau\,\varphi(\xi))} + F_\ell(\xi^2,0)\,, \qquad \text{for } \ell \geq 1.
\end{aligned} \tag{6.27}$$

In the second identity, we made use of

$$F_\ell(z^2(\xi,\tau),\tau) = \tilde{G}_\ell(\xi,\tau) + \frac{\delta_{\ell,1}}{2}\log\left(\xi + \tau\,\varphi(\xi)\right), \tag{6.28}$$

which is a trivial consequence of (6.15). Conversely, we can recover the free energy from the solutions (6.26) and (6.27) with

$$F_\ell(\alpha,\tau) = \tilde{G}_\ell(\xi(\sqrt{\alpha},\tau),\tau) + \delta_{\ell,1}\frac{\log\alpha}{4}. \tag{6.29}$$

6.2.1 The phase diagram

As a direct application of the previous formulas, one can read off the large-N expansion of the free energy in the weak-coupling phase,[4] taking as boundary condition the undeformed zero-flux partition function (2.140)

$$F_0(\alpha,0) = \frac{3}{4} + \frac{\alpha}{24} - \frac{\log\alpha}{2}, \tag{6.30}$$

$$F_1(\alpha,0) = -\frac{\alpha}{24}, \tag{6.31}$$

$$F_\ell(\alpha,0) = 0, \qquad \text{for } \ell \geq 2. \tag{6.32}$$

A peculiar feature of the undeformed theory is that at weak coupling, only F_0 and F_1 are nontrivial. As we will see in a moment, this property ceases to hold at finite τ.

To show this, we simply apply the algorithm previously described. The first step is to use (6.30) to compute

$$\varphi(\xi) = \mathcal{E}(\xi,0) = \frac{\xi}{12} - \frac{1}{\xi}. \tag{6.33}$$

We then plug this in (6.20) and find

$$\xi = 6\,\frac{z+\sqrt{z^2-\alpha_{\mathrm{w}}}}{\tau+12}, \tag{6.34}$$

which, in turn, from (6.21) and defining $\alpha_{\mathrm{w}} = -\tau(12+\tau)/3$, gives

$$\mathcal{E}(z,\tau) = \frac{z}{\tau+12} - 2\,\frac{z-\sqrt{z^2-\alpha_{\mathrm{w}}}}{\alpha_{\mathrm{w}}}. \tag{6.35}$$

Before computing the deformed large-N expansion of the free energy, we should discuss the bounds on the validity of the solution (6.35). A first bound comes from the fact that the initial condition we imposed so far holds in the weak-coupling phase, i.e. when $\alpha < \pi^2$ for the undeformed theory. Therefore, this initial condition can only be propagated in the region of parameters covered by characteristics that cross the $\tau = 0$ axis in the interval $\alpha \in (0,\pi^2)$. In other words, the characteristic

$$\alpha = \left[\pi + \tau\left(\frac{\pi}{12} - \frac{1}{\pi}\right)\right]^2 \tag{6.36}$$

[4] We have neglected inessential constant terms contributing to subleading orders in $1/N$.

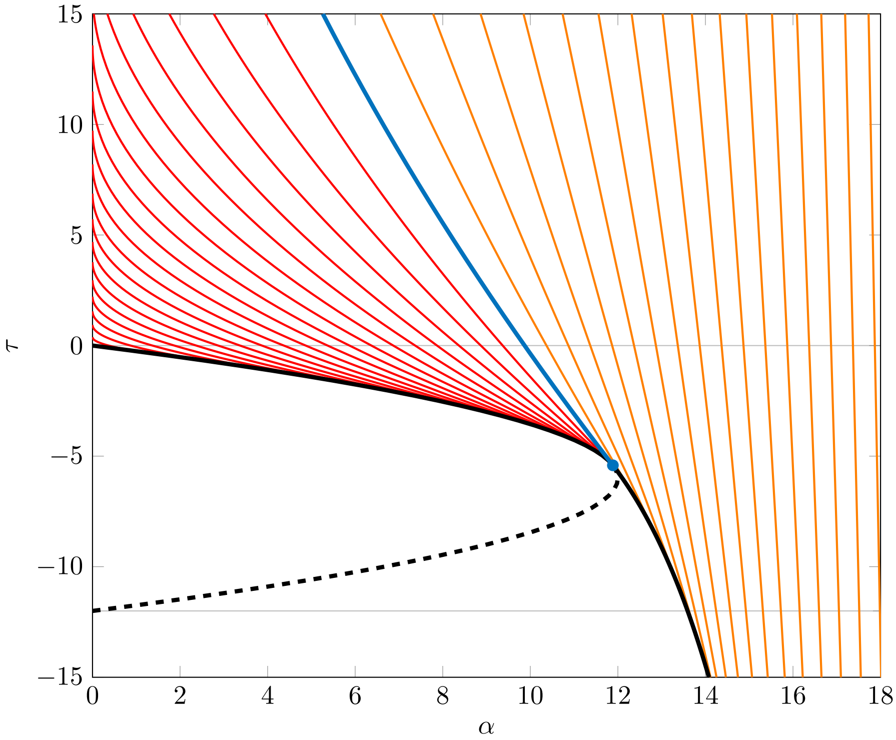


Figure 6.2: The diagram shows the system of characteristics associated with both the weak-coupling phase (red lines) and the strong-coupling phase (orange lines). The blue line is the characteristic that acts as a critical line between the two phases and crosses the $\tau = 0$ axis at $\alpha = \pi^2$. The critical line ends on the multicritical point $(\alpha_{\mathrm{mcp}}, \tau_{\mathrm{mcp}})$. The black parabola delimiting the weak-coupling phase is the envelope of the weak-coupling characteristics and corresponds to $\alpha = \alpha_{\mathrm{w}}$. The black line delimiting the strong-coupling phase is the envelope of the strong-coupling characteristics and has coordinates $(\alpha_{\mathrm{s}}, \tau_{\mathrm{s}})$.

represents a bound for the validity of the solution of the Burgers' equation (6.19) at weak coupling. We see that for $\tau > \tau_{\mathrm{max}}$, where

$$\tau_{\mathrm{max}} = \frac{12\pi^2}{12 - \pi^2}\,, \tag{6.37}$$

the theory is always in the strong-coupling phase for any value of α.

Furthermore, we notice that ξ and, as a consequence, $\mathcal{E}$ are real for $\alpha \geq \alpha_{\mathrm{w}}$. The set of points where the last inequality saturates is the envelope of the system of characteristics (6.35). This means that the parabola

$$\alpha - \alpha_{\mathrm{w}} = 0 \tag{6.38}$$

represents another bound for (6.19) at weak coupling. In the next section, we will see what the origin of said bound is and how to make sense of the deformed Yang–Mills partition function beyond the envelope.

As can be seen in Figure 6.2, the two parabolas (6.36) and (6.38) are tangent at the multicritical point

$$\alpha_{\mathrm{mcp}} = \left(\frac{24\pi}{12+\pi^2}\right)^2\,, \qquad \tau_{\mathrm{mcp}} = -\frac{12\pi^2}{12+\pi^2}\,. \tag{6.39}$$

We can now apply (6.26) and (6.27) to recursively generate any term in the large-N expansion of the deformed free energy. The first few terms read

$$F_0(\alpha,\tau) = \frac{3}{4} + \frac{\tilde{\alpha}}{24} - \frac{\log\tilde{\alpha}}{2} + \frac{\tau(12-\tilde{\alpha})^2}{288\tilde{\alpha}}\,, \tag{6.40}$$

$$F_1(\alpha,\tau) = -\frac{\tilde{\alpha}}{24} - \frac{1}{4}\log\left(1-\frac{\alpha_{\rm w}}{\alpha}\right)\,, \tag{6.41}$$

$$F_2(\alpha,\tau) = \frac{\tau^2}{72}\left(\frac{\tilde{\alpha}^2}{\tilde{\alpha}\alpha_{\rm w}-4\tau^2} + \frac{12\tilde{\alpha}\tau(5\tau-36)}{(\tilde{\alpha}\alpha_{\rm w}-4\tau^2)^2} + \frac{48\tau^2(21\tilde{\alpha}\alpha_{\rm w}-4\tau^2)}{(\tilde{\alpha}\alpha_{\rm w}-4\tau^2)^3} - \frac{108}{\tilde{\alpha}\alpha_{\rm w}+4\tau^2}\right)\,, \tag{6.42}$$

where

$$\tilde{\alpha} = \alpha\,\frac{1-\alpha_{\rm w}/2\alpha+\sqrt{1-\alpha_{\rm w}/\alpha}}{2(1+\tau/12)^2}\,. \tag{6.43}$$

Notice that the critical line $\tilde{\alpha}=\pi^2$ is not only a characteristic for the weak phase but also for the strong phase. This is simply due to the fact that $\mathcal{E}(z,0)$ is a continuous function at $z=\pi$, since the transition is of the third order. Therefore, the line of equation

$$z = \pi + \tau\varphi(\pi) \tag{6.44}$$

is a characteristic shared by both phases. We will discuss the transition in more detail in the next section.

We start now to explore the deformation of the original strong-coupling phase. The relevant characteristics can be obtained in implicit form. From (2.151), (2.152), and (2.153) we find

$$\varphi(\xi) = \frac{\xi}{12} - \frac{8(k+1)K^2}{3\xi^3} - \frac{16(k+1)^2K^4}{3\xi^5}\,, \tag{6.45}$$

where

$$\xi^2 = 4K(2E+(k-1)\,K)\,. \tag{6.46}$$

The corresponding curves are plotted in Figure 6.2. We see that, as it happens for the weak-coupling phase, the strong-coupling characteristics have an envelope for some range of negative values of τ. To find such a curve, we need to solve $1+\tau\dot{\varphi}(\xi)=0$ on the solutions of the characteristics equation (6.20). This is easily done in parametric form. In fact, while it is not possible to invert (6.46) in closed form, one can still use it to obtain $k'(\xi)$ as a function of k itself obtaining

$$\begin{aligned} \alpha_{\rm s} &= \left(\frac{384(k-1)^2K^4\xi}{\xi^6-32(k+1)K^2\xi^2+320(k-1)^2K^4}\right)^2\,, \\ \tau_{\rm s} &= -\frac{12\xi^6}{\xi^6-32(k+1)K^2\xi^2+320(k-1)^2K^4}\,, \end{aligned} \tag{6.47}$$

in terms of the parameter $k\in[0,1)$. The range corresponds to $\xi\in[\pi,\infty)$. The envelope has one extremum, namely the point at $k=0$, that coincides with the multicritical point (6.39): it connects nicely with the envelope for the weak-coupling characteristics, $\alpha=\alpha_{\rm c}$, thus creating a continuous line. It is also tangent to both the envelope for the weak-coupling characteristics and the critical line for the Douglas–Kazakov phase transition, as the three curves all share the same derivative at the multicritical point

$$\left.\frac{\mathrm{d}\alpha}{\mathrm{d}\tau}\right|_{\mathrm{mcp}} = 4 - \frac{96}{12+\pi^2}. \tag{6.48}$$

At large α, as already remarked in (2.155), the leading order of the undeformed free energy scales exponentially as $F_0(\alpha,0) \sim 2e^{-\alpha/2}$. Because

$$\mathcal{E}(z,0) = -z\,G_0(z,0) \sim -2z\,e^{-z^2/2} \tag{6.49}$$

is exponentially suppressed in z, the characteristic equation (6.20) reads $z \sim \xi$ for large z. In other words, as z increases, the $\mathcal{E}$ dependence on τ gets weaker and weaker, and the characteristics become, essentially, vertical lines in Figure 6.2.

In this regime, it is convenient to solve the characteristic equation by expanding ξ as a power series in τ and then by fixing the coefficients of the expansion order by order. The solution,

$$\begin{aligned}\xi = {} & z + 2z\tau\, e^{-z^2/2} + (z^5\tau - 4z^3\tau^2 - 6z^3\tau + 4z\tau^2 + 2z\tau)\, e^{-z^2} \\ & + (z^9\tau - 6z^7\tau^2 - 32z^7\tau/3 + 12z^5\tau^3 + 48z^5\tau^2 + 28z^5\tau - 28z^3\tau^3 - 60z^3\tau^2 \\ & - 16z^3\tau + 8z\tau^3 + 8z\tau^2 + 8z\tau)\, e^{-3z^2/2} + \dots\,,\end{aligned} \tag{6.50}$$

gives, in turn, a power-series expression for $\mathcal{E}(z,\tau)$. Upon integration, we find

$$\begin{aligned}F_0(\alpha,\tau) = {} & 2e^{-\alpha/2} + (\alpha^2/2 - 2\alpha - 1 - 2\alpha\tau)\, e^{-\alpha} \\ & + (\alpha^4/3 - 8\alpha^3/3 + 4\alpha^2 + 8/3 - 2\alpha^3\tau + 12\alpha^2\tau - 4\alpha\tau + 4\alpha^2\tau^2 - 4\alpha\tau^2)\, e^{-3\alpha/2} \\ & + \dots\,,\end{aligned} \tag{6.51}$$

which is the τ-deformed version of (2.155). Some comments are now in order to interpret the above result. The undeformed expression captures the Gross–Taylor string theory on the genus-zero target space [65], the leading order corresponding to connected covering maps of the type $S^2 \to S^2$. The exponential terms of the form $e^{-n\alpha/2}$ represent the contributions of coverings of degree n, while the associated polynomials are obtained by integrating over the positions of various types of singularities [66, 65]. In this regime, the τ deformation affects the polynomial part and acts as a perturbation of the original string expansion: one could conjecture that the deformation provides a refinement for the maps contributing to the string theory, similarly to the generalization induced by higher Casimirs [45, 29], but a precise interpretation of the new terms and their geometrical meaning are beyond the scope of the present paper.

6.3 The deformed Douglas–Kazakov phase transition

In the last section, we have seen that the deformed theory exhibits the critical line (6.36), separating the weak-coupling phase from the strong-coupling phase, which is the continuation of the Douglas–Kazakov critical point of the undeformed theory.[5] The associated phase transition

[5] In [116], the same phase transition was studied by considering the matrix-model of [35] with a τ-deformed potential.

remains of the third order. In fact, from (2.154) we see that near the critical line

$$\Delta\mathcal{E}(z,0) = 2z\,\partial_\alpha \Delta F_0(\alpha,0)\Big|_{\alpha=z^2} \tag{6.52}$$

$$= -\frac{z}{\pi^6}\,(z^2-\pi^2)^2 + O((z^2-\pi^2)^3)\,. \tag{6.53}$$

Let us consider the second derivative

$$\partial_z^2 \Delta\mathcal{E} = \Delta\ddot{\varphi}(\xi)\,(\partial_z\xi)^2 + \Delta\dot{\varphi}(\xi)\,\partial_z^2\xi\,, \tag{6.54}$$

and evaluate it on the characteristic with $\xi=\pi$. The second term vanishes since $\Delta\dot{\varphi}(\pi)=0$, and we are left with

$$\partial_z^2\Delta\mathcal{E}(z(\pi,\tau)) = \Delta\ddot{\varphi}(\pi)\,(1+\tau\dot{\varphi}(\pi))^{-2}$$

$$= -\frac{16}{\pi^3}\left(\frac{\tau_{\text{mcp}}}{\tau-\tau_{\text{mcp}}}\right)^2. \tag{6.55}$$

The discontinuity of the third derivative of the free energy on the critical line is easily obtained as

$$\operatorname{Disc}\partial_\alpha^3 F_0 = \frac{2}{\pi^6}\left(\frac{\tau_{\text{mcp}}}{\tau-\tau_{\text{mcp}}}\right)^2\left(\frac{\tau_{\text{max}}}{\tau-\tau_{\text{max}}}\right)^3, \tag{6.56}$$

which generalizes the undeformed result in (2.154). This expression diverges at both τ_{mcp} and τ_{max}, i.e. as one approaches both the multicritical point (6.39) and the limit value after which the theory is in the strong phase for any α.

As mentioned in the review 2.2.4, the Douglas–Kazakov phase transition of the undeformed theory is driven by instantons, and this fact was argued in [64] by computing the ratio (2.149). We now want to show that this property still holds in the deformed theory.

The first step is to obtain a convenient representation for the instanton contributions, suitable to compute the large-N limit of the relevant ratio. We found useful to express the Tricomi confluent hypergeometric through the following integral representation, that holds for $\operatorname{Re}a>0$ and $\operatorname{Re}z>0$,

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{d}t\,e^{-zt}\,t^{a-1}\,(t+1)^{b-a-1}\,. \tag{6.57}$$

The instanton partition function (6.11) can be recast as

$$z_{\mathfrak{m}}(\alpha,\tau) = C_N\,e^X\sum_{s=0}^\infty \frac{(-1)^s\,p_{\mathfrak{m},s}}{s!\,\Gamma(N^2/2+s)}\int_0^\infty \mathrm{d}t\,\frac{e^{-tW}}{t(t+1)}\left(\frac{tY}{t+1}\right)^{N^2/2+s}. \tag{6.58}$$

To evaluate its large-N limit, we use the Stirling approximation

$$\frac{C_N}{\Gamma(N^2/2+s)} \sim e^{\frac{5}{4}N^2}\left(\frac{2}{N^2}\right)^{N^2/2+s}, \tag{6.59}$$

and define

$$X \sim N^2 x\,, \qquad x = \frac{\alpha}{2(12+\tau)}\,, \tag{6.60}$$

$$Y \sim N^2 y\,, \qquad y = \frac{12+\tau}{24\tau}\,, \tag{6.61}$$

$$W \sim N^2 w\,, \qquad w = \frac{6\alpha}{\tau(12+\tau)}\,. \tag{6.62}$$

We then change the integration variable with

$$\frac{1}{u} = \frac{2ty}{t+1} \tag{6.63}$$

and by using (6.13) we obtain

$$\mathcal{z}_{\mathfrak{m}}(\alpha,\tau) \sim \int_{\frac{1}{2y}}^{\infty} \frac{\mathrm{d}u}{u}\, e^{-N^2\rho(u)}\, \frac{\mathcal{z}_{\mathfrak{m}}(u,0)}{\mathcal{z}_{\mathbf{0}}(u,0)}\,, \tag{6.64}$$

where

$$\rho(u) = \frac{w}{2uy-1} - x - \frac{5}{4} + \frac{1}{2}\log u\,. \tag{6.65}$$

The function $\rho(u)$ has minimum in $u = \tilde{\alpha}$. Moreover, this saddle point always falls within the integration range since $\tilde{\alpha} > 1/(2y)$ for $\tau \geq 0$.

As expected, the zero-flux sector (6.64) reproduces the result of the large-N leading order at weak coupling computed in (6.40). Namely, $\rho(\tilde{\alpha}) = -F_0(\alpha,\tau)$, so that

$$\mathcal{z}_{\mathbf{0}}(\alpha,\tau) \sim e^{N^2 F_0(\alpha,\tau)}\,. \tag{6.66}$$

For a generic $\mathfrak{m}$, we assume that in the large-N limit, the sum is always subleading with respect to the exponential, i.e. that the sum does not contribute to the fluctuations about the saddle. Under this assumption, one finds

$$\frac{\mathcal{z}_{\mathfrak{m}}(\alpha,\tau)}{\mathcal{z}_{\mathbf{0}}(\alpha,\tau)} \sim \frac{\mathcal{z}_{\mathfrak{m}}(\tilde{\alpha},0)}{\mathcal{z}_{\mathbf{0}}(\tilde{\alpha},0)}\,. \tag{6.67}$$

In other words, the deformed ratio coincides with the undeformed one upon replacing α with $\tilde{\alpha}$.[6]

Let us see how this works concretely in the case when $\mathfrak{m} = \mathbf{1}$. From (6.12), one can compute the coefficients for the one-flux sector, which turns out to be

$$p_{\mathbf{1},s} = (-1)^{N-1}(2\pi^2 N)^s\, {}_2F_1(-s,1-N;2;2)\,. \tag{6.68}$$

Then, the associated sum can be performed exactly by using the identity

$$\begin{aligned}\sum_{s=0}^{\infty} \frac{\xi^s}{s!}\, {}_2F_1(-s,a;b;x) &= e^{\xi}\, {}_1F_1(a;b;-\xi x) \\ &= e^{\xi}\, \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)}\, L_{-a}^{b-1}(-\xi x)\,,\end{aligned} \tag{6.69}$$

[6] An analogous result for $\mathfrak{m} = \mathbf{1}$ was obtained in [116] through a different approach.

and as a result, we find

$$\frac{z_1}{z_0} \sim (-1)^{N-1} e^{-2\pi^2 N/\tilde{\alpha}} L^1_{N-1}(4\pi^2 N/\tilde{\alpha}) , \tag{6.70}$$

which is the expected result. Therefore, we can simply invoke the argument of [64] and conclude that the partition function of the unit-flux sector is no longer suppressed in the large-N limit for $\tilde{\alpha} \geq \pi^2$. This confirms the observation of Section 6.2, where we obtained the same condition for the transition to the strong-coupling phase of the deformed theory. We remark that the above picture is a smooth deformation of the undeformed case. The nonperturbative contributions driving the transition are still instantons labeled by the quantized magnetic flux vector.

6.4 Envelopes and nonperturbative corrections

A puzzling feature of the phase diagram in Figure 6.2 is the emergence of an envelope of characteristics in both the weak-coupling and the strong-coupling phase. This phenomenon is similar to the emergence of a Douglas–Kazakov phase transition in that instantons drive both. However, while the Douglas–Kazakov transition is due to instantons in the effective 't Hooft coupling α, the novel phase transition is due to instantons in the deformation parameter τ.

To show this, let us first focus on the envelope at $\alpha = \alpha_{\mathrm{w}}$. In this case, the analysis is more straightforward since the envelope sits at the boundary of the weak-coupling phase where the zero-flux sector completely dominates the dynamics. We will show that, when N is large, the nonperturbative corrections in τ, typical of the deformation with $\tau < 0$, are suppressed only for $\alpha > \alpha_{\mathrm{w}}$.

Again, we use an integral representation for the Kummer confluent hypergeometric function to conveniently express our zero-flux partition function. For $\mathrm{Re}\, a > 0$,

$${}_1F_1(a;b;z) = \frac{1}{2\pi \mathrm{i}} \frac{\Gamma(b)\,\Gamma(a-b+1)}{\Gamma(a)} \int_0^{(1+)} \mathrm{d}t\, e^{zt}\, t^{a-1} (t-1)^{b-a-1} , \tag{6.71}$$

where the integral is taken over a contour starting and ending in 0 and encircling 1 in the positive sense. Armed with the above representation, we recast $z_0(\alpha,\tau)$ at finite N and $\tau < 0$ in terms of a contour integral and study its large-N limit using a saddle-point approximation. Starting from (6.11) and (6.12), we find

$$z_0(\alpha,\tau) = -\frac{\mathrm{i} C_N}{2\,\Gamma(N^2/2+1)} W e^X (-Y)^{N^2/2} \oint_\gamma \mathrm{d}s\, e^{Ws} \left(\frac{s}{s-1}\right)^{N^2/2} . \tag{6.72}$$

The choice of contour γ is shown in Figure 6.3. When N is large, we can write[7]

$$z_0(\alpha,\tau) \sim \frac{\mathrm{i} N e^{\frac{5}{4}N^2}}{\sqrt{\pi}} \frac{\alpha}{\alpha_{\mathrm{w}}} e^{-N^2 \frac{\tau}{6} \frac{\alpha}{\alpha_{\mathrm{w}}}} \left(-\frac{12+\tau}{12\tau}\right)^{N^2/2} \oint_\gamma \mathrm{d}s\, e^{-N^2 \phi(s)} , \tag{6.73}$$

where

$$\phi(s) = 2\frac{\alpha}{\alpha_{\mathrm{w}}} s - \frac{1}{2} \log \frac{s}{s-1} . \tag{6.74}$$

[7] For simplicity, we discard an irrelevant overall constant.

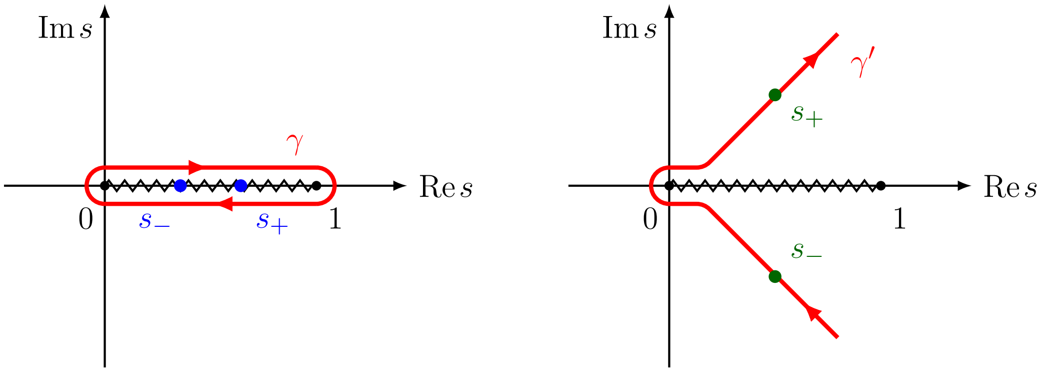


Figure 6.3: On the left, the integration contour for the original integral in (6.72). This choice is particularly convenient for $\alpha > \alpha_{\rm w}$, where both saddles are real (blue dots). On the right, the deformed contour is associated with the steepest descent approximation for $\alpha < \alpha_{\rm w}$, where both saddles are complex (green dots).

As $N \to \infty$ the integral will be dominated by the stationary points of $\phi(s)$,

$$s_\pm = \begin{cases} \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{\alpha_{\rm w}}{\alpha}}\right) & \text{for } \alpha > \alpha_{\rm w} \ , \\ \frac{1}{2}\left(1 \pm {\rm i}\sqrt{\frac{\alpha_{\rm w}}{\alpha} - 1}\right) & \text{for } \alpha < \alpha_{\rm w} \ . \end{cases} \tag{6.75}$$

For $\tau = 0$, the stationary points are at the endpoints $s = 0$ and $s = 1$ of the branch cut of $\phi(s)$. As τ decreases, these move towards $s = 1/2$ where they collide for $\alpha_{\rm w} = \alpha$. As τ decreases further, the stationary points acquire an opposite nonvanishing imaginary part and move away from the real axis.

When evaluated on the critical points, the second derivative of $\phi(s)$ reads

$$\phi''(s_\pm) = \begin{cases} \mp 8\, \frac{\alpha^2}{\alpha_{\rm w}^2} \sqrt{1 - \frac{\alpha_{\rm w}}{\alpha}} & \text{for } \alpha > \alpha_{\rm w} \ , \\ \mp 8{\rm i} \frac{\alpha^2}{\alpha_{\rm w}^2} \sqrt{\frac{\alpha_{\rm w}}{\alpha} - 1} & \text{for } \alpha < \alpha_{\rm w} \ . \end{cases} \tag{6.76}$$

When $\alpha > \alpha_{\rm w}$ and both $s_\pm$ sit on the real axis, $\phi''(s_+) < 0$, while $\phi''(s_-) > 0$. In other words, when applying the Laplace approximation method, we only consider the contribution coming from s_-, which is a minimum for $\phi(s)$ and corresponds to the perturbative saddle. On the other hand, the contribution coming from the nonperturbative saddle s_+ is suppressed. We already know what the large-N asymptotics in this regime is, as it is the result computed in Section 6.2.

The coalescence of the two saddle points is responsible for a critical behavior: when $\alpha < \alpha_{\rm w}$, the nonperturbative saddle is no longer suppressed and needs to be considered. The function $\phi(s)$ has the same real part when evaluated on both saddles. Specifically,

$$\phi(s_\pm) = \frac{\alpha}{\alpha_{\rm w}}\left(1 \pm {\rm i}\sqrt{\frac{\alpha_{\rm w}}{\alpha} - 1}\right) \mp {\rm i} \arctan\left(\sqrt{\frac{\alpha_{\rm w}}{\alpha} - 1}\right) \pm {\rm i}\frac{\pi}{2} \ . \tag{6.77}$$

The integral can be conveniently computed by deforming the original contour γ into a γ' that traverses the saddles along the associated steepest-descent path. As shown in Figure 6.3, γ'

crosses the saddle in s_+ with $\arg s = \pi/4$ and the saddle in s_- with $\arg s = 3\pi/4$. This gives the large-N asymptotics

$$\mathcal{Z}_{\mathbf{0}}(\alpha,\tau) \sim \frac{e^{5N^2/4}}{(\alpha_{\rm w}/\alpha-1)^{1/4}}\left(-\frac{12+\tau}{12\tau}\right)^{N^2/2}\exp\left(\frac{N^2\alpha(6+\tau)}{2\tau(12+\tau)}\right) \times\cos\left(N^2\left[\frac{\alpha}{\alpha_{\rm w}}\sqrt{\frac{\alpha_{\rm w}}{\alpha}-1}-\arctan\left(\sqrt{\frac{\alpha_{\rm w}}{\alpha}-1}\right)\right]-\frac{\pi}{4}\right). \tag{6.78}$$

We observe that the above expression does not have a definite sign. In fact, it oscillates rapidly when N is large: it is clear that the full theory cannot be dominated by just the zero-flux sector for $\alpha < \alpha_{\rm w}$. This regime is thus characterized by the presence of nonperturbative terms both in the effective 't Hooft coupling α and in the rescaled deformation parameter τ. We will denote this region of the phase diagram as the *mixed phase*.[8]

We can say more: when the system is in the weak-coupling phase and approaches the critical line at $\alpha = \alpha_{\rm c}$, we argue that it exhibits a behavior typical of a system in an ordered phase approaching a second-order phase transition. To this aim, in the following we loosely identify α as an inverse temperature. We can compute the "specific heat"

$$\begin{aligned} C &= \alpha^2\partial_\alpha^2 F_0(\alpha,\tau) \\ &= \frac{1}{2\sqrt{1-\alpha_{\rm c}/\alpha}}\,, \end{aligned} \tag{6.79}$$

to find that the associated critical exponent is 1/2.

This behavior is not specific to the weak-coupling phase, but rather it is typical of any envelope of characteristics of the Burgers equation (6.19). In fact, if we write C in the language of Section 6.2, we find

$$\begin{aligned} C &= \frac{z}{4}\Big(z\,\partial_z\mathcal{E}(z,\tau)-\mathcal{E}(z,\tau)\Big) \\ &= \frac{z}{4}\Big(z\,\dot\varphi(\xi(z,\tau))\,\partial_z\xi(z,\tau)-\varphi(\xi(z,\tau))\Big)\,. \end{aligned} \tag{6.80}$$

As mentioned in Section 6.2, the condition leading to an envelope of characteristics is that

$$\frac{1}{\partial_z\xi(z,\tau)} = 1+\tau\,\dot\varphi(\xi(z,\tau)) \tag{6.81}$$

should vanish as z approaches the critical value $z_{\rm c}$. As a consequence, on every envelope of characteristics, C diverges. Although its derivative is singular, ξ is finite on the envelope. We are therefore led to the ansatz

$$\xi(z,\tau) = \xi(z_{\rm c},\tau)+\xi_1(\tau)\,(z-z_{\rm c})^\gamma+\dots \tag{6.82}$$

where $0<\gamma<1$ and the dots represent subleading terms. Now we use (6.81) to fix the leading power in $z-z_{\rm c}$ and the associated coefficient. In particular, from

$$\frac{(z-z_{\rm c})^{1-\gamma}}{\gamma\,\xi_1(\tau)}+\dots = \tau\,\ddot\varphi(\xi(z_{\rm c},\tau))\,\xi_1(\tau)\,(z-z_{\rm c})^\gamma+\dots\,, \tag{6.83}$$

[8] The phase diagram at $\tau < 0$ and the role of the nonperturbative saddles was studied in [48], although the results therein do not quite agree with our findings.

we deduce that

$$\partial_z \xi(z,\tau) \sim \frac{(z-z_{\rm c})^{-1/2}}{\sqrt{2\tau\,\ddot{\varphi}(\xi(z_{\rm c},\tau))}}\,. \tag{6.84}$$

This, in turn, leads to

$$C \sim -\frac{z_{\rm c}^2}{4\tau\sqrt{2\tau\,\ddot{\varphi}(\xi(z_{\rm c},\tau))}}\,(z-z_{\rm c})^{-1/2}\,, \tag{6.85}$$

which, once again, gives $C \sim (\alpha-\alpha_{\rm c})^{-1/2}$. What we have just proven can be checked to reproduce exactly the result in the weak-coupling phase if we identify $\alpha_{\rm w} = z_{\rm c}^2$, but it applies also to the envelope associated with the strong-coupling characteristics upon identifying $\alpha_{\rm s} = z_{\rm c}^2$.[9] We conclude that the black envelope line in Figure 6.2 at the boundary of the mixed phase can thus be thought of as a single continuous critical line associated with a second order phase transition with critical exponent 1/2.

[9] We can either regard $\alpha_{\rm s}$ as a function of τ or assume that both $z_{\rm c}$ and τ are functions of k as in (6.47).

7 Conclusions and outlook

In this chapter, we finally draw the conclusions for this dissertation. For each of main contents studied, we summarize the most remarkable results achieved and we point out some possibile generalizations and outlooks. Section 7.1 is devoted to the perturbative expansions of correlators in JT gravity, Section 7.2 to JT gravity at finite cutoff, while Section 7.3 deals with $T\bar{T}$-deformed Yang-Mills theory. We finally add a Section 7.4 where we outline some related ongoing projects and directions we are going to pursue in the near future.

7.1 Wilson line perturbative expansion

In 3 we have considered JT bi-local correlators of operators with positive weight λ, on the disk and the trumpet topologies [1] . The perturbative series associated to these correlation functions is harder to obtain than in the parent case $\lambda \in -\mathbb{N}/2$, that was studied in [90] in the zero temperature limit. We have been able nevertheless to distill some aspects of the κ expansion of the two-point function.

We started from the exact non-perturbative expression for a boundary anchored Wilson line (2.189), that we derived in 2.2.6 in the context of first order gauge reformulation of gravity as a BF theory. In 2.2.6 we have also proven this is equivalent to a bilocal two-point function for an operator of weight λ in the Schwarzian theory. The semiclassical leading term and first order in the gravitational perturbative expansion (2.71) for this bilocal correlator were already known in the literature [86, 117] and in 2.1.3 we have reviewed how to obtain it for the disk topology, while in 3.1 we extended the perturbative analysis to the trumpet's less trivial case. We have then been able to check the agreement of the exact expression with the Schwarzian perturbation theory, at least for the known orders, for any value of β.

Moreover, for the particular case of $\lambda \in \mathbb{N}/2$, we derived an all-order formula for the perturbative contributions, that becomes particularly handful in the limit of infinite β. We have also shown that the exact expression for our bi-local correlators is closely related to the Mordell integral, a basic constituent in the theory of Mock-modular forms [23].

There are some lessons that we can draw from our computations and some directions that could be worth to study further. A feature that we may explore from the knowledge of the entire perturbative series is, for example, the nature of possible non-perturbative contributions to the full answer. For instance, let us consider the coefficient c_r in the case $\beta \to \infty$. We can easily read it from (3.66):

$$c_r = \frac{(2\lambda)!\,(2r-4\lambda-1)!!}{2^{2\lambda}}\frac{2^r\tau^{r-2\lambda}}{(2r)!}\left[\left(\sum_{k=0}^{2\lambda}\binom{2\lambda}{k}^2\mathcal{B}^{(4\lambda)}_{2r}(k+\lambda)\right)\right] \tag{7.1}$$

[1] Chapter 3 presents the results achieved in [59].

To understand its behavior for large value of r we need to know the behavior of the generalized Bernoulli polynomials in that limit. This aspect was discussed in detail in [83], where it was found that the dominant contribution is

$$\mathcal{B}^m_{2r}(z) \simeq -(2r)!\left[\beta^m_1 \frac{e^{2\pi i z}}{(2\pi i)^{2r}} + \beta^m_{-1}\frac{e^{-2\pi i z}}{(-2\pi i)^{2r}}\right] \tag{7.2}$$

where $\beta^m_k(n,z) \simeq \dfrac{(-1)^{m-1} n^{m-1}}{(m-1)!}$. At leading order these coefficients are independent of k and (7.2) collapses to

$$\mathcal{B}^m_{2r}(z) \simeq \frac{(2r)!\,(-1)^{m+r}\,(2r)^{m-1}\,2}{(2\pi)^{2r}\,(m-1)!}\cos(2\pi z)\,. \tag{7.3}$$

If we choose $m = 4\lambda$ the coefficient c_r for $r \to \infty$ takes the form

$$\begin{aligned} c_r \simeq& \frac{(2\lambda)!\,(2r-4\lambda-1)!!}{2^{2\lambda}}\frac{2^r\tau^{r-2\lambda}}{(2r)!}\frac{(2r)!\,(-1)^{4\lambda+r}\,(2r)^{4\lambda-1}\,2}{(2\pi)^{2r}\,(4\lambda-1)!}(-1)^{2\lambda}\sum_{k=0}^{2\lambda}\binom{2\lambda}{k}^2 = \\ =& \frac{(2\lambda)!\,(2r-4\lambda-1)!!}{2^{2\lambda}}\frac{2^r\tau^{r-2\lambda}}{(2r)!}\frac{(2r)!\,(-1)^{4\lambda+r}\,(2r)^{4\lambda-1}\,2}{(2\pi)^{2r}\,(4\lambda-1)!}(-1)^{2\lambda}\frac{4\lambda!}{(2\lambda!)^2}, \end{aligned} \tag{7.4}$$

where we have performed the sum over the square of the binomial coefficients. We can now easily complete our large $r-$expansion with the help of the Stirling formula. After some tedious algebra we find

$$c_r = \frac{4\lambda\,(-1)^{6\lambda}\,\tau^{r-2\lambda}e^{2\lambda}}{\sqrt{\pi}\,(2\lambda)!}\;\frac{(-1)^r\,r^{2\lambda-\frac{3}{2}}r!}{\pi^{2r}} \tag{7.5}$$

The coefficient grows as a power times $r!$ and its global sign alternates with the parity of r. Thus the perturbative series appears to be Borel-summable: in fact the leading pole appearing in the Borel-transform is located on the negative axis and thus one could argue that non-perturbative instanton-like configurations should not play any role here. On the one hand, there is no guarantee that the Borel resummation of a Borel summable series reconstructs the non-perturbative answer. There are sufficient conditions for this to be the case, which typically require strong analyticity conditions on the underlying non-perturbative function. On the other hand, in most of the examples of Borel summable series in quantum theories, Borel resummation does reconstruct the correct answer (see [50] for a lucid discussion of these topics). The actual determination of non-perturbative configurations, if any, remains therefore an important issue for future investigations, as well as to understand their possible physical meaning with respect to the boundary gravitons appearing in Schwarzian perturbation theory. This aspect would be really interesting also to possibly detect the contributions of higher topologies, that, as we know from (2.73), should indeed bring nonperturbative contributions in the gravitational coupling constant.

The obvious extension of the present work would consist in studying the perturbative series associated to general four-point correlators of bi-local operators. While the exact form on the disk and the trumpet is well known [95, 74], much less has been learned on its perturbative incarnation, due to the appearing in the out-of-time-ordered case of a very complicated vertex function inside the integrals. The relevant 6-j symbols involved there can be expressed through Wilson function and, in principle, one could try to perform an expansion using the analytical structure of the full amplitude. The success of such computation would certainly improve our understanding of the properties of the associated gravitational S-matrix.

Another generalization of our investigations would concern the perturbative aspects of two-point functions in presence of defects [92]. The trumpet correlators studied here are just a particular example within this class, being associated to a bi-local operator with the insertion of a hyperbolic defect in the bulk and computed without taking into account the winding sectors [92]. It could be interesting to extend our analysis to the winding case and to consider elliptic and parabolic defects too. The fate and the physics of bi-local correlators in presence of multiple defects [88] or for deformed JT gravity [132] could be also explored. It would be nice also to understand the character of perturbative contributions to bi-local correlators from boundary fluctuations in higher-genus geometry [10].

Another possible direction of work is to verify the $N = 1$ first subleading correction to the two-point function; in fact both the exact results and the perturbative leading order answer are known and contained in [90].

Finally we point out that the correlators studied here were obtained in [93] from boundary correlators of minimal Liouville string, exploiting a particular double-scaling limit. It would be interesting to see if the Mordell structure, underlying the exact form of the bi-local correlator on the disk, could be understood from a Liouville perspective.

7.2 JT gravity at finite cutoff

In 4 we have studied JT gravity as a model for holography at finite volume, considering the nonperturbative contributions coming from the cutoff scale. We assumed the conjectured holographic duality between a $T\bar{T}$-deformed theory and gravity in a finite patch of AdS space. In the case of JT gravity, this amounts to consider the Schwarzian theory deformed by a one-dimensional analog of the $T\bar{T}$ deformation, a mechanism that we reviewed in 2.3.3.

The same approach has been advocated in [73] where, among other things, crucial nonperturbative aspects arising from the radial cutoff have been explored. We investigated the appearance of exponentially-suppressed terms in the disk and trumpet partition functions through Borel resummation and resurgence, finding a nice relation between these terms and the analytic structure of the perturbative series in the deformation parameter. We then applied our findings to compute results for arbitrary topologies, exploiting the gluing procedure of [113], which we introduced in 2.1.4. The construction results in a consistent deformation of the Eynard–Orantin topological recursion relations, that we reviewed in 2.1.5 for the undeformed case, although we have not attempted to give a physical interpretation to the emerging deformed spectral density.

There are a certain number of open questions arising from our studies that could stimulate further investigations. In our opinion, a crucial one concerns the physical realization of the deformed holographic dual. As already stressed, the deformed spectral density associated with the topological recursion is not positive definite, and it certainly cannot originate from an ordinary (double-scaled) matrix model. It would be interesting to explore the resolution of this problem in a wider perspective: for example, naïve non-positive-definite spectral densities show up in super-matrix models. Actually, in the context of JT gravity, one could consider the supersymmetric version of the bulk theory [22, 44]. In this case, the holographic dual is a supersymmetric Schwarzian quantum mechanics: its disk partition function provides nevertheless a positive spectral density that allows considering a higher-genus completion, whose interpretation is given in terms of different matrix ensembles [120, 133]. The $T\bar{T}$ deformation (or some related integrable deformation) could arise from a radial cutoff in the bulk theory even for the supersymmetric JT gravity: we expect that exploring this direction would undoubtedly improve the understanding of the present construction and maybe provide a link with the matrix model approach proposed in [109]. A somewhat similar positivity problem appears in the context of JT gravity with defects [88] and was solved by an appropriate sum over quantum configurations, leading to a nontrivial modification of the theory [78].

Another research direction worth to be taken into account concerns a full first-principles derivation of the path integral for the bulk theory at finite cutoff. Besides providing a solid foundation for the gluing procedure used in Section 4.3, this would unambiguously fix the integration measure, which in [73] is assumed to be unaltered by the deformation. Such a choice is in sharp contrast with the approach of [121], where instead, a significant role was played by a particular class of paths in the nonperturbative regime, leading to very different results. In order to gain a better understanding of this discrepancy, one would perform a bulk path-integral calculation, taking properly into account configurations with non-constant Schwarzian action [100]: in principle, a semiclassical calculation could elucidate the contribution of such configurations to the path-integral measure and indicate other nonperturbative effects. This is the goal of a paper in preparation [104].

It would also be interesting to understand the behavior of the bulk theory at finite cutoff in the presence of matter: for example, the existence of a U(1) chiral current could provide a deformation of the Schwarzian quantum mechanics analogous to the $J\bar{T}$ deformation in two-dimensional CFTs [67]. In [21], such a deformation has been proposed leading, for some choice of the parameters, to a positive-definite spectral density.

As a final remark, we observe that the spectral form factor in the $T\bar{T}$-deformed theory shows some differences with respect to the original JT case, although certain universal aspects remain present. In particular, the appearance of an oscillating term in the ramp regime hints towards a different interpretation of the holographic picture.

7.3 $T\bar{T}$-deformed Yang-Mills theory

In 5 and 6 we have studied the effect of the $T\bar{T}$ deformation on 2d Yang-Mills theory on the sphere, which is exactly solvable with various methods, as we reviewed in the introduction 2.2.1.

The key strategy to make sense of the deformed partition function was to work at the level of the instaton expansion representation 2.2.2. In fact, for both signs of τ, our approach was based on the construction of solutions of the flow equation (5.1) for each flux sector of the theory. Then, since $\mathbf{F}_{\alpha,\tau}$ is linear, one can construct the full deformed partition function as a sum over individual deformed flux sectors $z_{\mathbf{m}}(\alpha,\tau)$ obeying $\mathbf{F}_{\alpha,\tau}\, z_{\mathbf{m}}(\alpha,\tau) = 0$.

There are two main features of the deformed theory that we have been able to address: these are associated with the two different sign choices for the deformation parameter μ. For $\mu > 0$, only a finite number of states in the deformed spectrum can be accessed by solving the relevant flow equation, the rest of the spectrum lying behind a divergence. If one insists on preserving the hierarchy of states of the undeformed theory, one should postulate that an infinite number of energy levels should decouple from the theory. Whitout postulating this feature, we have been able to demonstrate it (5.6), suggesting the truncation of the spectrum is due to a phenomenon of deconstructive interference between different flux sectors in this regime. For $\mu < 0$, to obtain a well-defined partition function (5.7), we have instead proven one has to incorporate instanton-like corrections in μ whose precise form is determined by imposing appropriate physical requirements.

There is ample reason to believe that (5.6) should directly generalize to arbitrary gauge groups and manifolds, with and without boundaries, in analogy with the case of $\mu = 0$. This is necessary in order to preserve the topological composition properties of the undeformed theory. In fact, these rely solely on the orthogonality of characters and are unaffected by the deformation of the Hamiltonian or by the finite range of the sum. The situation is quite different for the partition function (5.7). Specifically, it is less obvious how $\mathcal{R}$ should be modified to account for different groups and topologies. Furthermore, the nonperturbative terms appear to be incompatible with the gluing rules of undeformed Yang–Mills theory, at least in their simplest form. This inconsistency could be interpreted by invoking a breakdown of locality in the $\mu < 0$ regime. It would be interesting to investigate these points further, e.g. on the torus[2]

Furthermore, it is natural to employ our analysis of the flow equation to extend previous results at large N [116, 48]. In [57], we obtain the full $1/N$ expansion of the deformed theory by studying the differential equation governing the deformation of the free energy in the large-N limit. This, in turn, allows us to study the full phase diagram of the deformed theory.

Finally, an obvious line of inquiry concerns applying the ideas presented here outside the realm of gauge theories, for instance in the context of TT-deformed conformal field theories. These also exhibit pathologies involving the upper portion of their spectrum. However, contrary to the Yang–Mills case, these happen for $\mu < 0$ and in the form of a complexification of eigenvalues, as we have experienced for instance in $T\bar{T}$-deformed Schwarzian theory, the dual of finite cutoff JT gravity. While in our case the offending states naturally decouple as their contribution to the partition function vanishes through the associated Boltzmann weight, it is less clear for deformed conformal theories what the fate of the complex energies should be. In [28], it was advocated that these states should be removed from the spectrum, thus leading to a theory with a finite number of degrees of freedom, but a loss of unitarity has also been proposed as a possible interpretation of the phenomenon. Another point of analogy with our results is the presence of instanton-like corrections in μ for $\mu < 0$ [1]. This seems to be a general feature of the $T\bar{T}$ deformation, with the flow equation admitting a nonperturbative branch of solutions for negative deformation parameters. Outside the present work and [54], though, we are not aware of any precise computation in this direction. While we are able to address both points by studying each flux sector, there is no direct analogue of this for a general conformal field theory. Perhaps resorting to a semiclassical analysis as described earlier could provide a way forward.

The large N limit In 6 we have been able to extend the previous analysis to the large N regime of Yang-Mills theory in the presence of $T\bar{T}$ deformation. By studying the flow equation for the free energy at large N and exploiting the previous results at finite N, we have been able to derive the full phase diagram of the theory and explaining its main properties.

In particular, phase diagram of large-N Yang–Mills theory on the sphere displays an intriguing interplay between different types of nonperturbative contributions. In particular, the discovered second-order phase transition sharply deviates from the familiar third-order Douglas–Kazakov transition, signaling a genuine new effect due to the $T\overline{T}$ deformation. A

[2]For the undeformed torus partition function in a generic flux sector, see [51, 52, 60]

natural follow-up of the present investigations would consist in deriving an effective description of the mixed phase: in that region of parameters, we expect a behavior dominated by degrees of freedom quite different both from the Gross–Taylor string and from the perturbative gauge fluctuations, typical of the weak-coupling phase.[3]

The other obvious extension of our work concerns the study of the large-N Yang–Mills theory on the torus. The undeformed theory has been studied from different points of view over the years. In particular, it admits an accurate string description in the Gross–Taylor approach [61, 66], and it is equivalent to a topological string theory on a non-compact toric manifold [127]. It would certainly be interesting to understand how these properties are deformed along the $T\overline{T}$ flow and if a string-theory picture survives after the deformation. The torus topology also offers a possible connection with the well-studied case of $T\overline{T}$-deformed conformal field theories: it is well known that Yang–Mills theory on the torus has a large-N description in terms of an interacting compact boson [33] with subtle modular properties [30, 102]. We expect that the $T\overline{T}$ deformation could be implemented and studied as some nontrivial interaction potential in this effective theory.

Finally, the large-N theory on the sphere has a dual description in terms of a vicious walkers model [49]. The Douglas–Kazakov phase transition has been studied in this context [43]. It would be nice to extend this duality along the $T\overline{T}$ flow, possibly gaining new understanding of the second-order phase transition.

7.4 Other related directions

There are various aspects and directions it is worth deepening about quantum gravity and holography in lower dimensions, expecially in the context of JT gravity. An ongoing project in this direction is the possibility of localizing the JT gravity partition function directly from the bulk theory, not passing through the dual Schwarzian theory on the boundary [119]. Localization is a very powerful tool to reduce the infinite dimensional path integral to a finite dimensional one, given by local contributions around fixed points.

We have explained in 2.2.5 how the exact computation of the JT gravity partition function from the gravity side strongly relies on its rephrasal as a gauge theory and on the $\mathrm{SL}(2,\mathbb{R})$ representation data [74]. However, the link between gravity and fine theoretical structure of the gauge theory is not directly obvious, as explained in [12] and treated in A.7. We are firmly convinced that these aspects could be implemented in an elegant way within a supersymmetric localization argument for the BF formulation of JT gravity, by imposing some suitable boundary conditions on the relevant fields. we have already made some concrete progress in this direction, by finding the correct localizing term and computing the one-loop determinant corresponding to the quadratic fluctuations around the locus; this will be the content of our future paper [53]. The goal is to complete this derivation, possibly extending it to a boundary anchored Wilson line, which in this language should be implemented by adding a chiral hypermultiplet in the bulk. Furthermore, there is a concrete possibility to generalize this construction to higher spin JT gravity [81], by promoting tha gauge group from $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SL}(\mathbb{N},\mathbb{R})$.

Furthermore, it would be illuminating to promote all these fascinating problems to one dimension more, i.e. to three-dimensional gravity, where the gauge reformulation as a double

[3]We already noticed in [55] how the expression for the individual flux sector becomes ill-defined below a certain bound at $\tau_{\min}$.

Chern-Simons is amaenable to localization and exact computations too. Remarkably, recent studies [91] have also proposed how an effective description of three-dimensional gravity, corresponding to the high temperature regime of the dual CFT, could arise from a doubled-version of JT gravity. In this framework, well-known properties of Wilson lines in JT gravity could be exploited to gain information about their three-dimensional uplift. The latter ones are very notable objects [18] because, in the conventional Chern-Simons formulation, they can be constructed to represent the geometry of two-sided black holes and to possibly describe an emergent specetime from entanglement entropy in the bulk.

Furthermore, because of the close analogy between JT gravity and almost-topological gauge theories 2.2.5, another fascinating direction of work consists in studying Yang-Mills theory on general Riemann surfaces of genus g, with possibile insertions of boundaries. Specifically, we have argued in 2.2.2 that the localization computation of 2d Yang-Mills performed by Witten in [130] includes contributions from unstable flux sectors, which account for the nonperturbative sector of the theory. Nevertheless, while the non-exponentially suppressed part of the partition function, accounting for the intersection numbers on the moduli space of flat connections, is precisely computed, the localized contribution from the vicinity of instantons is not worked out explicitly. Instead, its asymptotic dependence on the physical gauge coupling is only inferred on general grounds.

As a first step, we have already performed an exact Poisson resummation of the heat-kernel sum for rank-one gauge groups, which led us to find the precise form of these instanton terms. Our goal for the future is understanding their structure directly from localization: the emerging one-loop determinant around these higher critical points is singular, so a key point would be to interpret it correctly in the approach of non-abelian localization A.2. Thanks to some toy-model example computation, we already have some intuition about what type of regularization is needed for these singular effective actions; these are the contents of the future paper [58] in preparation. Hopefully, since the cohomological multiplet is available in any dimension, this procedure may shed light on localization of higher dimensional Yang-Mills, where the exact result is not known. Finally, a natural extension of this project is the inclusion of Wilson loops, which, in the vanishing area limit, should be interpreted as some point-like operator insertion probing the topological sector of the theory.

Appendix A

A.1 The DH theorem

We use equivariant localization to sketch a proof the DH localization formula, mainly following the notations of [119]. We are given a symplectic manifold M with coordinates x^i and with a Hamiltonian H that generates a $U(1)$ symmetry via Poisson brackets $\delta x^i = v^i = \omega^{ij}\partial_j H$, where ω^{ij} is the inverse of the symplectic form. Our task is to evaluate the integral

$$Z = \int_M \mathrm{d}^n x \; \mathrm{d}^n \psi \; \exp\left[\frac{1}{2}\omega_{ij}\psi^i\psi^j + H(x)\right] = \int \mathrm{d}^n x \; \mathrm{d}^n \psi \; \exp\left[-S(x,\psi)\right] \tag{A.1}$$

Integrating out the ψ's give the usual symplectic measure on the symplectic manifold M. The action in (A.1) enjoys a fermionic symmetry, since it is closed under the action of a fermionic charge $\mathcal{Q}$, i.e. $\mathcal{Q}S(x,\psi)$, which acts on the coordinates as

$$\mathcal{Q}x^i = \psi^i \qquad \mathcal{Q}\psi^i = v^i \tag{A.2}$$

We recognize $\mathcal{Q}$ as the equivariant differential

$$\mathcal{Q} = \mathrm{d} + i_v = \psi^i \frac{\partial}{\partial x^i} + v^i \frac{\partial}{\partial \psi^i} \tag{A.3}$$

One can further check that $\mathcal{Q}^2$ acts as the generator of the $U(1)$ symmetry. It follows that we can add $t\mathcal{Q}V$ to the action, for any $U(1)$-invariant V and the integral will be deformed by an exact term that will not change its cohomology class. According to equivariant localization, the integral will hence not be affected by the deformation [1]. To prove the DH formula, it suffices to choose a particular $V = g_{ij}v^i\psi^j$ where g_{ij} is any $U(1)$-invariant metric. In fact

$$\mathcal{Q}V = \psi^k \frac{\partial g_{ij}}{\partial x^k} v^i \psi^j + g_{ij}\psi^k \frac{\partial v^i}{\partial x^k}\psi^j + g_{ij}v^i v^j \tag{A.4}$$

Since the integral is indipendent of the coefficient t, we choose large t and so from (A.4) this localizes the integral to the points where $\psi^i = v^i = 0$, which corresponds to the critical points of the Hamiltonian. When expanding $\mathcal{Q}V$ around the critical points of H, we immeditely note the first and second term are cubic in the fluctuation and so can be discarded in evaluating the quadratic integral. One can check the Gaussian integral of the last term indeed induces the correct measure for the one-loop integral about those points to prove the DH formula.

[1] the integral of an equivariantly exact form is zero.

A.2 Non-abelian localization

Let X be a compact closed manifold acted on by a compact connected Lie group G, with Lie algebra $\mathcal{G}$ of elements T_a. Every genenerator T_a is represented by a vector field V_a on X.

We define the following integral

$$\oint_X \alpha = \frac{1}{\mathrm{vol}(G)} \int_{\mathcal{G}\times X} \frac{\mathrm{d}\phi_1 \mathrm{d}\phi_2 \cdots \mathrm{d}\phi_s}{(2\pi)^s} \alpha \; e^{-\frac{\epsilon}{2}(\phi,\phi)} \tag{A.5}$$

where

- $\phi_1, \phi_2 \ldots \phi_s$ are Euclidean coordinates of a vector $\phi \in \mathcal{G}$
- $\alpha \in \Omega^*(X)$ is an equivariantly closed differential form on X, i.e. $D\alpha = 0$ where $D = d - i\sum_a \phi_a j_{V_a}$ [2] is the equivariant exterior derivative.
- $(\ ,\ )$ is positive definite invariant quadratic form on $\mathcal{G}$, i.e. $(\phi,\phi) = \sum_a \phi_a \phi_a$

The integral above depends only on the cohomology class of D. This means we can always deform an equivariantly closed form with an an equivariantly exact one [3] and we will get the same answer for integration, i.e.

$$\oint_X \alpha \equiv \oint_X \alpha e^{tD\lambda} \tag{A.6}$$

where λ belongs to the G invariant subspace of $\Omega^*(X)$, where the Lie derivative with respect to V vanishes, $D^2 = -i\mathcal{L}_V = 0$. The deformed integral must be independent of the parameter t. Writing the above more explicitly we get

$$\oint_X \alpha = \frac{1}{\mathrm{vol}(G)} \int \frac{\mathrm{d}\phi_1 \mathrm{d}\phi_2 \cdots \mathrm{d}\phi_s}{(2\pi)^s} \alpha \; \exp\left(t\mathrm{d}\lambda - it\sum_a \phi_a \lambda(V_a) - \frac{\epsilon}{2}\sum_a \phi_a^2 \right) \tag{A.7}$$

We now perform the Gaussian integral over each ϕ_s and we obtain

$$\oint_X \alpha = \frac{1}{\mathrm{vol}(G)\,(2\pi\epsilon)^{s/2}} \int_X \alpha \; \exp\left(t\mathrm{d}\lambda - \frac{t^2}{2\epsilon}\sum_a (\lambda(V_a))^2 \right) \tag{A.8}$$

As $t \to \infty$ the integral is localized to a sum of local contributions Z_σ

$$\oint_X \alpha = \sum_{\sigma\in S} Z_\sigma \tag{A.9}$$

coming from the set S where

$$\lambda(V_a) = 0 \tag{A.10}$$

[2] j_{V_a} is the contraction of a generic form η with the vector V_a, also denoted with the inner product $\eta(V_a)$

[3] The integral of an exact equivariant form on X is zero $\int_X D\eta = 0$.

A.2.1 The case where X is a symplectic manifold

We suppose now that X is a symplectic manifold, with symplectic form ω, and that the G action on X has a moment map μ, defined by the moment equation $\mathrm{d}\mu_a = -j_{V_a}\omega$. Moreover we define $I = (\mu, \mu)$. We pick an almost complex structure J related to ω through

$$g(u,v) = \omega(u, Jv) \tag{A.11}$$

where g is a positive definite metric on X. We define λ in this way

$$\lambda = \frac{1}{2} J(\mathrm{d}I) \tag{A.12}$$

Witten proved that, with this choice of λ, the integral localizes to the critical points of I, i.e. the condition $\lambda(V_a) = 0$ is mapped into $\mathrm{d}I = 0$.

We now consider a specific equivariantly closed form α, which is the equivariant extension of the symplectic form ω:

$$\alpha = \exp\left(\omega - i\sum_a \phi_a \mu_a\right) \tag{A.13}$$

Let's check it is closed under the action of D:

$$D\alpha = \exp\left(\omega - i\sum_a \phi_a \mu_a\right)\left(-i\ \phi_a j_{V_a}\omega - i\phi_a \mathrm{d}\mu_a\right) = 0 \tag{A.14}$$

where we used $\mathrm{d}\omega = 0$ and the moment map equation. Let's specify the integral introduced above (A.5) in this case:

$$\oint_X \alpha = \frac{1}{\mathrm{vol}(G)} \int_{\mathcal{G}\times X} \frac{\mathrm{d}\phi_1 \mathrm{d}\phi_2 \cdots \mathrm{d}\phi_s}{(2\pi)^s} \exp\left(\omega - i\sum_a \phi_a \mu_a - \frac{\epsilon}{2}\sum_a \phi_a \phi_a\right) \tag{A.15}$$

We perfom the integral over ϕ and expand $\exp(\omega)$:

$$\oint_X \alpha = \frac{1}{\mathrm{vol}(G)\,(2\pi\epsilon)^{s/2}} \int_X \frac{\omega^n}{n!} \exp\left(-\frac{I}{2\epsilon}\right) \tag{A.16}$$

where $I = (\mu, \mu)$ and $2n$ is the dimension of X.

Yang-Mills The partition function of two dimensional quantum Yang-Mills theory on a surface Σ is formally given by the Feynman path integral:

$$Z(\epsilon) = \frac{1}{\mathrm{vol}(G)} \int_{\mathcal{A}} DA\ \exp\left(-\frac{1}{2\epsilon}(F,F)\right) \tag{A.17}$$

where DA is the symplectic form on the infinite dimensional space of connections $\mathcal{A}$. The moment map is given by the map

$$\mu(A) = -\frac{F}{4\pi^2} \tag{A.18}$$

from the connection A to the curvature two-form $F = \mathrm{d}A + A\wedge A$. On the other hand, since (F,F) is the norm of the moment map with respect to an invariant metric on the Lie algebra, the integral (A.17) is precisely of the form of the integrals (A.16) governed by the non-abelian localization principle exposed above. Accordingly, the path integral will therefore be localized on the critical points of the square of moment map, corresponding either to $F = 0$ of $\mathrm{d}F = 0$, i.e. to the flat connections and Yang-Mills connections respectevely.

A.3 tr (ϕ^2) reproduces the Schwarzian

We fix the component A_τ of the gauge field along the boundary and thus easily use the equations of motion to solve for the value of ϕ along I. We set

$$A_\tau|_{\text{bdy}} = \omega l_0 + \sqrt{\frac{\Lambda}{2}} e_+ l_+ + \sqrt{\frac{\Lambda}{2}} e_- l_- \tag{A.19}$$

where one defines

$$l_0 = i\mathrm{P}_0 \qquad l_+ = -\mathrm{P}_2 - i\mathrm{P}_1 \qquad l_- = \mathrm{P}_2 - i\mathrm{P}_1 \tag{A.20}$$

with commutation relations

$$[l_\pm, l_0] = \pm l_\pm \qquad [l_+, l_-] = 2l_0 \tag{A.21}$$

The components of the gauge field on the boundary are given by

$$\omega = -i\omega_\tau|_{\text{bdy}} \qquad e_+ = \left.\frac{ie^1_\tau - e^2_\tau}{2}\right|_{\text{bdy}} \qquad e_- = \left.\frac{ie^1_\tau + e^2_\tau}{2}\right|_{\text{bdy}} \tag{A.22}$$

The equation of motion for ϕ near the boundary is

$$D_\tau \phi = \partial_\tau \phi + \left[A_\tau|_{\text{bdy}}, \phi\right] = 0 \tag{A.23}$$

It is convenient to relate the parametrizations of the boundary and of the defect I through the function $u(\tau)$, i.e. finding the solution $\phi(u)$ as a function of u instead of τ. Equation (A.23) will then be

$$D_u \phi = \partial_u \phi + [A_u, \phi] = 0 \Longrightarrow \partial_u \phi + \left[A_\tau|_{\text{bdy}} \tau', \phi\right] = 0 \tag{A.24}$$

with $\tau' = \frac{\partial \tau(u)}{\partial u}$. It can be easily checked that the solution to the above is

$$\begin{aligned}\phi(u) &= \phi_0(u) l_0 + \phi_+(u) l_+ + \phi_-(u) l_- \\ &= \frac{1}{e}\left[2\left(\omega\tau' - \frac{\tau''}{\tau'}\right) l_0 + \sqrt{2\Lambda}\left(e_+\tau' + \frac{\tau'''}{\Lambda e_- \tau'^2} - \frac{\omega\tau''}{\Lambda e_- \tau'} - \frac{\omega\tau''^2}{\Lambda e_- \tau'^3}\right) l_+ + \sqrt{2\Lambda} e_- \tau' \; l_-\right]\end{aligned} \tag{A.25}$$

The next step is to sobstitute the last expression into the defect action (2.167) [4]:

$$S_{\mathrm{I}} = -\frac{1}{e}\int_0^\beta \mathrm{d}u \; \frac{\tau'''}{\tau'} - \frac{3}{2}\left(\frac{\tau''}{\tau'}\right)^2 + 2\tau'^2\left(\frac{2\Lambda e_+ e_- - \omega^2}{4}\right) \tag{A.26}$$

where in the first piece we recognize the Schwarzian derivative and in the second the determimant of A_τ in the fundamental representation. [5] Therefore (A.26) reads as

$$S_{\mathrm{I}} = -\frac{1}{e}\int_0^\beta \mathrm{d}u \; \{\tau(u), u\} + 2\tau'^2 \det(A_\tau) \tag{A.27}$$

[4] Exploiting the trace relations $\mathrm{tr}\,(l_0 l_0) = 1 \qquad \mathrm{tr}\,(l_0 l_\pm) = 0 \qquad \mathrm{tr}\,(l_\pm l_\mp) = 1 \qquad \mathrm{tr}\,(l_\pm l_\pm) = 0$

[5] In fact

$$\det(A_\tau) = \det\begin{pmatrix} \frac{1}{2}\omega & \sqrt{\frac{\Lambda}{2}} e_+ \\ -\sqrt{\frac{\Lambda}{2}} e_- & -\frac{1}{2}\omega \end{pmatrix} = \frac{2\omega_\tau^2 - \Lambda\gamma_{\tau\tau}}{8} \qquad \gamma_{\tau\tau} = -4e_+e_- = e^1_\tau e^1_\tau + e^2_\tau e^2_\tau$$

Finally, taking advantage of the same change of variables performed in (2.29)

$$F(u) = \tan\left(\sqrt{\det(A_\tau)}\tau(u)\right)$$

we recover exactly the Schwarzian action as in (2.26)

$$S_{\mathrm{I}} = -\frac{1}{e}\int_0^\beta \{F(u), u\} \tag{A.28}$$

with the identification $1/e = \frac{\phi_r}{8\pi G} = \frac{C}{2}$.

A.4 Schwarzian propagator

We use the Cauchy formula

$$f(z) = \oint \frac{f(t)}{t-z}dt \tag{A.29}$$

to rexepress the propagator as

$$\langle\delta(0)\delta(u)\rangle = \frac{2}{C}\sum_{n\neq 0,\pm 1}\oint_{\gamma_n}\frac{ds}{e^{2\pi i s}-1}\frac{e^{isu}}{s^2\,(s^2-1)} \tag{A.30}$$

where γ_n are infinitesimal circles of integration around every integer n and the denominator $e^{2\pi i s}-1$ reproduces the factor $t-z$ in (A.29) when we expand around every pole n

$$e^{2\pi i s}-1 \approx 2\pi i\,(s-n) \quad s\to n \tag{A.31}$$

To evaluate (A.30), we deform the contour of integration to a big circle Γ and a small circle γ characterized by a ray R and r in the complex plane respectevely. If we send $R\to\infty$ the integrand goes to zero as $1/R^4$ and so we can neglect the contribution of the integral along Γ. The other circle γ is driven counterclockwise and so its contribution amounts of the residues in the poles $0, 1, -1$ contained within it, i.e.

$$\begin{aligned}\mathrm{Res}_{s=1} &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{e^{isu}}{s\,(s-1)}\bigg|_{s=1} \\ &= \frac{iue^{isu}}{s^2\,(s+1)}\bigg|_{s=1} - \frac{e^{isu}}{s^2\,(s+1)^2}\bigg|_{s=1} - \frac{2e^{isu}}{s^3\,(s+1)}\bigg|_{s=1} = iu\frac{e^{iu}}{2} - \frac{5}{4}e^{iu}\end{aligned} \tag{A.32}$$

The same calculation holds for $\mathrm{Res}_{s=-1}$ which yields

$$\mathrm{Res}_{s=-1} = -iu\frac{e^{-iu}}{2} - \frac{5}{4}e^{-iu} \tag{A.33}$$

They combine to give the therm

$$\mathrm{Res}_{s=1} + \mathrm{Res}_{s=-1} = u\ \sin(u) - \frac{5}{2}\cos(u) \tag{A.34}$$

that appears in (2.44). After expanding around $s=0$, we will instead get a third order pole, that we can compute by

$$\lim_{s\to 0}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(s^3 f(s)\right) \tag{A.35}$$

Putting all together we recover the full result (2.44).

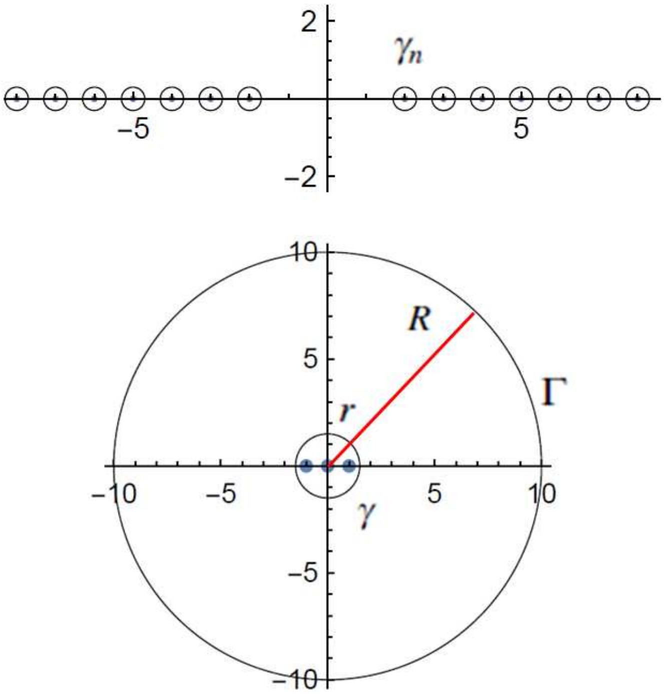


Figure A.1: Top: Circuits of integration γ_n $n \in Z$ around every integer except for $0, 1, -1$. Bottom: the contour of integration is deformed and the poles in $z = 0, -1, 1$ are shown.

A.5 Gauss law constraint

Using integration by parts and ciclicity of the trace, the BF Lagrangian can be rewritten as

$$\mathcal{L}_{\text{BF}} = \text{tr}\left(\phi \partial_0 A_1 + A_0 \left(\partial_1 \phi + i\left[A_1, \phi\right]\right)\right) \tag{A.36}$$

where A_0 can be thought as a Lagrange multiplier in imposing the constraint

$$D_1 \phi^a = 0 \qquad D_1 = \partial_1 + i\left[A_1, \ \right] \tag{A.37}$$

Besides from (A.36) the canonical momentum conjugate to the space component of the gauge field A_1^a is ϕ^a, i.e.

$$\pi_{A_1} = \frac{\delta L}{\delta \partial_0 A_1^a} = \phi^a \tag{A.38}$$

In canonical quantization therefore we impose

$$\left[A_1^a, \phi^b(y)\right] = \delta^{ab} \delta^2(x - y) \tag{A.39}$$

Since in canonical quantization the momentum π_i associated with a generalized coordinate q_i (satisfying the equal time algebra (A.39)) is defined as $\pi_i = -\frac{\partial}{\partial q_i}$, in this case the generalized coordinate is a field $A_1^a(x)$ (the discrete index i becomes continuos x) and therefore the same formula holds with the partial derivative sobstituted by a functional derivative

$$\pi_{A_1^a(x)} = \phi^a(x) = -\frac{\delta}{\delta A_1^a(x)} \tag{A.40}$$

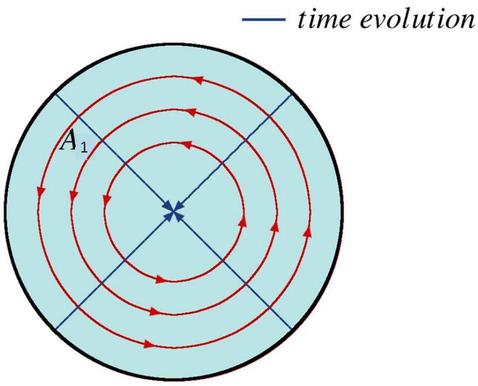


Figure A.2: Radial quantization: the wavefunction depends on the holonomy (represented in red) around a circle of constant ray $\psi(\mathcal{P}\exp\oint A_1^a T^a)$ while time flows along the radial direction.

(A.37) then translates into a functional constraint

$$D_1 \frac{\delta}{\delta A_1^a(x)} = 0 \tag{A.41}$$

on all possible wavefunctionals $\psi(A_1^a(x))$ of the A_1 component of the gauge field. Expressing the commutator inside the covariant derivative D through the structure constants $f^{ab}{}_c$, we have

$$D_1 \frac{\delta}{\delta A_1^a(x)}\psi = \left(\partial_1 \frac{\delta}{\delta A_1^a(x)} + f^{ab}{}_c A_1^b(x) \frac{\delta}{\delta A_1^c(x)}\right)\psi = 0 \tag{A.42}$$

The condition(A.42) is called Gauss law constraint because it is solved by wavefunctionals of the form

$$\psi(A_1^a(x)) = \psi\left[\mathcal{P}\exp\oint A_1^a T^a\right] \tag{A.43}$$

i.e they depend only on the holonomy $g = \mathcal{P}\ \exp\oint A_1^a T^a$ of the angular component A_1 of the gauge field on a spacial circle (uo to conjugation).

A.6 Boundary particle approach

A different approach to quantize JT gravity [134] uses a direct rewriting of the GHY boundary term of the gravity path integral exploiting the 2d Gauss-Bonnet theorem to relate the extrinsic curvature to an integral over the bulk:

$$\int_{\partial\mathcal{M}} du\ \sqrt{g}\kappa = 2\pi\chi(\mathcal{M}) - \frac{1}{2}\int_{\mathcal{M}} R \tag{A.44}$$

The regularized action then is

$$I = -\frac{\phi_r}{\varepsilon}\int_{\partial\mathcal{M}} du\ \sqrt{g}(\kappa - 1) = -\frac{\phi_r}{\varepsilon}\left(2\pi\chi(\mathcal{M}) - \frac{1}{2}\int_{\mathcal{M}} R + \int_{\partial\mathcal{M}} du\ \sqrt{g}\right) \tag{A.45}$$

The area term $(R = -2)$, expressed in Poincare coordinates as $\int \frac{dtdz}{z^2}$, can be rewritten using Stokes theorem as a line integral of the one form gauge field $A_t = -\frac{1}{z}dt$ [6]. The perimeter term

[6] The field strenght in fact becomes $F_{tz} = \frac{1}{z^2}$

is just the integrated line element along the boundary [7]. We thus end up with the action of a non-relativistic particle in AdS_2 in the presence of a magnetic field [8]:

$$I = \int_0^\beta du \left(\frac{t'^2 + z'^2}{z^2} - q\frac{t'}{z} \right) - \frac{1}{2}\int du \left(\frac{1}{4} - q^2 \right) \qquad q = \frac{\phi_r}{\varepsilon} \tag{A.46}$$

where the last term was added for convenience and just represents a shift of the ground state energy. Performing the Legendre transform, it is easy to write down the Hamiltonian, where we recognize the minimal coupling of the momentum of the particle with the gauge field:

$$H = \frac{z^2}{2}\left[\left(p_t - \frac{iq}{z} \right)^2 + p_z^2 \right] + \frac{q^2}{2} - \frac{1}{8} \tag{A.47}$$

We wish to solve the Schroedinger equation $H\psi(t,z) = E\psi(t,z)$; we parametrize $E = j(1-j)$ and we exploit separation of variables. We diagonalize the p_t momentum exploiting plane waves e^{ikt}, so we replace p_t with its eigenvalue $p_t \to k$. We are left with the differential equation:

$$\psi''(z) - \left[k^2 - \frac{2kb}{z} - \frac{\frac{1}{4} + s^2}{z^2} \right] \psi(z) = 0 \tag{A.48}$$

where we have further set $q = ib$ and $j = \frac{1}{2} + s$ for convenience and [9] we recognize the above as the Whittacker equation. The normalized solution (such that the integral over AdS_2 yields 1) is thus

$$\psi_{k,s}(z,t) = \left(\frac{s \sinh 2\pi s}{4\pi^3 |k|} \right)^{\frac{1}{2}} \left| \Gamma\left(\frac{1}{2} + is - b \right) \right| e^{-ikt}\, W_{b,is}\left(2\,|k|\, z \right) \tag{A.49}$$

where $s^2/2$ labels the energy eigenvalue.
The path integral can be computed through the Euclidean termal trace:

$$\begin{aligned} Z &= e^{2\pi q}\, \mathrm{tr}\left(e^{-\beta H} \right) \\ &= e^{2\pi q} \int_{AdS_2} \int_{-\infty}^{+\infty} dk \int_0^{+\infty} ds e^{-\beta s^2/2} \left| \psi_{k,s}\left(z,t \right) \right|^2 \end{aligned} \tag{A.50}$$

where in the second step we inserted a completeness on the eigenfunction basis and integrated over all possible starting points of the loop in AdS_2. The integral over k can be performed exploiting parity and the following property [24]

$$\int_0^{+\infty} \frac{dx}{x}\, W_{\beta,\mu}(x)^2 = \pi \frac{\psi\left(\frac{1}{2} + \mu - \beta \right) - \psi\left(\frac{1}{2} - \mu - \beta \right)}{\sin\left(2\pi\mu \right) \Gamma\left(\frac{1}{2} + \mu - \beta \right) \Gamma\left(\frac{1}{2} - \mu - \beta \right)} \tag{A.51}$$

where $\psi = d\frac{\log\Gamma(x)}{dx}$ is the Digamma function. From this we get [10]

$$Z = e^{2\pi q} \int_0^{+\infty} ds e^{-\beta s^2/2}\, \frac{s}{2\pi^2}\, \mathrm{Im}\left[\psi\left(\frac{1}{2} + is - iq \right) \right] \tag{A.52}$$

[7] In fact $q\int du\sqrt{g} = \int du q^2 = \int du\, \frac{t'^2 + z'^2}{z^2}$ once we fix the length of the boundary curve inserting into the path integral the constraint $\delta\left(\frac{t'^2 + z'^2}{z^2} - q^2 \right)$.

[8] We use that $\chi(\mathcal{M}) = 1$ and keep out a factor $e^{2\pi q}$ from the path integral.

[9] $j = \frac{1}{2} + s$ indeed represent the continuos series representation of $SL(2,\mathbb{R})$.

[10] We neglect the infinite volume of $\mathcal{M}$ since it can be reabsorbed.

We now use the following identity

$$\mathrm{Im}\left[\psi\left(\frac{1}{2}+is-iq\right)\right]=\frac{\pi\sinh(2\pi s)}{2(\cosh(2\pi q)+\cosh(2\pi s))} \tag{A.53}$$

to get the final answer

$$Z=\frac{e^{2\pi q}}{4\pi}\int_0^{+\infty} ds\ s\ e^{-\beta s^2/2}\ \frac{\sinh(2\pi s)}{\cosh(2\pi q)+\cosh(2\pi s)} \tag{A.54}$$

A.7 SL(2, ℝ) and SL$^+$(2, ℝ) representation theory

In this section we propose a brief review of SL(2, ℝ) and SL$^+$(2, ℝ) representation theory, mainly following [12]. We choose the carrier space of the spin-j representation, on which the Casimir evaluates to $j(j+1)$, to be $L^2(\mathbb{R})$. The action of a group element g on functions $f(x)$ is defined as

$$(g\circ f)(x)=|bx+d|^{2j} f\left(\frac{ax+b}{cx+d}\right) \tag{A.55}$$

The realization of the generators is

$$J_0=x\partial_x-j \qquad J^-=\partial_x \qquad J^+=-x^2\partial_x+2jx \tag{A.56}$$

and they satisfy the $\mathfrak{sl}(2,\mathbb{R})$ algebra . One can easily check the normalized wavefunctions of J^- and J^+ with eigenvalues ν and λ respectevely are [11]

$$\langle x|\nu\rangle=\frac{e^{i\nu x}}{\sqrt{2\pi}} \qquad \langle x|\lambda\rangle=\frac{|x|^{2j}\,e^{i\lambda/x}}{\sqrt{2\pi}} \tag{A.57}$$

Exploiting the Gauss decomposition of the group element $g=e^{\gamma_-J_-}e^{2\phi J_0}e^{\gamma_+J_+}$, we deduce the matrix elements [12]

$$\begin{aligned} R_{k,\nu,\lambda}(g)\equiv\langle\nu|g|\lambda\rangle&=e^{i\gamma_-\nu}e^{i\gamma_+\lambda}\left\langle\nu|e^{2\phi J_0}|\lambda\right\rangle\\ &=\frac{e^{i\gamma_-\nu}e^{i\gamma_+\lambda}e^{2j\phi}}{2\pi}\int dx e^{i\nu e^{-2\phi}x}\ |x|^{2ik-1}\,e^{i\lambda/x}\\ &=\frac{e^{i\gamma_-\nu}e^{i\gamma_+\lambda}e^{2j\phi}}{2\pi}\cosh(\pi k)\left(\frac{\lambda}{\nu e^{-2\phi}}\right)^{ik}\ K_{2ik}\left(\sqrt{\nu\lambda}e^{-\phi}\right)\\ &=\frac{e^{i\gamma_-\nu}e^{i\gamma_+\lambda}e^{-\phi}}{2\pi}\cosh(\pi k)\left(\frac{\lambda}{\nu}\right)^{ik}\ K_{2ik}\left(\sqrt{\nu\lambda}e^{-\phi}\right) \end{aligned} \tag{A.58}$$

[11] An easy computation shows the antihermiticity of the generators with respect to the measure dx requires $j=-\frac{1}{2}+ik$.

[12] We used $\int_{-\infty}^{+\infty}dxe^{i\nu x}\ |x|^{2ik-1}\,e^{i\lambda/x}=\int_0^{+\infty}dxe^{i\nu x}\ x^{2ik-1}e^{i\lambda/x}+\int_0^{+\infty}dxe^{-i\nu x}\ x^{2ik-1}e^{-i\lambda/x}$, exploited the integral representation of the modified Bessel function of the second kind: $\int_0^{+\infty}dx\ x^{2ik-1}e^{-\nu x}e^{-\lambda/x}=\left(\frac{\lambda}{\nu}\right)^{ik}\ K_{2ik}\left(\sqrt{\nu\lambda}\right)$ and analitically continued $\nu\to e^{i\pi/2}\nu,\lambda\to e^{-i\pi/2}\lambda$ for the first integral, while $\nu\to e^{-i\pi/2}\nu,\lambda\to e^{i\pi/2}\lambda$ for the second one.

Exploiting the orthogonality relation

$$\int_0^{+\infty} \frac{dx}{x}\, K_{2i\mu}(x) K_{2i\nu}(x) = \frac{\pi^2}{8\mu \sinh(2\pi\mu)} \delta(\mu - \nu) \tag{A.59}$$

we obtain

$$\int \mathrm{d}g\, R_{k,\nu\lambda}(g)^* \, R_{k',\nu'\lambda'}(g) = \frac{\pi^2}{2k \tanh(\pi k)} \delta(k-k')\delta(\nu-\nu')\delta(\lambda-\lambda') \tag{A.60}$$

where we implemented the Haar measure $dg = \frac{1}{2} e^{2\phi} d\phi d\gamma_+ d\gamma_-$ of $\mathrm{SL}(2,\mathbb{R})$. From the above can easily read off the Plancherel measure $\rho(k) = \frac{2k\tanh(\pi k)}{(2\pi)^2}$ for $\mathrm{SL}(2,\mathbb{R})$.

One can also study harmonic analysis on $\mathrm{SL}(2,\mathbb{R})$. It turns out that the Casimir eigenfunctions are

$$\begin{aligned} &\nu\lambda > 0 \qquad e^{i\gamma_- \nu} e^{i\gamma_+ \lambda} e^{-\phi}\, K_{2ik}\left(\sqrt{\nu\lambda}\, e^{-\phi}\right) \\ &\nu\lambda < 0 \qquad e^{i\gamma_- \nu} e^{i\gamma_+ \lambda} e^{-\phi}\, J_{2ik}\left(\sqrt{-\nu\lambda}\, e^{-\phi}\right) \end{aligned} \tag{A.61}$$

The first solution can be interpreted as the continuous series representation matrices, while the second case can be interpreted as the discrete series representation matrices. A quick way to understand this is that, for the discrete representations where $\nu\lambda < 0$, we demand single-valuedness of the representation matrix element when $\phi \to \phi + 2i\pi$ and, since the BesselJ function is generically a multi-valued function of its argument, except when the index is an integer, we set $2j \in -\mathbb{N}$. [13]

A.7.1 SL$^+$(2,ℝ)

Since the elements of an $\mathrm{SL}^+(2,\mathbb{R})$ matrix are all positive, we choose as a carrier space of the spin-j representation of $\mathrm{SL}^+(2,\mathbb{R})$ $L^2\left(\mathbb{R}^+\right)$ with inner product $\int_0^{+\infty} f(x)^* g(x)$. The big difference between $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}^+(2,\mathbb{R})$ is that, while in the former case the parabolic eigenfunctions do form a basis, in the latter case they do not. This is because the only $\mathfrak{sl}(2,\mathbb{R})$ generator that is antihermitian on $\mathbb{R}^+$ is the hyperbolic generator J_0: the parabolic generators $J_\pm$ are not. Correspondingly, the eigenfunctions of $J_\pm$ are not delta-function normalizable on $\mathbb{R}^+$.. Indeed, these are

$$\langle x|\nu\rangle = e^{-\nu x} \qquad \langle x|\lambda\rangle = |x|^{2j} e^{-\lambda/x} \tag{A.62}$$

for J_+ and J_- respectevely, with eigenvalues ν and λ. These two sets of eigenfunctions do not satisfy orthogonality relations on $\mathbb{R}^+$.

Prooceding as before, the former modifications make the mixed parabolic matrix element read as

$$R_{k,\nu,\lambda}(g) = \frac{e^{-\gamma_- \nu} e^{-\gamma_+ \lambda} e^{-\phi}}{2\pi} \left(\frac{\lambda}{\nu}\right)^{ik} K_{2ik}\left(\sqrt{\nu\lambda}\, e^{-\phi}\right) \tag{A.63}$$

[13] Actually the $\mathrm{SL}(2,\mathbb{R})$ group manifold is actually covered by four patches, which just provide an addictional factor of 4. Therefore the Plancherel measure is indeed $\rho(k) = \frac{2k\tanh(\pi k)}{(2\pi)^2}$

The factor $\cosh(\pi k)$ in (A.58) has disappeared, because we immediately obtain the integral representation of the modified Bessel function of the second kind, without need to perform any analytic continuations. This fact is crucial because the orthogonality relation (A.59) now immediately provides the Plancherel measure for $\mathrm{SL}^+(2,\mathbb{R})$, i.e.

$$\rho(k) = k \sinh(2\pi k) \tag{A.64}$$

One can study harmonic analysisalso when restricted to the subsemigroup $\mathrm{SL}^+(2,\mathbb{R})$. This merely requires setting $\lambda \to i\lambda$ and $\nu \to -i\nu$ with $\nu, \lambda > 0$. This means we only have the case where $\nu\lambda > 0$, so we notice that the discrete representation matrices cannot be found in the regime $\nu\lambda > 0$ relevant for the subsemigroup.

A.7.2 Connection with boundary particle approach

The spectral density in (A.54) can be rewritten in a more suggestive way as a geometric series

$$s\frac{\sinh(2\pi s)}{\cosh(2\pi q) + \cosh(2\pi s)} = \frac{1}{\pi} s e^{2\pi q} \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-2\pi q n} \sinh(2\pi s n) \tag{A.65}$$

where the different terms in the sum can be interpreted as different instanton sectors, corresponding to classical solutions associated with the particle winding n times around the circle. This higher sectors should not appear in the context of JT gravity, since we want to impose the boundary not to self-intersect to have a physical bulk gravity theory. We will now comment on two different regimes of the above:

- in the limit $q \to \infty$, corresponding to an infinite imaginary magnetic field, all higher instanton sectors, except for $n = 1$, are exponentially suppressed and so one is left with the partition function of JT gravity. This is the physical interpretation of the real extension of $\mathrm{SL}(2,\mathbb{R})$ performed by [74].

- in the limit $q \to 0$ the sum above does not converge, but can be analitically continued to the origin

  $$s \tanh(\pi s) = \frac{1}{\pi} \sum_{n=1}^{+\infty} (-1)^{n-1} s \sinh(2\pi s n) \tag{A.66}$$

  We note that in this regime we obtain the Plancherel measure of $\mathrm{SL}(2,\mathbb{R})$, while restricting to $n = 1$ gives directly the $\mathrm{SL}^+(2,\mathbb{R})$ Plancherel measure. A clear physical interpretation thus emerges: choosing $\mathrm{SL}^+(2,\mathbb{R})$ [12] automatically projects out all higher sectors, which correspond to replicated geometries $\mathrm{SL}^n(2,\mathbb{R})$, or conical singularities in the bulk with angular deficit $2\pi n$.

A.8 Evaluation of the Residue

Most of our results can be expressed in terms of the so-called generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\ell)}(x;\mu)$. If $\ell \in \mathbb{N}$, they are defined through the generating function:

$$\left(\frac{t}{\mu e^t - 1}\right)^{\ell} e^{xt} = \sum_{n=\ell}^{\infty} \mathcal{B}_n^{(\ell)}(x;\mu) \frac{t^n}{n!}. \tag{A.67}$$

This definition implies that $\mathcal{B}_n^{(\ell)}(x;\mu)=0$ for $n=0,\ldots,\ell-1$. The generalized Apostol-Bernoulli numbers $\mathcal{B}_n^{(\ell)}(\mu)$ are then given by

$$\mathcal{B}_n^{(\ell)}(\mu) \equiv \mathcal{B}_n^{(\ell)}(0;\mu). \tag{A.68}$$

The familiar Bernoulli polynomials are recovered when we set $\ell=1$ and $\mu=1$. The explicit form of this polynomials can be obtained as follows. First we consider the combination $\left(\frac{t}{\mu e^t-1}\right)^\ell$ and write its formal expansion in power of e^t-1.

$$\begin{aligned}\left(\frac{t}{\mu e^t-1}\right)^\ell =&\frac{t^\ell}{(\mu-1)^\ell}\left(\frac{\mu-1}{\mu e^t-1}\right)^\ell = \frac{t^\ell}{(\mu-1)^\ell}\left(1+\frac{\mu}{\mu-1}(e^t-1)\right)^{-\ell} = \\ =&t^\ell\sum_{k=0}^{\infty}\binom{k+\ell-1}{k}\frac{(-\mu)^k}{(\mu-1)^{k+\ell}}(e^t-1)^k\end{aligned} \tag{A.69}$$

Next we use that

$$(e^t-1)^k = k!\sum_{r=k}^{\infty}S(r,k)\frac{t^r}{r!} \tag{A.70}$$

where $S(r,k)$ denotes the Stirling numbers of the second kind. Thus

$$\begin{aligned}\left(\frac{t}{\mu e^t-1}\right)^\ell =&\sum_{k=0}^{\infty}\sum_{r=k}^{\infty}\binom{k+\ell-1}{k}\frac{k!(-\mu)^k}{(\mu-1)^{k+\ell}}S(r,k)\frac{t^{r+\ell}}{r!} = \\ =&\sum_{r=0}^{\infty}\frac{t^{r+\ell}}{r!}\sum_{k=0}^{r}\binom{k+\ell-1}{k}\frac{k!(-\mu)^k}{(\mu-1)^{k+\ell}}S(r,k) = \\ =&\sum_{r=0}^{\infty}\frac{t^{r+\ell}}{(r+l)!}\ell!\sum_{k=0}^{r}\binom{r+\ell}{r}\binom{k+\ell-1}{k}\frac{k!(-\mu)^k}{(\mu-1)^{k+\ell}}S(r,k)\end{aligned} \tag{A.71}$$

From eq. (A.71) we can immediately extract a representation for the generalized Apostol-Bernoulli numbers $\mathcal{B}_n^{(\ell)}(\mu)$ by setting $r=n-\ell$.

$$\begin{aligned}\mathcal{B}_n^{(\ell)}(\mu) =&\ell!\sum_{k=0}^{n-\ell}\binom{n}{n-\ell}\binom{k+\ell-1}{k}\frac{k!(-\mu)^k}{(\mu-1)^{k+\ell}}S(n-\ell,k) = \\ =&\ell!\sum_{k=0}^{n-\ell}\binom{n}{\ell}\binom{k+\ell-1}{k}\frac{k!(-\mu)^k}{(\mu-1)^{k+\ell}}S(n-\ell,k).\end{aligned} \tag{A.72}$$

Given the generalized Apostol-Bernoulli numbers $\mathcal{B}_n^{(\ell)}(\mu)$, it is easy to write down the expansion for the polynomial

$$\mathcal{B}_n^{(\ell)}(x;\mu) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}^{(\ell)}(\mu)x^k. \tag{A.73}$$

The case $\mu=1$ are simply known as generalized Bernoulli polynomials and we shall denote them as $\mathrm{B}_n^{(\ell)}(x)$. Obviously we can also introduce the generalized Bernoulli numbers, $\mathrm{B}_n^{(\ell)} \equiv \mathrm{B}_n^{(\ell)}(0)$. These polynomials are not simply obtained by taking the limit for $\mu\to 1$ of the previous explicit expressions. The latter are in fact divergent in this limit.

In the following we show that the residue appearing in the computation of the bi-local correlator when 2λ is an integer can be expressed in terms of these generalized quantities. We start by observing that

$$\frac{1}{\left(\cosh\frac{v}{2}-\cosh\frac{u}{2}\right)^n} = -(-2)^{n-1}e^{\frac{1}{2}nv}\left(\frac{2}{y}\right)^{2n}\underbrace{\frac{\left(\frac{y}{2}\right)^n e^{\frac{1}{4}ny}}{\left(e^{\frac{y}{2}}-1\right)^n}}_{\text{(A)}}\underbrace{\frac{\left(\frac{y}{2}\right)^n e^{\frac{1}{4}ny}}{\left(e^{v+\frac{y}{2}}-1\right)^n}}_{\text{(B)}} =$$

$$= -(-2)^{n-1}e^{\frac{1}{2}nv}\left(\frac{2}{y}\right)^n\sum_{\ell=0}^{\infty}\left(\frac{y}{2}\right)^{\ell}\sum_{j=0}^{\ell}\frac{\mathrm{B}_{\ell-j}^{(n)}\left(\frac{n}{2}\right)\mathcal{B}_{j+n}^{(n)}\left(\frac{n}{2};e^v\right)}{(\ell-j)!(n+j)!}, \tag{A.74}$$

where we have introduced $y=u-v$ to keep a compact notation. We have expanded the factor (A) in terms of generalized Bernoulli polynomials, while the remaining factor (B) has been expressed as series whose coefficients are the generalized Apostol-Bernoulli polynomials for $\lambda=e^v$. If we use the property

$$\mathcal{B}_k^{(n)}(n-x,\mu) = \frac{(-1)^k}{\mu^n}\mathcal{B}_k^{(n)}(x,\mu^{-1}) \tag{A.75}$$

we find that

$$\mathrm{B}_k^{(n)}\left(\frac{n}{2}\right) = (-1)^k\mathrm{B}_k^{(n)}\left(\frac{n}{2}\right) \qquad e^{\frac{1}{2}nv}\mathcal{B}_k^{(n)}\left(\frac{n}{2},e^v\right) = (-1)^k e^{-\frac{1}{2}nv}\mathcal{B}_k^{(n)}\left(\frac{n}{2},e^{-v}\right). \tag{A.76}$$

The first identity implies that $\mathrm{B}_k^{(n)}\left(\frac{n}{2}\right)$ vanishes for odd k. Next we observe that the combination $u\exp\left(-\alpha u^2\right)$ can be written as

$$ue^{-\alpha u^2} = \frac{1}{2}\sum_{n=0}^{\infty}(-2)^n\frac{\alpha^{\frac{n-1}{2}}}{n!}\mathcal{H}_{n+1}(\sqrt{\alpha}v)e^{-\alpha v^2}\left(\frac{y}{2}\right)^n \tag{A.77}$$

where $\mathcal{H}_n(u)$ stand for the usual Hermite polynomials. Thus we find the following Laurent expansion for the function $f(u)=\frac{ue^{-\alpha u^2}}{\left(\cosh\frac{v}{2}-\cosh\frac{u}{2}\right)^n}$:

$$f(u) = -(-2)^{n-1}\left(\frac{2}{y}\right)^n e^{-\alpha v^2+\frac{n}{2}v}\sum_{p=0}^{\infty}\left(\frac{y}{2}\right)^p e^{-\alpha v^2}\times$$

$$\times\sum_{\ell=0}^{p}(-2)^{p-\ell}\frac{\alpha^{\frac{p-\ell-1}{2}}}{(p-\ell)!}\mathcal{H}_{p-\ell+1}(\sqrt{\alpha}v)\sum_{j=0}^{\ell}\frac{\mathrm{B}_{\ell-j}^{(n)}\left(\frac{n}{2}\right)\mathcal{B}_{j+n}^{(n)}\left(\frac{n}{2};e^v\right)}{(\ell-j)!(n+j)!}. \tag{A.78}$$

It is a trivial exercise to extract the relevant residue form (A.78). We get

$$\mathrm{Res}[f(u)]_{u=v} = e^{-\alpha v^2+\frac{n}{2}v}\sum_{\ell=0}^{n-1}\frac{(-2)^{2n-\ell-1}\alpha^{\frac{n-\ell-2}{2}}\mathcal{H}_{n-\ell}(\sqrt{\alpha}v)}{(n-\ell-1)!}\sum_{j=0}^{\ell}\frac{\mathrm{B}_{\ell-j}^{(n)}\left(\frac{n}{2}\right)\mathcal{B}_{j+n}^{(n)}\left(\frac{n}{2};e^v\right)}{(\ell-j)!(n+j)!}. \tag{A.79}$$

A.9 Some useful expansions for generalized Apostol-Bernoulli Polynomials

In ref [85, 84] they provide the following expansion for the generalized Apostol-Bernoulli polynomials in terms of the generalized Bernoulli polynomials (i.e. $\mu = 1$):

$$\mathcal{B}_j^{(n)}(x,\mu) = e^{-x\log\mu}\sum_{k=0}^{\infty}\binom{j+k-n}{k}\binom{j+k}{k}^{-1}\mathrm{B}_{k+j}^{(n)}(x)\frac{(\log\mu)^k}{k!} \tag{A.80}$$

This expansion suggests that it is possible to expand the generalized Apostol-Bernoulli polynomials at a given $\mu=\mu_1\mu_2$ in terms of the same polynomials at $\mu=\mu_1$. In fact, exploiting the properties of logarithms

$$\mathcal{B}_j^{(n)}(x,\mu_1\mu_2) = e^{-x(\log\mu_1+\log\mu_2)}\sum_{k=0}^{\infty}\binom{j+k-n}{k}\binom{j+k}{k}^{-1}\mathrm{B}_{k+j}^{(n)}(x)\frac{(\log\mu_1+\log\mu_2)^k}{k!} =$$

$$= e^{-x(\log\mu_1+\log\mu_2)}\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\binom{j+k-n}{k}\binom{j+k}{k}^{-1}\mathrm{B}_{k+j}^{(n)}(x)\frac{1}{k!}\binom{k}{\ell}(\log\mu_1)^{\ell}(\log\mu_2)^{k-\ell} \tag{A.81}$$

We can disentangle the two sums by setting $k=\ell+m$. Then the two sums becomes independent:

$$= e^{-x(\log\mu_1+\log\mu_2)}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\binom{j+m+\ell-n}{m+\ell}\binom{j+m+\ell}{m+\ell}^{-1}\mathrm{B}_{m+\ell+j}^{(n)}(x)\frac{(\log\mu_1)^{\ell}}{\ell!}\frac{(\log\mu_2)^m}{m!}. \tag{A.82}$$

Let use rearrange the binomials coefficient as follows and perform the sum over ℓ:

$$= e^{-x(\log\mu_1+\log\mu_2)}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{\binom{j+m-n}{m}\binom{j+\ell+m-n}{\ell}}{\binom{j+m}{m}\binom{j+l+m}{\ell}}\mathrm{B}_{m+\ell+j}^{(n)}(x)\frac{(\log\mu_1)^{\ell}}{\ell!}\frac{(\log\mu_2)^m}{m!} =$$

$$= e^{-x\log\mu_2}\sum_{m=0}^{\infty}\frac{\binom{j+m-n}{m}}{\binom{j+m}{m}}\mathcal{B}_{m+j}^{(n)}(x,\mu_1)\frac{(\log\mu_2)^m}{m!}. \tag{A.83}$$

Therefore we have shown

$$\mathcal{B}_j^{(n)}(x,\mu_1\mu_2) = e^{-x\log\mu_2}\sum_{m=0}^{\infty}\frac{\binom{j+m-n}{m}}{\binom{j+m}{m}}\mathcal{B}_{m+j}^{(n)}(x,\mu_1)\frac{(\log\mu_2)^m}{m!}. \tag{A.84}$$

If we apply this result to our specific case we get

$$e^{\frac{nv}{2}}\mathcal{B}_{j+n}^{(n)}\left(\frac{n}{2};e^{v-\frac{2\pi i\tau}{\beta}}\right) = \sum_{m=0}^{\infty}\frac{\binom{j+m}{m}}{\binom{j+m+n}{m}}\mathcal{B}_{m+j}^{(n)}\left(\frac{n}{2};e^{-\frac{2\pi i\tau}{\beta}}\right)\frac{v^m}{m!}. \tag{A.85}$$

A.10 The case of generic n: the details

After performing the shift $v\mapsto v+\frac{2\pi i(\beta-\tau)}{\beta}$ in (3.43) to move the gaussian center around $v=0$, we find

$$\langle\mathcal{O}^{(\lambda)}(\tau)\rangle_{\beta}^{\text{disk}} = -\frac{i\pi^2\Gamma(2\lambda)\kappa^{2\lambda-\frac{3}{2}}\beta^{\frac{3}{2}}}{2^{2\lambda+3}\pi^{\frac{7}{2}}\tau^{\frac{3}{2}}(\beta-\tau)^{\frac{3}{2}}}\int_{-\infty}^{+\infty}\mathrm{d}v\,\left(v-\frac{2\pi i\tau}{\beta}\right)f\left(v+\frac{2\pi i(\beta-\tau)}{\beta}\right)\,e^{-\frac{\beta v^2}{4\kappa\tau(\beta-\tau)}}. \tag{A.86}$$

Using the result (A.85), we can immediately expand $f\left(v+\frac{2\pi i(\beta-\tau)}{\beta}\right)$ in powers of v

$$f\left(v+\frac{2\pi i(\beta-\tau)}{\beta}\right)= \tag{A.87}$$
$$=(-1)^{2\lambda}\sum_{m=0}^{\infty}\frac{v^m}{m!}\sum_{\ell=0}^{2\lambda-1}\frac{(-2)^{4\lambda-\ell-1}\alpha^{\frac{2\lambda-\ell-2}{2}}\mathcal{H}_{2\lambda-\ell}\left(\sqrt{\alpha}\left(v+\frac{2\pi i(\beta-\tau)}{\beta}\right)\right)}{(2\lambda-\ell-1)!}c^{(\lambda)}_{\ell,m}(\beta,\tau),$$

where $c^{(\lambda)}_{\ell,m}(\beta,\tau)$ is given in (3.46):

$$c^{(\lambda)}_{\ell,m}(\beta,\tau)=e^{-\frac{2\pi i\lambda\tau}{\beta}}\sum_{j=0}^{\ell}\frac{(j+m)!}{(\ell-j)!(2\lambda+j+m)!j!}\,\mathrm{B}^{(2\lambda)}_{\ell-j}(\lambda)\,\mathcal{B}^{(2\lambda)}_{j+2\lambda+m}\left(\lambda;e^{-\frac{2\pi i\tau}{\beta}}\right). \tag{A.88}$$

Plugging the expansion (A.87) into the integral (A.86), we get a series representation for our correlator

$$\langle\mathcal{O}^{(\lambda)}(\tau)\rangle^{\text{disk}}_{\beta}=-\frac{i\pi^2(-1)^{2\lambda}\Gamma(2\lambda)\kappa^{2\lambda-\frac{3}{2}}\beta^{\frac{3}{2}}}{2^{2\lambda+3}\pi^{\frac{7}{2}}\tau^{\frac{3}{2}}(\beta-\tau)^{\frac{3}{2}}}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{\ell=0}^{2\lambda-1}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}c^{(\lambda)}_{\ell,m}(\beta,\tau)\times \tag{A.89}$$
$$\times\,\alpha^{\frac{2\lambda-\ell-2}{2}}\int_{-\infty}^{+\infty}\mathrm{d}v\;v^m\mathcal{H}_{2\lambda-\ell}\left(\sqrt{\alpha}\left(v+\frac{2\pi i(\beta-\tau)}{\beta}\right)\right)\left(v-\frac{2\pi i\tau}{\beta}\right)\,e^{-\frac{\beta v^2}{4\kappa\tau(\beta-\tau)}}.$$

To single out the dependence on κ, we scale our variable of integration as follows $v\to 2\sqrt{\frac{\kappa\tau(\beta-\tau)}{\beta}}$ and we get

$$\langle\mathcal{O}^{(\lambda)}(\tau)\rangle^{\text{disk}}_{\beta}=-\frac{i\pi^2(-1)^{2\lambda}\Gamma(2\lambda)\kappa^{2\lambda-\frac{3}{2}}\beta^{\frac{3}{2}}}{2^{2\lambda+3}\pi^{\frac{7}{2}}\tau^{\frac{3}{2}}(\beta-\tau)^{\frac{3}{2}}}\sum_{m=0}^{\infty}\frac{2^{m+1}(\kappa\tau(\beta-\tau))^{\frac{m+1}{2}}}{\beta^{\frac{m+1}{2}}m!}\sum_{\ell=0}^{2\lambda-1}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}c^{(\lambda)}_{\ell,m}(\beta,\tau)\times$$
$$\times\left(\frac{1}{4\kappa(\beta-\tau)}\right)^{\frac{2\lambda-\ell-2}{2}}\int_{-\infty}^{+\infty}\mathrm{d}v\;v^m\mathcal{H}_{2\lambda-\ell}\left(\sqrt{\frac{\tau}{\beta}}v+\frac{i\pi\sqrt{\beta-\tau}}{\beta\sqrt{\kappa}}\right)\left(2\sqrt{\frac{\kappa\tau(\beta-\tau)}{\beta}}v-\frac{2\pi i\tau}{\beta}\right)\,e^{-v^2}. \tag{A.90}$$

Next we exploit a simple rule holding for Hermite polynomials with shifted argument

$$\mathcal{H}_{2\lambda-\ell}\left(\sqrt{\frac{\tau}{\beta}}v+\frac{i\pi\sqrt{\beta-\tau}}{\beta\sqrt{\kappa}}\right)=\sum_{k=0}^{2\lambda-\ell}\binom{2\lambda-\ell}{k}\mathcal{H}_k\left(\sqrt{\frac{\tau}{\beta}}v\right)\left(\frac{2\pi i\sqrt{\beta-\tau}}{\beta\sqrt{\kappa}}\right)^{2\lambda-\ell-k}, \tag{A.91}$$

to rearrange our correlator in the form

$$\langle\mathcal{O}^{(\lambda)}(\tau)\rangle^{\text{disk}}_{\beta}=-\frac{i\pi^2(-1)^{2\lambda}\Gamma(2\lambda)\kappa^{2\lambda-\frac{3}{2}}\beta^{\frac{3}{2}}}{2^{2\lambda+3}\pi^{\frac{7}{2}}\tau^{\frac{3}{2}}(\beta-\tau)^{\frac{3}{2}}}\sum_{m=0}^{\infty}\frac{2^{m+1}(\kappa\tau(\beta-\tau))^{\frac{m+1}{2}}}{\beta^{\frac{m+1}{2}}m!}\sum_{\ell=0}^{2\lambda-1}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}c^{(\lambda)}_{\ell,m}(\beta,\tau)\times$$
$$\times\left(\frac{1}{4\kappa(\beta-\tau)}\right)^{\frac{2\lambda-\ell-2}{2}}\sum_{k=0}^{2\lambda-\ell}\binom{2\lambda-\ell}{k}\left(\frac{2\pi i\sqrt{\beta-\tau}}{\beta\sqrt{\kappa}}\right)^{2\lambda-\ell-k}\left(\sqrt{\frac{4\kappa\tau(\beta-\tau)}{\beta}}\mathrm{P}_{k,m+1}-\frac{2\pi i\tau}{\beta}\mathrm{P}_{k,m}\right), \tag{A.92}$$

where

$$\mathrm{P}_{k,m}=\int_{-\infty}^{\infty}dv\;v^m\mathcal{H}_k\left(\sqrt{\frac{\tau}{\beta}}v\right)e^{-v^2}. \tag{A.93}$$

This integral yields a polynomial of order k in $\sqrt{\frac{\tau}{\beta}}$ and its explicit expression in terms of the associated Legendre function is given in (A.104). Obviously $\mathrm{P}_{k,m}$ is different from zero only when $m+k$ is an even number. We shall use this selection rule to rearrange the two contributions proportional to $\mathrm{P}_{k,m+1}$ [(A)] and $\mathrm{P}_{k,m}$ [(B)] respectively. For the former the selection rule is $m+k+1=2p$ with $p=1,\cdots,\infty$. We can use this result to replace the sum over m with a sum over p

$$(A) = -\frac{i\pi^2(-1)^{2\lambda}\Gamma(2\lambda)}{2^{2\lambda}\pi^{\frac{7}{2}}}\sum_{p=1}^{\infty}\frac{\kappa^{p+\ell}}{(2p-k-1)!}\left(\frac{4\kappa\tau(\beta-\tau)}{\beta}\right)^{p+\ell-2\lambda}\times \tag{A.94}$$
$$\times\sum_{\ell=0}^{2\lambda-1}\sum_{k=0}^{\min(2\lambda-\ell,2p-1)}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}c^{(\lambda)}_{\ell,2p-k-1}(\beta,\tau)\binom{2\lambda-\ell}{k}\left(\frac{4\pi i\sqrt{\tau}(\beta-\tau)}{\beta^{\frac{3}{2}}}\right)^{2\lambda-\ell-k}\left(\frac{\tau}{\beta}\right)^{\frac{2\lambda-\ell-2}{2}}\mathrm{P}_{k,2p-k}.$$

Next we introduce a new index $n\equiv k+\ell$, which simply counts the power of the coupling constant

$$(A) = \frac{(-\pi i)^{2\lambda+1}\Gamma(2\lambda)}{(2\beta)^{2\lambda}\pi^{\frac{5}{2}}}\sum_{n=1}^{\infty}\left(\frac{4\kappa\tau(\beta-\tau)}{\beta}\right)^{n}\sum_{\ell=0}^{\min(n,2\lambda-1)}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}\times \tag{A.95}$$
$$\times\sum_{k=0}^{\min(2\lambda-\ell,2n-2\ell-1)}\frac{c^{(\lambda)}_{\ell,2n-2\ell-k-1}(\beta,\tau)}{(2n-2\ell-k-1)!}\binom{2\lambda-\ell}{k}\left(\frac{\beta^2}{4\pi i\tau(\beta-\tau)}\right)^{\ell+k}\left(\frac{\tau}{\beta}\right)^{\frac{k}{2}-1}\mathrm{P}_{k,2n-2\ell-k}.$$

The same analysis is done for the latter contribution, taking into account the constraint $m+k=2p$ with $p=0,1,\cdots,\infty$:

$$(B) = \left(\frac{2\pi i\tau}{\beta}\right)\frac{i\pi^2(-1)^{2\lambda}\Gamma(2\lambda)}{2^{2\lambda}\pi^{\frac{7}{2}}}\sum_{p=0}^{\infty}\frac{\kappa^{p+\ell}}{(2p-k)!}\left(\frac{4\kappa\tau(\beta-\tau)}{\beta}\right)^{p+\ell-2\lambda}\times \tag{A.96}$$
$$\times\sum_{\ell=0}^{2\lambda-1}\sum_{k=0}^{\min(2p,2\lambda-\ell)}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}c^{(\lambda)}_{\ell,2p-k}(\beta,\tau)\binom{2\lambda-\ell}{k}\left(\frac{4\pi i\sqrt{\tau}(\beta-\tau)}{\beta^{\frac{3}{2}}}\right)^{2\lambda-\ell-k}\left(\frac{\tau}{\beta}\right)^{\frac{2\lambda-\ell-2}{2}}\mathrm{P}_{k,2p-k}.$$

Setting again $n=p+\ell$, we get

$$(B) = -\left(\frac{2\pi i\tau}{\beta}\right)\frac{(-\pi i)^{2\lambda+1}\Gamma(2\lambda)}{(2\beta)^{2\lambda}\pi^{\frac{5}{2}}}\sum_{n=0}^{\infty}\left(\frac{4\kappa\tau(\beta-\tau)}{\beta}\right)^{n}\times \tag{A.97}$$
$$\times\sum_{\ell=0}^{\min(n,2\lambda-1)}\sum_{k=0}^{\min(2n-2\ell,2\lambda-\ell)}\frac{(-2)^{4\lambda-\ell-1}c^{(\lambda)}_{\ell,2n-2\ell-k}(\beta,\tau)}{(2n-2\ell-k)!(2\lambda-\ell-1)!}\binom{2\lambda-\ell}{k}\left(\frac{\beta^2}{4\pi i\tau(\beta-\tau)}\right)^{\ell+k}\left(\frac{\tau}{\beta}\right)^{\frac{k}{2}-1}\mathrm{P}_{k,2n-2\ell-k}.$$

Combining the two contributions, we obtain the final expansion

$$\langle\mathcal{O}^{(\lambda)}(\tau)\rangle^{\mathrm{disk}}_{\beta} = \frac{\pi^{2\lambda}}{\beta^{2\lambda}\sin^{2\lambda}\frac{\pi\tau}{\beta}}+\frac{(-\pi i)^{2\lambda+1}\Gamma(2\lambda)}{(2\beta)^{2\lambda}\pi^{\frac{5}{2}}}\sum_{n=1}^{\infty}\left(\frac{4\kappa\tau(\beta-\tau)}{\beta}\right)^{n}\sum_{\ell=0}^{\min(n,2\lambda-1)}\frac{(-2)^{4\lambda-\ell-1}}{(2\lambda-\ell-1)!}\times$$
$$\times\left[\sum_{k=0}^{\min(2n-2\ell-1,2\lambda-\ell)}\frac{c^{(\lambda)}_{\ell,2n-2\ell-k-1}(\beta,\tau)}{(2n-2\ell-k-1)!}\binom{2\lambda-\ell}{k}\left(\frac{\beta^2}{4\pi i\tau(\beta-\tau)}\right)^{\ell+k}\left(\frac{\tau}{\beta}\right)^{\frac{k}{2}-1}\mathrm{P}_{k,2n-2\ell-k}-\right.$$
$$\left.-2\pi i\sum_{k=0}^{\min(2n-2\ell,2\lambda-\ell)}\frac{c^{(\lambda)}_{\ell,2n-2\ell-k}(\beta,\tau)}{(2n-2\ell-k)!}\binom{2\lambda-\ell}{k}\left(\frac{\beta^2}{4\pi i\tau(\beta-\tau)}\right)^{\ell+k}\left(\frac{\tau}{\beta}\right)^{\frac{k}{2}}\mathrm{P}_{k,2n-2\ell-k}\right]. \tag{A.98}$$

A.11 Gaussian integrals of Hermite polynomials

We consider the following integral

$$I_k = \int_{-\infty}^{\infty} dv\ \mathcal{H}_k(av)\, e^{-v^2+xv}. \tag{A.99}$$

To find its expression for general k we construct the following generating functional

$$\begin{aligned} G(t) =& \sum_{k=0}^{\infty} \frac{t^k}{k!} I_k = \int_{-\infty}^{\infty} dv\ e^{-v^2+xv+2atv-t^2} = \sqrt{\pi} e^{\left(a^2-1\right)t^2+atx+\frac{x^2}{4}} = \\ =& \sqrt{\pi} e^{-\left(\sqrt{1-a^2}t\right)^2+2\frac{a}{\sqrt{1-a^2}}(\sqrt{1-a^2}t)\frac{x}{2}+\frac{x^2}{4}} = \sqrt{\pi} e^{\frac{x^2}{4}} \sum_{k=0}^{\infty} \frac{t^k(1-a^2)^{\frac{k}{2}}}{k!} \mathcal{H}_k\left(\frac{ax}{2\sqrt{1-a^2}}\right), \end{aligned} \tag{A.100}$$

where we used

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{H}_n(av) = e^{2atv-t^2}. \tag{A.101}$$

Thus

$$I_k = \sqrt{\pi}(1-a^2)^{\frac{k}{2}} \mathcal{H}_k\left(\frac{ax}{2\sqrt{1-a^2}}\right) e^{\frac{x^2}{4}}. \tag{A.102}$$

The integral P_{km} defined in (A.93) can be computed by taking the m^{th} derivative with respect to x of I_k and then setting $x=0$

$$\begin{aligned} \mathrm{P}_{km} =& \sqrt{\pi}\, \partial_x^m I_k|_{x=0} = \sum_{j=0}^{m} \binom{m}{j} (1-a^2)^{\frac{k}{2}} \partial_x^j \mathcal{H}_k\left(\frac{ax}{2\sqrt{1-a^2}}\right) \partial_x^{m-j} e^{\frac{x^2}{4}} \Bigg|_{x=0} = \\ =& \sqrt{\pi} \sum_{j=0}^{m} \binom{m}{j} \frac{k! a^j \left(1-a^2\right)^{\frac{k-j}{2}} H_{k-j}\left(\frac{ax}{2\sqrt{1-a^2}}\right)}{(k-j)!} e^{\frac{x^2}{4}} \left(-\frac{i}{2}\right)^{m-j} H_{m-j}\left(\frac{ix}{2}\right) \Bigg|_{x=0} = \\ =& \pi^{\frac{3}{2}} \sum_{j=0}^{m} \binom{m}{j} \frac{(-i)^{m-j}\Gamma(k+1) a^j \left(1-a^2\right)^{\frac{k-j}{2}} 2^{k-j}}{\Gamma(k-j+1)\Gamma\left(\frac{1}{2}(j-k+1)\right)\Gamma\left(\frac{1}{2}(j-m+1)\right)}. \end{aligned} \tag{A.103}$$

This sum can be easily evaluated in terms of hypergeometric functions once we extend the range of j to infinity. In fact the generic term, once written in terms of $\Gamma-$function, vanishes for $j \geq m+1$. We find

$$\begin{aligned} \mathrm{P}_{km} =& \pi^{3/2} 2^k (-i)^m \left(1-a^2\right)^{\frac{k}{2}} \left(\frac{{}_2F_1\left(-\frac{k}{2}, -\frac{m}{2}; \frac{1}{2}; \frac{a^2}{a^2-1}\right)}{\Gamma\left(\frac{1-k}{2}\right)\Gamma\left(\frac{1-m}{2}\right)} + \frac{2ia\, {}_2F_1\left(\frac{1-k}{2}, \frac{1-m}{2}; \frac{3}{2}; \frac{a^2}{a^2-1}\right)}{\sqrt{1-a^2}\Gamma\left(-\frac{k}{2}\right)\Gamma\left(-\frac{m}{2}\right)} \right) = \\ =& (-i)^m 2^{\frac{k-m-1}{2}} \pi \left(1-a^2\right)^{\frac{k-m-1}{4}} \mathcal{P}_{\frac{1}{2}(k-m-1)}^{\frac{1}{2}(k+m+1)} \left(-\frac{ia}{\sqrt{1-a^2}}\right), \end{aligned} \tag{A.104}$$

where $\mathcal{P}_\nu^\mu(x)$ is the Associated Legendre Function.

A.12 Perturbative expansion from Bell polynomials

In this appendix, we show how to obtain the coefficients A_n defined in (4.10). These determine the t-expansion of the disk and the trumpet partition functions. As in Section 4.1.1, we begin by decomposing the exponential term as

$$e^{-I(u,t;s)} = e^{-us^2/2} \exp\left(us^2 \left(\frac{1}{2} - \frac{1}{\sqrt{1-ts^2}+1} \right) \right) , \tag{A.105}$$

and define A_n as the coefficients in the expansion of the second term about $t=0$,

$$\exp\left(us^2 \left(\frac{1}{2} - \frac{1}{\sqrt{1-ts^2}+1} \right) \right) = \sum_{n=0}^{\infty} A_n t^n , \tag{A.106}$$

where

$$A_n = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \exp\left(us^2 \left(\frac{1}{2} - \frac{1}{\sqrt{1-ts^2}+1} \right) \right) \Bigg|_{t=0} . \tag{A.107}$$

To compute the n-th derivative above, we interpret the exponential as a composite function and make repeated use of the Faà di Bruno formula

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} (f \circ g)(t) = \sum_{k=0}^{n} f^{(k)}(g(t)) \, \mathrm{B}_{n,k}(g'(t), g''(t), \ldots, g^{(n-k+1)}(t)) \tag{A.108}$$

involving Bell polynomials $\mathrm{B}_{n,k}$.

We apply (A.108) a first time, with the square root in (A.107) playing the role of the function g. This gives

$$A_n = \frac{1}{n!} \sum_{k=0}^{n} C_k \, \mathrm{B}_{n,k} \left(\left\{ -\frac{(2j-3)!! \, s^{2j}}{2^j (1-ts^2)^{j-1/2}} \right\}_{j=1,\ldots,n-k+1} \right) \Bigg|_{t=0} , \tag{A.109}$$

where

$$C_k = \frac{\mathrm{d}^k}{\mathrm{d}z^k} \exp\left(us^2 \left(\frac{1}{2} - \frac{1}{z+1} \right) \right) \Bigg|_{z=1} . \tag{A.110}$$

We then apply (A.108) again to determine C_k. This time, the function g is identified with the term within parenthesis inside the exponential,

$$C_k = \sum_{l=0}^{k} \frac{\mathrm{d}^l}{\mathrm{d}y^l} e^{us^2 y} \bigg|_{y=0} \mathrm{B}_{k,l} \left(\left\{ \frac{(-1)^{j+1} j!}{(z+1)^{j+1}} \right\}_{j=1,\ldots,k-l+1} \right) \Bigg|_{z=1} . \tag{A.111}$$

Bell polynomials enjoy the identity

$$\mathrm{B}_{n,k}(abx_1, ab^2 x_2, \ldots, ab^{n-k+1} x_{n-k+1}) = a^k b^n \, \mathrm{B}_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) . \tag{A.112}$$

We can make use of the above to rewrite the expressions for A_n and C_k in terms of known combinations of Bell polynomials,

$$A_n = \frac{1}{n!} \left(\frac{s^2}{2} \right)^n \sum_{k=0}^{n} (-1)^k C_k \, \mathrm{B}_{n,k} \left((-1)!!, 1!!, 3!! \ldots, (2(n-k)-1)!! \right) , \tag{A.113}$$

$$C_k = \frac{1}{(-2)^k} \sum_{l=0}^{k} \left(-\frac{us^2}{2} \right)^l \mathrm{B}_{k,l} \left(1!, 2!, \ldots, (k-l+1)! \right) . \tag{A.114}$$

Specifically,

$$\mathrm{B}_{n,k}(1!,2!,3!,(n-k+1)!) = \binom{n-1}{k-1}\frac{n!}{k!} \tag{A.115}$$

are *Lah numbers*, while [107]

$$\mathrm{B}_{n,k}((-1)!!,1!!,3!!,(2(n-k)-1)!!) = [2(n-k)-1]!!\binom{2n-k-1}{2(n-k)}\,. \tag{A.116}$$

By plugging these identities in the expressions above, we arrive at

$$A_n = \frac{1}{2^{2n}n!}\sum_{k=1}^{n}\frac{(2n-k-1)!}{(n-k)!}\sum_{l=1}^{k}\binom{k}{l}\left(-\frac{u}{2}\right)^l\frac{s^{2(l+n)}}{(l-1)!}\,. \tag{A.117}$$

It is convenient to exchange the two sums with

$$A_n = \frac{1}{2^{2n}n!}\sum_{l=1}^{n}\left(-\frac{u}{2}\right)^l\frac{s^{2(l+n)}}{(l-1)!}\sum_{k=l}^{n}\frac{(2n-k-1)!}{(n-k)!}\binom{k}{l}\,. \tag{A.118}$$

The final expression (4.11) is obtained by using

$$\sum_{k=l}^{n}\frac{(2n-k-1)!}{(n-k)!}\binom{k}{l} = \frac{(2n)!\,(n-1)!}{(n+l)!\,(n-l)!}\,, \tag{A.119}$$

and

$$\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{(-x)^l}{(n+l)!} = -\frac{(n-1)!}{(2n)!}\,x\,L_{n-1}^{n+1}(x)\,. \tag{A.120}$$

A.13 Directional Laplace transforms of hypergeometric functions

In this appendix, we collect some useful results concerning the directional Laplace transforms of the hypergeometric functions appearing in the t-expansion of the disk and the trumpet partition functions.

We start from the following identity,

$$\int_0^\infty \mathrm{d}\zeta\, e^{-\beta\zeta}\,{}_2F_1\!\left(\frac{1}{2}-m,\frac{1}{2}+m;1;-\alpha\zeta\right) = \frac{e^{\beta/2\alpha}}{\sqrt{\pi\alpha\beta}}\,K_m\!\left(\frac{\beta}{2\alpha}\right), \tag{A.121}$$

which holds for $\alpha>0$, $\mathrm{Re}\,\beta>0$, and $m\in\mathbb{Z}$. Over the integration range, the hypergeometric is evaluated on the negative real axis. When continuing this result to negative values of α, one should be careful about the fact that the Gauss hypergeometric function has a branch cut on the positive real axis. Approaching the branch cut from above and below gives the lateral Laplace transforms

$$\begin{aligned}\int_0^{e^{\pm i0}\infty}\mathrm{d}\zeta\, e^{-\beta\zeta}\,{}_2F_1\!\left(\frac{1}{2}-m,\frac{1}{2}+m;1;\gamma\zeta\right) &= \frac{e^{-\beta/2\gamma}}{\sqrt{-\pi\gamma\beta\mp i0}}\,K_m\!\left(-\frac{\beta}{2\gamma}\pm i0\right)\\ &= \frac{e^{-\beta/2\gamma}}{\sqrt{\pi\gamma\beta}}\left[\pi\, I_m\!\left(\frac{\beta}{2\gamma}\right)\pm(-1)^m i\,K_m\!\left(\frac{\beta}{2\gamma}\right)\right],\end{aligned} \tag{A.122}$$

where $\gamma > 0$. The discontinuity in the directional Laplace transform is reflected in the discontinuity of the square root and of the modified Bessel function K_m, the both having a branch cut along the negative real axis. In the last steps of the identity above we used

$$K_m(-x \pm \mathrm{i}0) = (-1)^m K_m(x) \mp \mathrm{i}\pi\, I_m(x)\,, \tag{A.123}$$

which holds for $m \in \mathbb{Z}$ and $x > 0$.

The difference between the two lateral Laplace transforms can also be obtained through an integral over a Hankel-like contour wrapping the branch cut, which, in turn, amounts to taking the Laplace transform of the discontinuity of the hypergeometric function. We use the identity

$$\begin{aligned} &{}_2F_1(a,b;c;x+\mathrm{i}0) - {}_2F_1(a,b;c;x-\mathrm{i}0) \\ &\quad = \frac{2\pi\mathrm{i}\,\Gamma(c)\,\theta(x-1)}{\Gamma(a)\,\Gamma(b)\,\Gamma(1+c-a-b)}\,(x-1)^{c-a-b}\,{}_2F_1(c-a,c-b;1+c-a-b;1-x)\,, \end{aligned} \tag{A.124}$$

which holds for $x > 1$, to write

$$\begin{aligned} &\int_0^\infty \mathrm{d}x\, e^{-\beta x}\left[{}_2F_1\left(\frac{1}{2}-m,\frac{1}{2}+m;1;\gamma x+\mathrm{i}0\right) - {}_2F_1\left(\frac{1}{2}-m,\frac{1}{2}+m;1;\gamma x-\mathrm{i}0\right)\right] \\ &\quad = (-1)^m 2\mathrm{i}\int_{1/\gamma}^\infty \mathrm{d}x\, e^{-\beta x}\,{}_2F_1\left(\frac{1}{2}-m,\frac{1}{2}+m;1;1-\gamma x\right) \\ &\quad = (-1)^m 2\mathrm{i}e^{-\beta/\gamma}\int_0^\infty \mathrm{d}x\, e^{-\beta x}\,{}_2F_1\left(\frac{1}{2}-m,\frac{1}{2}+m;1;-\gamma x\right) \\ &\quad = (-1)^m 2\mathrm{i}\,\frac{e^{-\beta/2\gamma}}{\sqrt{\pi\gamma\beta}}\,K_m\left(\frac{\beta}{2\gamma}\right). \end{aligned} \tag{A.125}$$

A.14 Confluent hypergeometric functions

The confluent hypergeometric differential equation

$$z\,\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (b-z)\,\frac{\mathrm{d}w}{\mathrm{d}z} - aw = 0 \tag{A.126}$$

has solution

$$w = c_1\,{}_1F_1(a;b;z) + c_2\,U(a,b,z)\,, \tag{A.127}$$

where the functions

$${}_1F_1(a;b;z) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k}\frac{z^k}{k!}\,, \tag{A.128}$$

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}\,{}_1F_1(a;b;z) + \frac{\Gamma(b-1)}{\Gamma(a)}\,z^{1-b}\,{}_1F_1(a-b+1;2-b;z)\,, \tag{A.129}$$

are referred to, respectively, as Kummer and Tricomi confluent hypergeometric functions. Notice that ${}_1F_1(a;b;z)$ does not exist when b is a nonpositive integer, and that (A.129) holds for b noninteger. One can extend the definition of the ${}_1F_1$ by using

$$\lim_{b\to -n}\frac{{}_1F_1(a;b;z)}{\Gamma(b)} = \frac{(a)_{n+1}}{(n+1)!}\,z^{n+1}\,{}_1F_1(a+n+1;n+2;z)\,, \tag{A.130}$$

for n nonnegative integer. The function $U(a,b,z)$ has a branch cut in the complex z-plane along $z \in (-\infty, 0]$. For $x<0$,

$$\begin{aligned} U(a,b,x+\mathrm{i}0) &= U(a,b,x)\,, \\ U(a,b,x-\mathrm{i}0) &= e^{2\mathrm{i}b\pi}\,U(a,b,x) - \frac{2\mathrm{i}\pi e^{\mathrm{i}b\pi}}{\Gamma(a-b+1)\,\Gamma(b)}\,{}_1F_1(a;b;x)\,. \end{aligned} \tag{A.131}$$

The Kummer confluent hypergeometric function can be recast in terms of generalized Laguerre polynomials as

$${}_1F_1(a;b;z) = \frac{\Gamma(1-a)\,\Gamma(b)}{\Gamma(b-a)}\,L_{-a}^{b-1}(z)\,. \tag{A.132}$$

Two asymptotic behaviors are particularly relevant for the present work: for $x \to +\infty$,

$$U(a,b,x) \sim x^{-a}\,, \tag{A.133}$$

$${}_1F_1(a;b;-x) \sim \frac{\Gamma(b)}{\Gamma(b-a)}\,x^{-a}\left(1+O\!\left(\frac{1}{x}\right)\right) + e^{-x}\ldots\,. \tag{A.134}$$

The confluent hypergeometric functions enjoy the two following integral representations. For $\mathrm{Re}\,a>0$ and $\mathrm{Re}\,z>0$,

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{d}t\; e^{-zt}\,t^{a-1}\,(t+1)^{b-a-1}\,. \tag{A.135}$$

For $\mathrm{Re}\,a>0$,

$${}_1F_1(a;b;z) = \frac{1}{2\pi\mathrm{i}}\,\frac{\Gamma(b)\,\Gamma(a-b+1)}{\Gamma(a)}\int_0^{(1+)} \mathrm{d}t\; e^{zt}\,t^{a-1}\,(t-1)^{b-a-1}\,. \tag{A.136}$$

The last integral is taken over a contour starting and ending in 0 and encircling 1 in the positive sense.

For $\mathrm{Re}\,y>1/2$, the two multiplication theorems hold,

$$\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{1}{y}-1\right)^{j} U(a-j,b,x) = e^{x(1-y)}\,y^{b-a}\,U(a,b,xy)\,, \tag{A.137}$$

$$\sum_{j=0}^{\infty}\frac{(b-a)_j}{j!}\left(1-\frac{1}{y}\right)^{j} {}_1F_1(a-j;b;x) = e^{x(1-y)}\,y^{b-a}\,{}_1F_1(a;b;xy)\,. \tag{A.138}$$

A.15 Some useful identities

In this appendix, we prove various identities that are used in Section 5.4.2. We start from (5.68) and (6.12). The sum over s can be performed by using the integral representation

$$z\,{}_1F_1(a+1;2;z) = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{\mathrm{d}u}{u}\,e^{\frac{zu}{u+y}}\left(-\frac{y}{u}\right)^{-a}\,, \tag{A.139}$$

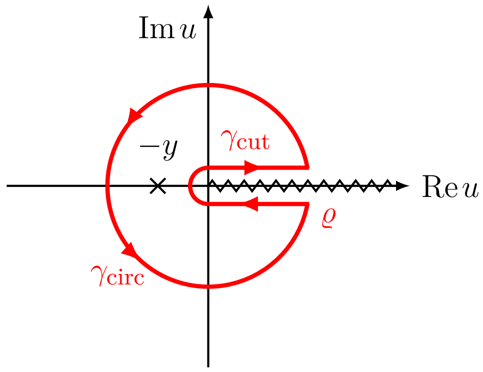


Figure A.3: The integration contour for (A.139). The integrand has a branch cut along the positive real u-axis if a is not integer, and an essential singularity at $u=-y$. The contour γ is the sum of a circle $\gamma_{\rm circ}$ of radius $\varrho > y$ and of a Hankel-like contour $\gamma_{\rm cut}$ wrapping the cut.

which is obtained directly from (A.136) through integration by parts and with the change of integration variable

$$t = \frac{u}{u+y}\,, \tag{A.140}$$

with $y>0$. The integral is performed over the contour γ depicted in Figure A.3.

For our purposes, it is convenient to use $y=-2NY$. This gives[14]

$$\begin{aligned} z^{+}_{\mathbf{m}}(\alpha,\tau) = \frac{(-1)^{\nu}\pi^{\frac{N}{2}}e^{X}}{N!\,G^2(N+1)}\frac{\mathrm{i}}{2}\oint_{\gamma}\frac{\mathrm{d}u}{u}\,e^{\frac{uW}{u-2NY}}\,(-u)^{\frac{N^2}{2}}\,\mathbf{V}\Bigg[(\pi^2|\mathbf{m}|^2)^{\nu} \\ \times\Bigg({}_1\tilde{F}_2\Big(1;\tfrac{N^2}{2},\nu+1;-\pi^2|\mathbf{m}|^2u\Big)-\pi|\mathbf{m}|\sqrt{-u}\,{}_1\tilde{F}_2\Big(1;\tfrac{N^2+1}{2},\nu+\tfrac{3}{2};-\pi^2|\mathbf{m}|^2u\Big)\Bigg)\Bigg] \end{aligned} \tag{A.141}$$

$$\begin{aligned} z^{-}_{\mathbf{m}}(\alpha,\tau) = \frac{(-1)^{\nu}\pi^{\frac{N}{2}}e^{X}}{N!\,G^2(N+1)}\frac{\mathrm{i}}{2}\oint_{\gamma}\frac{\mathrm{d}u}{u}\,e^{\frac{uW}{u-2NY}}\,(-u)^{\frac{N^2}{2}} \\ \times\mathbf{V}\Bigg[-(\pi^2|\mathbf{m}|^2)^{\nu}\sum_{s=1-\frac{N^2}{2}}^{-1/2}\frac{(-\pi^2|\mathbf{m}|^2u)^{s}}{\Gamma(s+1+\nu)\,\Gamma(s+N^2/2)}\Bigg]\,. \end{aligned} \tag{A.142}$$

Notice that the terms obtained by summing over half-integer s's in (5.68), namely (A.142) and the term with the second regularized hypergeometric in (A.141), are free from branch cuts along the positive real u-axis. These are, in fact, the terms associated with an integer a in (A.139).

Let us focus first on $z^{+}_{\mathbf{m}}(\alpha,\tau)$, with the goal of proving Eq. (5.81). We separate the contributions coming from $\gamma_{\rm cut}$ and $\gamma_{\rm circle}$ and denote them as $z^{+}_{\mathbf{m}}(\alpha,\tau)=I_{\rm cut}+I_{\rm circ}$. As mentioned, the integral over the Hankel-like contour receives contributions only from the first regularized hypergeometric. We use (5.75) and the same argument of Section (5.4.1) to write

$$I_{\rm cut} = \frac{\pi e^{X}}{N!\,G^2(N+1)}\frac{\mathrm{i}}{2}\int_{\gamma_{\rm cut}}\frac{\mathrm{d}u}{u}\,e^{\frac{uW}{u-2NY}}\sqrt{-u}\,\mathbf{V}\Big[|\mathbf{m}|^{1-\frac{N}{2}}\,u^{\frac{N}{4}}\,J_{\frac{N}{2}-1}(2\pi|\mathbf{m}|\sqrt{u})\Big]\,. \tag{A.143}$$

[14]The absence of the $(-1)^m$ factor coming from (6.12) is merely due to the fact that m is always even if N is chosen to be odd.

The counterterms that appear in (5.81) come from the integral over the circle. We can split the exponential in the integrand in terms of its power expansion in $1/u$ as

$$e^{\frac{uW}{u-2NY}} = \mathcal{E}_1 + \mathcal{E}_2 \,, \tag{A.144}$$

where

$$\mathcal{E}_1 = e^W + We^W \sum_{k=1}^{k_{\max}} \frac{1}{k} \left(\frac{2NY}{u}\right)^k L^1_{k-1}(-W) \,, \tag{A.145}$$

$$\mathcal{E}_2 = We^W \sum_{k=k_{\max}+1}^{\infty} \frac{1}{k} \left(\frac{2NY}{u}\right)^k L^1_{k-1}(-W) \,, \tag{A.146}$$

and $k_{\max} = (N^2-1)/2$. With an obvious notation we denote $I_{\text{circ}} = I_{\text{circ},1} + I_{\text{circ},2}$. We notice that in

$$\begin{aligned} I_{\text{circ},1} = {}& \frac{\pi e^X}{N!\,G^2(N+1)} \frac{\mathrm{i}}{2} \int_{\gamma_{\text{circ}}} \frac{\mathrm{d}u}{u}\, \mathcal{E}_1 \sqrt{-u}\, \mathbf{V}\left[|\mathbf{m}|^{1-\frac{N}{2}}\, u^{\frac{N}{4}}\, J_{\frac{N}{2}-1}(2\pi|\mathbf{m}|\sqrt{u})\right] \\ & + \frac{(-1)^\nu \pi^{2\nu+\frac{N}{2}+1} e^X}{N!\,G^2(N+1)} \frac{\mathrm{i}}{2} \int_{\gamma_{\text{circ}}} \mathrm{d}u\, \mathcal{E}_1\, u^{k_{\max}}\, \mathbf{V}\left[|\mathbf{m}|^{2\nu+1}\, {}_1\tilde{F}_2\left(1; \tfrac{N^2+1}{2}, \nu+\tfrac{3}{2}; -\pi^2|\mathbf{m}|^2 u\right)\right] \end{aligned} \tag{A.147}$$

the second line vanishes, as it is the integral of an analytic function over a closed contour. Moreover, we can use the fact that now the integrand in the first line does not have an essential singularity at $u = 2NY$, and it is possible to shrink the contour around the cut. In doing so, one can combine the above with (A.143) to generate the appropriate counterterms. In taking $\varrho \to \infty$, we see that $I_{\text{cut}} + I_{\text{circ},1}$ reproduces (5.81). What is left to show now is that, in the same limit, the remaining term $I_{\text{circ},2}$ vanishes. To this end, we use

$${}_1\tilde{F}_2\left(1; \tfrac{N^2+1}{2}, \nu+\tfrac{3}{2}; -z^2\right) = z^{\frac{N}{2}-N^2}\, J_{1-\frac{N}{2}}(2z) + \sum_{s=1-\frac{N^2}{2}}^{-\frac{1}{2}} \frac{(-z^2)^{s-\frac{1}{2}}}{\Gamma(s+\nu+1)\,\Gamma(s+N^2/2)} \,. \tag{A.148}$$

The two Bessel functions combine in a Hankel function of the first kind with

$$(-1)^\nu J_{\frac{N}{2}-1}(2z) + \mathrm{i}J_{1-\frac{N}{2}}(2z) = (-1)^\nu H^{(1)}_{\frac{N}{2}-1}(2z) \,. \tag{A.149}$$

The asymptotic behavior of the Hankel function, namely

$$H^{(1)}_\nu(2z) \sim \sqrt{1/(\pi z)}\, e^{\mathrm{i}(2z-\nu\pi/2-\pi/4)} \qquad \text{for } z \to \infty \,, \tag{A.150}$$

is sufficient to show that

$$\begin{aligned} I_{\text{circ},2} = {}& \frac{\pi^{N/2} e^X}{N!\,G^2(N+1)} \frac{\mathrm{i}}{2} \int_{\gamma_{\text{circ}}} \frac{\mathrm{d}u}{u}\, \mathcal{E}_2 \sqrt{-u} \\ & \times \mathbf{V}\left[(\pi|\mathbf{m}|)^{1-\frac{N}{2}} (\mathrm{i}\sqrt{-u})^{\frac{N}{2}}\, H^{(1)}_{\frac{N}{2}-1}(2\mathrm{i}\pi|\mathbf{m}|\sqrt{-u}) + \sum_{s=1-\frac{N^2}{2}}^{-\frac{1}{2}} \frac{(-\pi^2|\mathbf{m}|^2)^{s+\nu}\, u^{s+k_{\max}}}{\Gamma(s+\nu+1)\,\Gamma(s+N^2/2)}\right] \end{aligned} \tag{A.151}$$

vanishes in the $\varrho \to \infty$ limit for the Jordan's lemma. This concludes the proof of (5.81).

We are now in the position to comment on the convergence of the sum over all flux sectors at $\tau < 0$. In fact, from (A.148) we see that $z^{-}_{\mathfrak{m}}(\alpha,\tau)$ cancels the finite sum generated by the second regularized hypergeometric in $z^{+}_{\mathfrak{m}}(\alpha,\tau)$ and one is left with an expression with the sole Hankel function, namely

$$z_{\mathfrak{m}}(\alpha,\tau) = \frac{\pi\, e^{X}}{N!\, G^2(N+1)} \frac{\mathrm{i}}{2} \oint_\gamma \frac{\mathrm{d}u}{u}\, e^{\frac{uW}{u-2NY}} \sqrt{-u}\, \mathbf{V}\left[|\mathfrak{m}|^{1-\frac{N}{2}} (\mathrm{i}\sqrt{-u})^{\frac{N}{2}} H^{(1)}_{\frac{N}{2}-1}(2\mathrm{i}\pi|\mathfrak{m}|\sqrt{-u}) \right]. \tag{A.152}$$

By closing the contour γ around the essential singularity at $u = 2NY$, the Hankel function generates, according to (A.150), an exponential term of the form $e^{-2\pi|\mathfrak{m}|\sqrt{-2NY}}$ that ensures the convergence of the sum over $\mathfrak{m}$.

A.16 Some useful tools

In this appendix, we follow the notation of [16].

Let us consider the N-dimensional integral

$$I_{\nu,\varepsilon} = \int\limits_{\Omega\setminus B_\varepsilon} \mathrm{d}\mathbf{x}\, \frac{f(\mathbf{x})}{|\mathbf{x}|^\nu}\,, \tag{A.153}$$

where Ω is some region of $\mathbb{R}^N$ that contains the origin, and f is sufficiently differentiable. The integral $I_{\nu,0}$ converges for $\mathrm{Re}\,\nu < N$. The *Hadamard finite-part integral*

$$⨎_{\Omega} \mathrm{d}\mathbf{x}\, \frac{f(\mathbf{x})}{|\mathbf{x}|^\nu} = \lim_{\varepsilon\to 0}\Big(I_{\nu,\epsilon} - \mathbf{H}_{\nu,\varepsilon} f(\mathbf{0}) \Big)\,, \tag{A.154}$$

is the analytic continuation of $I_{\nu,0}$ to $\nu \in \mathbb{C}\setminus(\mathbb{N}+N)$ defined through the subtraction generated the differential operator

$$\mathbf{H}_{\nu,\varepsilon} = \sum_{n=0}^{\lfloor \mathrm{Re}\,\nu - N\rfloor} \frac{1}{n!} \int\limits_{\mathbb{R}^d\setminus B_\varepsilon} \mathrm{d}\mathbf{x}\, \frac{(\mathbf{x}\cdot\boldsymbol{\nabla})^n}{|\mathbf{x}|^\nu}\,. \tag{A.155}$$

For $\nu \in (\mathbb{N}+N)$, we define instead

$$\mathbf{H}_{\nu,\varepsilon} = \sum_{n=0}^{\nu-N-1} \frac{1}{n!} \int\limits_{\mathbb{R}^d\setminus B_\varepsilon} \mathrm{d}\mathbf{x}\, \frac{(\mathbf{x}\cdot\boldsymbol{\nabla})^n}{|\mathbf{x}|^\nu} + \frac{1}{(\nu-N)!} \int\limits_{B_1\setminus B_\varepsilon} \mathrm{d}\mathbf{x}\, \frac{(\mathbf{x}\cdot\boldsymbol{\nabla})^{\nu-N}}{|\mathbf{x}|^\nu}\,, \tag{A.156}$$

which amounts to dropping logarithmic and power-law divergences in $I_{\nu,\varepsilon}$.

The *Epstein zeta function*

$$\mathrm{Z}\left|\begin{smallmatrix}\mathbf{y}\\ \mathbf{p}\end{smallmatrix}\right|(\nu) = \sum_{\mathbf{x}\neq\mathbf{y}} \frac{e^{-2\pi\mathrm{i}\mathbf{p}\cdot\mathbf{x}}}{|\mathbf{x}-\mathbf{y}|^\nu}\,, \tag{A.157}$$

is smooth in $\mathbf{p} \in \mathbb{R}^N \setminus \mathbb{Z}^N$. By subtracting the singularity in $\mathbf{p} = \mathbf{0}$ one can define the regularized Epstein zeta function

$$\mathrm{Z}^{\mathrm{reg}} \left| \begin{smallmatrix} \mathbf{y} \\ \mathbf{p} \end{smallmatrix} \right| (\nu) = e^{2\pi \mathrm{i} \mathbf{p}\cdot\mathbf{y}}\, \mathrm{Z} \left| \begin{smallmatrix} \mathbf{y} \\ \mathbf{p} \end{smallmatrix} \right| (\nu) - \frac{\Gamma(N/2 - \nu/2)}{\pi^{N/2-\nu}\,\Gamma(\nu/2)} \, |\mathbf{p}|^{N-\nu} \,, \tag{A.158}$$

which is analytic in $\mathbf{p} = \mathbf{0}$.

Let us now consider a polynomial P. The sum

$$S_\nu = \sum_{\mathbf{x}\neq\mathbf{0}} \frac{P(\mathbf{x})}{|\mathbf{x}|^\nu} \tag{A.159}$$

is well-defined for $\mathrm{Re}(\nu) > N + \deg P$. A meromorphic continuation of S_ν outside the region where it converges is given by [15]

$$S_\nu = \lim_{\beta\to 0} \left(\sum_{\mathbf{x}\neq\mathbf{0}} e^{-\beta|\mathbf{x}|^2} \frac{P(\mathbf{x})}{|\mathbf{x}|^\nu} - \int\!\!\!\!\!\!-_{\mathbb{R}^N} \mathrm{d}\mathbf{x}\, e^{-\beta|\mathbf{x}|^2} \frac{P(\mathbf{x})}{|\mathbf{x}|^\nu} \right) . \tag{A.160}$$

Alternatively, the same continuation can be obtained by taking derivatives of the regularized Epstein zeta function, i.e.

$$S_\nu = P\!\left(\frac{\mathrm{i}\boldsymbol{\nabla}_{\mathbf{p}}}{2\pi} \right) \mathrm{Z}^{\mathrm{reg}} \left| \begin{smallmatrix} \mathbf{0} \\ \mathbf{p} \end{smallmatrix} \right| (\nu) \Big|_{\mathbf{p}=0} \,. \tag{A.161}$$

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