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Quantum aspects of one-dimensional sectors in ABJ(M) theory

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Sommario

In questa tesi vengono investigate le proprietà quantistiche di settori unidimensionali nella teoria ABJ(M), i.e. difetti rappresentati dalla linea topologica (libera) e dalla linea di Wilson 1/2-BPS. Nel primo caso, costruiamo il settore topologico di ABJ(M) operando il twist tra il gruppo conforme sulla linea e un sottogruppo di R-simmetria. Calcoliamo le funzioni di correlazione dell'operatore superprimario in questo settore, portando i conti in teoria delle perturbazioni a due loop per la funzione a due punti, comparandoli con le predizioni ottenute da un modello di matrici con deformazioni massive e la sua relazione congetturata con la carica centrale della teoria. Nel secondo caso, anche avendo la stessa algebra di simmetria della linea libera, le interazioni del loop di Wilson 1/2-BPS implicano la ricombinazione dei supermultipletti. Questo ci forza a studiare più approfonditamente la struttura generale del loop, partendo dalla sua forma in termini di supermatrici. Generalizziamo le supercariche rappresentandole sullo spazio delle supermatrici, costruiamo il supermultipletto dell'operatore di dislocamento e calcoliamo la funzione di Bremsstrahlung partendo dalle funzioni di correlazione di operatori inseriti nel loop. Infine, calcoliamo la dimensione anomala per gli operatori superprimari del nuovo multipletto lungo.

Abstract

In this thesis we investigate the quantum properties of one-dimensional sectors in the ABJ(M) theory, namely the topological (free) line and the 1/2-BPS Wilson Line defects. In the first case, we build the topological sector twisting the conformal line with a subgroup of the R-symmetry. We compute the correlation functions of the superprimary operator in this sector, pushing the perturbation theory up to two-loops for the two-point function, matching localization predictions from a mass deformed Matrix Model and its conjectured relation with the central charge of the theory. In the second case, despite having the same symmetry algebra of the free line, the 1/2-BPS Wilson Loop interactions lead to supermultiplet recombination. This force us to investigate more the general structure of the loop, starting from its supermatrix form. We generalize the supercharges representating them on the supermatrix space, we build the dispalcement supermultiplet and we compute the Bremsstrahlung function by correlation function of operator insertions. We then compute the anomalous dimension of the new long multiplet superprimary.

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[1]Details can be found at http://creativecommons.org/licenses/by/4.0/.

To my family

Acknowledgements

If I have seen further, it is by standing
upon the shoulders of Giants.

Sir Isaac Newton

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Introduction

The aesthetic appeal of symmetry has been a guide, sometimes a tyrannic one, for philosophers of nature since the dawn of science.

Di Francesco, Mathieu, Senechal [1]

For a theoretical physicist, beautiful means exact. From the very first days, physicists have been striving to find exact results. In this puzzle, symmetries are a useful tool to reduce the complexity in a description of a system. They constrain the problems, reaching simplifications and in particular cases allowing exact results. The prototypical example is a three-dimensional isotropic system, whose physical observables are reduced to be functions of only one variable. Furthermore, in quantum mechanics, symmetries are realized as invariance of the action under group transformation. This requirement dictates the shape of the action allowed by symmetries.

Quantum Field Theories (QFT) are built to be invariant under the space-time symmetry represented by the Poincaré group $\mathbb{R}^{1,3} \rtimes SO(1,3)$. In particular, in the *Standard Model* (SM) of particle physics, three of the four fundamental forces arise as connections on a principal bundle, whose fibers are preserved by the action of the Lie group $U(1) \times SU(2) \times SU(3)$ (respectively, electromagnetism, weak and strong interactions). Some "accidental symmetries", e.g. the baryon number conservation are present.

Nevertheless, within the SM the best only viable technique is perturbation theory in the weakly coupled regime. The number of perturbative (Feynman) diagrams grows factorially with the loop order and in the present days, with some advanced amplitudology, computations can be pushed up to five loops. Moreover, perturbative results are blind to some additional effects, called indeed non-perturbative effects. Besides, perturbative results are written in terms of asymptotic series and people are studying techniques to *resurge* non-perturbative aspects from asymptotic series, and vice versa [2].

Supersymmetry (SUSY) was introduced to overcome these difficulties. Developed as a SM extension, it introduces a supersymmetric partner for every particle.

The spectrum of the theory is then an equal number of bosons and fermions, organized in supermultiplets. This symmetry between particles often leads to divergence cancellation in quantum corrections. In certain cases, we have enough symmetries to implement the supersymmetric localization, which permits us to reduce the Path Integral to a finite-dimensional integral. In particular, localization produces exact results. Phenomenologically, it represents an interesting way to solve longstanding issues such as the hierarchy problem or the origin of dark matter. However, we have no direct proof that supersymmetry is realized in Nature yet, at least at the energy scales we are able to explore (but it will be very awkward if we do not find anything until the Plank scale).

From the renormalization point of view, we see that QFTs at their fixed point display conformal symmetry. Conformal transformations are those that leave the metric invariant up to an overall coefficient, called scale factor. Remarkably, the Poincaré group coincides with the conformal group isometries. Therefore, we can consider the conformal group as an extension of Poincaré symmetry. Conformal Field Theories (CFT) are so constrained that in principle we can compute all the data just by consistency conditions coming from the symmetries. This approach is called *Conformal Bootstrap* and, in the most general case, it is hopeless to solve analytically unless the spectrum of the theory is finite. An interesting question is then whether the theory admits or not subsectors closed under Operator Product Expansion (OPE) leading to a solvable truncation of the Bootstrap equation. In [3], the authors argue that a subsector like this can be realized by the topological one, with the additional simplification of not having coordinates dependence.

An astonishing feature of conformal symmetry is its perfect compatibility with supersymmetry. At the price of doubling the SUSY generators, we obtain a closed superalgebra, that we called *superconformal algebra*. Theories constrained by this algebra are called SuperConformal Field Theories (SCFT). The amount of symmetry present here is often enough to allow exact results. Moreover, we can directly see this effect in perturbation theory, since the symmetries prevent correlation functions to receive divergent quantum corrections.

Moreover, the conformal group in d-dimensions is the same symmetry group of the Anti-De Sitter space in $d+1$-dimensions. A conjecture about the duality between conformal gauge theories in d-dimension and string theories on Anti-De Sitter space in d+1-dimension has been formulated [4]; the duality implies that strongly coupled gauge theories are dual to weakly coupled string theories and vice versa. It is known as *holographic duality* or *AdS/CFT correspondence*. Although the correspondence passes too many non-trivial tests, a direct proof is still missing. To cite the most relevant examples, we know that $\mathcal{N}=4$ Super Yang-Mills in four dimensions is dual to type IIB string theory on $AdS_5 \times S^5$; ABJ(M), that is a $\mathcal{N}=6$

Chern–Simons (C–S) matter theory in three dimensions, is dual to an M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, k being the C–S level.

A central role in the AdS/CFT correspondence is played by the Wilson Loop, known to be dual to the fundamental string in the gravity theory. Wilson Loops are the most general objects that can be defined in a gauge theory. Its importance was understood within the Quantum ChromoDynamics (QCD) context since its VEV measures the interaction between a quark-antiquark pair and it has been used as the order parameter between the confined and free phases of QCD. In particular, we are interested in Wilson Loops in ABJ(M) because of its dual description inside the M-theory. Thus, the study of Wilson Loops in ABJ(M) can give us invaluable insights on still mysterious objects inside M-theory.

Aside from the correspondence, Wilson Loops and operators insertions on them define defect Conformal Field theories. Wilson Loops in SCFT are conformal defects, adding more structures to the theory in which they are defined.

In this thesis we focus on the gauge part of the holographic duality, working in the context of ABJ(M) theory. We construct the one-dimensional topological sector and compute quantum correction to the two-point function of its superprimary operator at two loops. The reason to find and study the topological sector is, as we already stressed above, that the topological sector can realize a solvable truncation of the bootstrap equation and it can provide quantum information useful for the application of other techniques, either exact or perturbative. We then consider a mass deformed Matrix Model (MM), conjectured to capture integrated correlation functions of the same operators integrated on the line. The weak coupling expansion of the mass deformed MM coincides with the perturbative result. As a by-product, since our superprimary is related to the stress-energy tensor by Ward identities, we can compute the central charge of ABJ(M) at two loops.

Subsequently, we turn our attention to the 1/2-BPS Wilson Loop. We consider the contour to be an infinite straight line or a circle. The principal reason is to generalize the topological sector on the defect CFT living on the Wilson Line. The supermatrix structure imposed by the fermionic loop is still not completely understood and precludes a direct generalization of the twisting procedure used in the free case. We then deeper investigate the supermatrix structure, finding that even the supercharges have to be represented as supermatrices. We study the vacuum expectation value in the line and circle configurations: already at one loop they are not equivalent, in contrast with conformal symmetry. Line and circle are conformal equivalents (related by the inversion) and we expected that correlation functions can be mapped between each other. This phenomenon can be related to the presence of a conformal anomaly, or simply because in the infinite straight line we are cutting-off IR divergences, breaking gauge invariance. We investigate the last option, considering a cut-offed version of the Wilson Circle.

We then derive the expression for the displacement operator, both from deformations of the loop and the supermultiplet construction. In particular, the coefficient of the two-point function of the superprimary of the displacement supermultiplet is the Bremsstrahlung function. We show that the Bremsstrahlung is protected at one loop and the two-loop correction is not completed yet; still, we show the involved Feynman diagrams. Lastly, we show the two-point function of another superprimary (believed to be protected by supersymmetry) is divergent at one-loop and we compute its anomalous dimension. The anomalous dimension is the signature of multiplet recombination. Furthermore, we investigate the recombination phenomenon by a group theory analysis, trying to understand how the Wilson Loop affects the system compared to the free case.

General features of the fermionic Wilson Loop are still obscure, in particular its supermatrix structure. We try making some progress in this direction. Moreover, we provide the first computation of the Bremsstrahlung function from operators insertion in the defect CFT.

Outline

The thesis is organized as follows. Part I is devoted to introducing the cornerstones of our work. In Chapter 1, we review the construction of conformal and superconformal field theories. We pay attention to the bootstrap program since its analytic solution is the ultimate goal of most of the present works. In Chapter 2, we introduce conformal defects and we see that an analog bootstrap equation arises. We then introduce the Wilson Loop showing that it provides a nice example of defect Conformal Field Theory. This is why we decided to include it in the defects chapter even if it would probably deserve an entire dedicated book.

The original work is presented in part II. In Chapter 3, based on paper [5], we build the one-dimensional topological sector of ABJ(M) and loop computations are presented. In Chapter 4, we investigate the dCFT properties of the fermionic loop in ABJ(M). Remarkably, we will find unexpected results, different from what happens in four dimensions. The article about this last topic will be submitted soon [6].

In the Appendix, technical tools used throughout the thesis can be found.

I

Background

1

(Super)Conformal Field Theories

If one is working from the point of view of getting beauty into one's equation, [...] one is on a sure line of progress.

P. A. M. Dirac [7]

In the last fifty years, Conformal Field Theory has found an enormous range of applications in all areas of theoretical physics, from statistical physics to condensed matter to string theory, being an inspiration for pure mathematicians as well. There are many examples in which the conformal symmetry appears: for instance, in the phase transitions in statistical ensembles, at the fixed point of the renormalization group flow (where the Poincaré symmetry is enhanced to conformal symmetry as a consequence of the vanishing beta function) or at high energies in QCD.

Perhaps the most striking property of CFTs, consequence of the power of the conformal symmetry, is the Bootstrap Equation, formulated in [8, 9]. It is obtained by consistency conditions and in principle allows us to compute every data of the theory without the need of local action. With the bootstrap approach, even non-Lagrangian theories can be studied. But the conformal bootstrap approach is as beautiful as hopeless unless the number of fields in the theory is finite. Nonetheless, new progress on this direction were made by the pioneering works [10, 11, 12] that found numerical application in [13]. It has been exploited successfully in statistical problems, computing critical exponent for the 3D Ising Model [14, 15]. In this sense, the actual computing power gives us the possibility to consider a very large amount of constraints but, still being in a finite number, only the extraction very precise bound on the spectrum of the CFTs is possible.

The conformal algebra can be nicely extended including supersymmetry, yielding SuperConformal Field theories (SCFT). There are examples for which the presence of supersymmetry protects the conformal symmetry even at the quantum level. This is the case of $\mathcal{N} = 4$ Super Yang-Mills in four dimensions [16].

Last but not definitely not least, CFTs are known to be dual to gravity theories in one more dimension, through the holographic correspondence [4]. The conjecture relates strongly coupled field theory in flat space to weakly coupled string theory on Anti de Sitter space, by means of the partition function. Although a rigorous proof of the correspondence is still missing, it has passed a huge number of non-trivial tests.

The cornerstone of the gauge/gravity duality is that the space-time symmetry group of AdS in $d+1$-dimensions coincides with the conformal group in d-dimensions. Therefore, exploiting this property, we can consistently apply conformal techniques to study the S-matrix of QFT defined on AdS, even in presence of mass deformation, for every value of the coupling [17]. Bootstrapping the S-matrix allows us to compute the 1-loop determinant of the theories. This machinery has been applied for the $O(N)$ and Gross-Neveu model and can be applied for gravity [18].

In this chapter we introduce the conformal symmetry in dimension $d \neq 2$, which is finite-dimensional, building its representations on fields. We will define the Operator Product Expansion and derive the bootstrap equation. We present how to combine the conformal algebra with supersymmetry and we then introduce the ABJ(M) theory, which is the main character of this thesis.

1.1 Conformal Field Theory

1.1.1 Conformal Symmetry

Let $g_{\mu\nu}$ be the metric of a d-dimensional spacetime. A *Conformal Transformation* (CT) is a diffeomorphism $x \to x'(x)$ such that

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x) \tag{1.1}$$

Isometries form a subgroup of CT corresponding to $\Omega^2 = 1$. Furthermore, if we consider $g_{\mu\nu} = \eta_{\mu\nu}$, then the isometry group coincides with Poincaré group. Dilatations correspond constant Ω^2. This class of transformation are called conformal since they leave the angles between curves unchanged. The finite transformations are:

- Poincaré group: $\begin{cases} translation: \ x'^{\mu} = x^{\mu} + a^{\mu} \\ Lorentz: \ x'^{\mu} = M^{\mu}{}_{\nu} x^{\nu} \end{cases}$

- Dilatations: $x'^{\mu} = \alpha x^{\mu}$
- Special Conformal Transformations: $x'^{\mu} = \dfrac{x^{\mu} - b^{\mu} x^2}{1 - 2(b \cdot x) + x^2 b^2}$.

Another type of CT, discrete and not connected to the identity, is the *inversion* I

$$I : x^{\mu} \to x'^{\mu} = \frac{x^{\mu}}{x^2}, \qquad I^2 = \mathbb{1}$$

However, it's easy to show that inversion after a SCT is

$$I(x'^{\mu}) = \frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu}.$$

In other words

$$SCT = I \circ translation \circ I. \tag{1.2}$$

The infinitesimal forms of the generators are

$$\begin{aligned}
&P_{\mu} = -i\partial_{\mu} && \text{translation} \\
&D = -ix^{\mu}\partial_{\mu} && \text{dilatation} \\
&L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) && \text{Lorenz} \\
&K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu}) && \text{special conformal}
\end{aligned} \tag{1.3}$$

that give us a representation of the conformal algebra, satisfying the following commutation relations

$$\begin{aligned}
&[D, P_{\mu}] = iP_{\mu}, && [D, K_{\mu}] = -iK_{\mu} \\
&[K_{\mu}, P_{\nu}] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}), && [L_{\mu\nu}, P_{\rho}] = -i(\eta_{\mu\rho}P_{\nu} - \eta_{\nu\rho}P_{\mu}) \\
&[L_{\mu\nu}, K_{\rho}] = -i(\eta_{\mu\rho}K_{\nu} - \eta_{\nu\rho}K_{\mu}), && [D, L_{\mu\nu}] = 0 \\
&[P_{\mu}, P_{\nu}] = 0, \qquad [K_{\mu}, K_{\nu}] = 0, && [D, D] = 0, \\
&[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\nu\rho}L_{\mu\sigma}).
\end{aligned} \tag{1.4}$$

If we define

$$\begin{aligned}
&J_{\mu,\nu} = L_{\mu\nu}, && J_{-1,0} = D \\
&J_{0,\mu} = \frac{1}{2}(P_{\mu} + K_{\mu}) && J_{-1,\mu} = \frac{1}{2}(P_{\mu} - K_{\mu})
\end{aligned} \tag{1.5}$$

with the property $J_{a,b} = -J_{b,a}$, and this J would obey the $SO(2,d)$ commutation relations in a d-dimensional Minkoswki spacetime. From this we see that the conformal algebra has dimension

$$\frac{(d+1)(d+2)}{2}.$$

We note from (1.4) that the generators $L_{\mu\nu}$ and P_{μ} form a closed subgroup (Poincaré subgroup); the set $\{L_{\mu\nu}, P_{\mu}, D\}$ form a closed subgroup too.

Conformal Invariance and the Stress-Energy Tensor

In QFT, translation invariance implies the conservation of the stress-energy tensor $T^{\mu\nu}$; it can be made symmetric if we have also rotation invariance. Under an infinitesimal transformation of the coordinates, the action change as

$$\delta S = -\frac{1}{2}\int d^d x\, T^{\mu\nu}\left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right).$$

The definition (1.1) for infinitesimal transformation in flat spacetime requires

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x)\eta_{\mu\nu},$$

then

$$\delta S = -\frac{1}{2}\int d^d x f(x) T^{\mu\nu}\eta_{\mu\nu} = -\frac{1}{2}\int d^d x f(x) T^{\mu}{}_{\mu}$$

and imposing $\delta S = 0$ we have (paying attention to the form of $f(x)$)

$$T^{\mu}{}_{\mu} = 0$$

so a traceless stress-energy tensor implies conformal invariance, but the converse is not always true[1]

1.1.2 Representations on fields

From now on, we use the terms "field" and "operator" interchangeably. In particular, the term "field" make contact with ordinary QFT, while "operator" is more general and does not imply an underlying Lagrangian description.

In CFT, the spectrum of a field is given by the eigenvalue Δ of the dilatation operator, called *scaling dimension*. The commutator of D with $L_{\mu\nu}$ says that spin l and scaling dimension Δ are good quantum numbers that we can use to label our states in the Hilbert space.

From the commutation relations with D in (1.4), we can think of P_μ and K_μ as ladder operators for the scaling dimension. Indeed, if we consider an operator $\mathcal{O}_\Delta$, then we have

$$D\left(P_\mu \mathcal{O}_\Delta\right) = \left[D, P_\mu\right]\mathcal{O}_\Delta + i\Delta P_\mu \mathcal{O}_\Delta = i(\Delta+1)P_\mu \mathcal{O}_\Delta$$

and we conclude that $P_\mu \mathcal{O}_\Delta$ is again an eigenvector of the dilatation generator but with eigenvalue $\Delta + 1$. Similarly, for K we have

$$D\left(K_\mu \mathcal{O}_\Delta\right) = \left[D, K_\mu\right]\mathcal{O}_\Delta + i\Delta K_\mu \mathcal{O}_\Delta = i\left(\Delta-1\right)K_\mu \mathcal{O}_\Delta.$$

[1] In two dimensions, under certain assumptions, the converse is also true [19].

Then it will exist a $\tilde{\Delta}$ such that

$$K_\mu \mathcal{O}_{\tilde{\Delta}} = 0 \tag{1.6}$$

Such kind of operators are called *primary* operators; those constructed applying n-times P_μ on a primary are called *descendant* operators.

In a CFT, fields are homogeneous functions, with homogeneity degree given by their *scaling dimension* Δ; this means that, under a scale transformation, we have

$$\Phi(\lambda x) = \lambda^{-\Delta}\Phi(x). \tag{1.7}$$

where Φ is a generic field. If we consider the general conformal transformation $x \to x'$ such that (1.1) holds, than the field changes as

$$\Phi'(x') = \Omega^{-\Delta} R[M_{\mu\nu}]\Phi(x) \tag{1.8}$$

where R is the irreducible representation of the orthogonal group in which the field Φ is defined. For scalars, (1.8) reduces to (1.7).

We want to find the matrix representations of the conformal algebra generators. We start considering the subgroup of the isotropy of the origin (the transformations that leave the origin invariant), spanned by the generators $D, K_\mu, L_{\mu\nu}$. Let $\Phi_a(x)$ be a field (we consider a multicomponent field, but the scalar case is straightforward). Then we define the action of the generators in the origin, translating the results for general x. We define

$$[D, \Phi_a(0)] = i\Delta\Phi_a(0) \tag{1.9}$$

$$[L_{\mu\nu}, \Phi_a(0)] = i\left(S_{\mu\nu}\right)_{ab}\Phi_b(0). \tag{1.10}$$

where $S_{\mu\nu}$ is the spin matrix constructed by the gamma matrices. For primary fields, eq, (1.6) is translated in

$$[K_\mu, \Phi_a(0)] = 0, \tag{1.11}$$

so Φ_a is a primary field if it is annihilated by K_μ at the origin.

In order to generalize these results at generic x, we compute all the commutators using the following relation

$$\Phi_a(x) = e^{-iP\cdot x}\Phi_a(0)e^{iP\cdot x}$$

using (1.4) and (1.9)-(1.11). The action of the conformal algebra on a field in a generic point is then given by

$$[P_\mu, \Phi_a(x)] = i\partial_\mu\Phi_a(x) \tag{1.12}$$

$$[D, \Phi_a(x)] = i\left(\Delta + x^\mu \partial_\mu\right)\Phi_a(x) \tag{1.13}$$

$$[L_{\mu\nu}, \Phi_a(x)] = i\left(S_{\mu\nu}\right)_{ab}\Phi_b(x) - i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)\Phi_a(x) \tag{1.14}$$

and

$$[K_\mu, \Phi_a(x)] = 2ix_\mu\Delta\Phi_a(x) + 2ix^\rho\left(S_{\rho\mu}\right)_{ab}\Phi_b(x) + i\left(2_\mu x^\nu\partial_\nu\Phi_a(x) - x^2\partial_\mu\mu\Phi_a(x)\right). \tag{1.15}$$

1.1.3 Radial Quantization and OPE

If we write the metric of $\mathbb{R}^d$ in spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega_{d-1} = r^2\left[\frac{dr^2}{r^2} + \Omega_{d-1}\right] \tag{1.16}$$

whre r is the radius of the sphere S^{d-1} whose metric on it is Ω_{d-1}. The dilatation map in this coordinates is $D = r\partial_r$ and maps concentric circles into each other. Then the radius in the r.h.s. of (1.16) can be view as a rescaling parameter. Thus CFTs on $\mathbb{R}^d$ are equivalent to CFTs on $\mathbb{R}\times S^{d-1}$. In particular, defining $t = \log r$, the dilatation change as

$$D = r\partial_r \qquad \rightarrow \qquad D = \partial_t \tag{1.17}$$

In this sense, the generator D has an energetic interpretation, like to the Hamiltonian in quantum mechanics, and justify why the scaling dimensions have such a central role in the study of CFTs.

Radial Quantization and State-Operator Correspondence

In ordinary QFT, we foliate the spacetime in surfaces of equal time (time slices), and in every time slice lives a Hilbert space. These Hilbert spaces are connected by the time evolution operator $U = e^{iH\Delta t}$. We can create "in" states $|\psi_{in}\rangle$ on a certain time slice by insertions of operators before that time slice

$$|\psi_{in}\rangle = \mathcal{O}_n \cdots \mathcal{O}_1 |0\rangle\,;$$

the "out" states $\langle\psi_{out}|$ are created by operator insertion after that time slice

$$\langle\psi_{out}| = \langle 0|\,\mathcal{O}_1 \cdots \mathcal{O}_n.$$

The correlator between these states is the bracket

$$\langle\psi_{out}|\psi_{in}\rangle = \langle\psi_{out}|\,U\,|\psi_{in}\rangle\,.$$

In CFT, we foliate spacetimes using concentric spheres of different radii. The "in" state $|\psi_{in}\rangle$ on the surface of a sphere is created inserting operators inside this sphere, and "out" state $\langle\psi_{out}|$ is created inserting operators outside the sphere. The Hilbert spaces living on the surface of each leaf are connected with the dilation operator $U = e^{iD\Delta\tau}$, with $\tau = \log(r)$, and we choose $|\Delta\rangle$, eigenvectors of D, as basis for our Hilbert spaces; in addition, we choose them as irreducible representation of $SO(D)$ with spin l

$$D\,|\Delta\rangle = i\Delta\,|\Delta\rangle\,, \qquad L_{\mu\nu}\,|l,\Delta\rangle_a = i(S_{\mu\nu})_a{}^b\,|l,\Delta\rangle_b\,.$$

The vacuum $|0\rangle$ corresponds to no insertions.

Operator Product Expansion

From radial quantization, we can construct states by operator insertions. The eigenstates of D correspond to insertions of operators at the origin

$$|\Delta\rangle = \Phi_1(0)\cdots\Phi_n(0)\,|0\rangle\,.$$

With this definition, $|\Delta\rangle$ is automatically a primary. Starting from it, we can construct all the descendants by acting with P. In general, $|\Delta\rangle$ can be either a primary or a descendant (like $P_\mu\,|\Delta-1\rangle$) or a linear combination of these. However, both primaries and descendants are eigenvectors of D and thus they form a complete set.

States in CFT also satisfy associativity. The product of two operators

$$\mathcal{O}(x)\mathcal{O}(0)\,|0\rangle$$

can be written as a linear combination of operators at the origin. From the state-operator correspondence, we have in general

$$|\psi\rangle = \phi_1(x)\phi_2(0)\,|0\rangle$$

and this state can be expanded on the basis of D eigenvectors

$$|\psi\rangle = \sum_{\Delta} c_\Delta(x)\,|\Delta\rangle$$

and $|\Delta\rangle$ itself is a linear combination of primaries and descendants. Thus, we have

$$\phi_1(x)\phi_2(0)\,|0\rangle = \sum_{\phi\text{ primaries}} C_\Delta(x,\partial)\Phi_\Delta(0)\,|0\rangle \tag{1.18}$$

where the sum is only over the primaries and the descendants are obtain with the action of

$$C(x,\partial) = c_1 + c_2^\mu\partial_\mu + c_3x^\mu x^\nu\partial_\mu\partial_\nu + \ldots$$

Equation (1.18) is called *Operator Product Expansion* (OPE) and it is a convergent series. The form of c_1 is fixed by dilatation invariance up to an overall factor

$$c_1 \sim \frac{\lambda_\Delta}{|x|^k} \qquad k = \Delta_1 + \Delta_2 - \Delta_O \tag{1.19}$$

The factors λ_Δ are called *OPE coefficients*.

1.1.4 Correlation functions

At the classical level, conformal symmetry forbids the presence of mass terms in the Lagrangian (if there is any). Generally speaking, the commutator between dilatation and translation in (1.4) leaves us with only two mutually exclusive possibilities: either we have all masses vanishing or we have a continuous spectrum. For this reason, the particle interpretation of QFT is no longer viable and the observables in a CFT are the correlation functions.

In particular, conformal symmetry is strong enough to completely fix the two- and three-point functions, while the four-point correlator is the first not completely fixed function.

Let's consider scalar fields $\phi_n(x)$, with associated scaling dimension Δ_n, and we start from the Vacuum Expectation Value $\langle\phi(x)\rangle$. Translation set it to be a constant, but the only dilation invariant constant is zero, therefore

$$\langle\phi(x)\rangle = 0 \tag{1.20}$$

The two-point function $\langle\phi_1(x_1)\phi_2(x_2)\rangle$ is constrained by translation and dilatation to be only a homogeneous function of the distance $|x_1 - x_2| = |x_{12}|$. Special conformal symmetry set the correlation function to be non-vanishing only if $\Delta_1 = \Delta_2 = \Delta$, therefore

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \frac{\mathcal{C}}{|x_{12}|^{2\Delta}} \tag{1.21}$$

where $\mathcal{C}$ is a normalization constant and can be dependent on the coupling.

For the same reasons, the three-point function is set to be

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{c_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}} \tag{1.22}$$

where c_{123} are physical coefficients called *conformal data*. A CFT is completely determined once the conformal data are given and they are precisely the coefficients λ_Δ appearing in the OPE expansion.

The four-point correlation function is not fixed by conformal symmetry, because with four point we can construct two independent conformal invariants

$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \tag{1.23}$$

called the *cross ratios*. Then, 4-pt correlation function takes the form

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{\mathcal{G}(u,v)}{\prod_{1=i<j}^4 x_{ij}^{\gamma_{ij}}} \tag{1.24}$$

and scale invariance gives the constraint

$$\sum_{i\neq j}\gamma_{ij} = \Delta_i.$$

1.1.5 Bootstrap

Exploiting the OPE is possible to reduce higher-point correlation function to the fixed two- and three-point functions: for instance, let's take the four-point function of four scalar fields with scaling dimensions Δ_i

$$\langle \phi_1(x_1)\phi_(x_2)\phi_(x_3)\phi_(x_4)\rangle \tag{1.25}$$

From (1.18), it is convinient to extract a coefficient from the function $C_\Delta(x,\partial)$ as

$$\phi_1(x)\phi_2(0) \sim \sum_\Delta c_\Delta \tilde{C}_\Delta(x,\partial)\Phi_\Delta(0) \tag{1.26}$$

and the coefficients λ_Δ are called *OPE coefficients*. Inside (1.25), we expand the pair 12 and 34 using OPE

$$\phi(x_1)\phi(x_2) \sim \sum_\Delta c_\Delta C_\Delta(y,\partial_y)\phi_\Delta(y)\big|_{y=\frac{x_1+x_2}{2}} \tag{1.27}$$

$$\phi(x_3)\phi(x_4) \sim \sum_{\tilde{\Delta}} c_{\tilde{\Delta}} C_{\tilde{\Delta}}(z,\partial_z)\phi_{\tilde{\Delta}}(z)\big|_{z=\frac{x_3+x_4}{2}} \tag{1.28}$$

that we can rewrite as

$$\langle \phi_1(x_1)\phi_(x_2)\phi_(x_3)\phi_(x_4)\rangle = \sum_\Delta c_\Delta^2 \left[C_\Delta(y,\partial_y)C_\Delta(z,\partial_z)\langle\phi_\Delta(y)\phi_\Delta(z)\rangle\right] \tag{1.29}$$

since the 2-point function forces the exchanged operator to have the same scaling dimension. It's worth noting that the expression inside the square brackets is completely fixed by conformal invariance; they are called *conformal partial waves*, Therefore, eq. (1.29) is called *conformal partial waves expansion*. This expression can be easily generalized including exchanged spinnig operators. It's convenient to rewrite the expression in square brackets as a function of the conformal ratios, such as

$$\langle \phi_1(x_1)\phi_(x_2)\phi_(x_3)\phi_(x_4)\rangle = \sum_{\Delta,\ell} c_{\Delta,\ell}^2 \frac{G_{\Delta,\ell}(u,v)}{x_{12}^{\frac{1}{2}(\Delta_1+\Delta_2)}x_{34}^{\frac{1}{2}(\Delta_2+\Delta_3)}} \tag{1.30}$$

where ℓ is the spin of the exchanged operator. The functions $G_{\Delta,\ell}$ are called *conformal blocks* and they can be computed exactly [10, 11, 12]. They can have different forms depending on the space-time dimension, but in general they are hypergeometric functions.

Large N expansion and anomalous dimension

The expansion in conformal blocks is very handy when it comes to compute perturbative quantities as the anomalous dimension, especially when the theory is non-Lagrangian and no Path-Integral description is available. Let's take, for instance, the four-point correlation function of four identical scalar operators

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta_\phi}x_{34}^{2\Delta_\phi}} \tag{1.31}$$

and we assume to know the explicit form of $\mathcal{G}$. We also know its expansion in conformal blocks, taking out explicitly a power of u

$$\mathcal{G}(u,v) = \sum_{\Delta,\ell} c^2_{\Delta,\ell}u^{\frac{\Delta-\ell}{2}}g_{\Delta,\ell}(u,v) \tag{1.32}$$

and in this case the sum over the spin is only on even value of ℓ. If we have an additional parameter, as N in gauge theory, we can think to make a large N expansion of both side of (1.32) using

$$\mathcal{G}(u,v) = \mathcal{G}^{(0)}(u,v) + \frac{1}{N}\mathcal{G}^{(1)}(u,v) \tag{1.33}$$

$$c^2_{\Delta,\ell} = a_{\Delta,\ell} = a^{(0)}_{\Delta,\ell} + \frac{1}{N}a^{(1)}_{\Delta.\ell} \tag{1.34}$$

$$\Delta = \Delta^{(0)} + \frac{1}{N}\gamma^{(1)}_{\Delta,\ell} \tag{1.35}$$

$$g_{\Delta^{(0)}+\frac{1}{N}\gamma,\ell}(u,v) = g_{\Delta^{(0)},\ell}(u,v) + \mathfrak{G}(u,v) \tag{1.36}$$

such that

$$\mathcal{G}^{(0)}(u,v) + \frac{1}{N}\mathcal{G}^{(1)}(u,v) = \sum_{\Delta,\ell}\left(a^{(0)}_{\Delta,\ell} + \frac{1}{N}a^{(1)}_{\Delta.\ell}\right)u^{\frac{\Delta^{(0)}+\frac{1}{N}\gamma^{(1)}-\ell}{2}}g_{\Delta^{(0)}+\frac{1}{N}\gamma^{(1)},\ell}(u,v) \tag{1.37}$$

Collecting the N^{-1} terms in the r.h.s.

$$\mathcal{G}^{(0)} + \frac{1}{N}\mathcal{G}^{(1)} = \sum_{\Delta,\ell} a^{(0)}u^{\frac{\Delta^{(0)}-\ell}{2}}g_{\Delta^{(0)},\ell} + \frac{1}{N}\sum_{\Delta,\ell}\left(a^{(1)} + \frac{a^{(0)}\gamma}{2}\log(u)\right)u^{\frac{\Delta^{(0)}-\ell}{2}}g_{\Delta^{(0)},\ell} + \mathfrak{G} \tag{1.38}$$

From here we see that is very easy to extract the anomalous dimension just looking at the coefficient of the $\log(u)$.

Bootstrap equation

We can use OPE to reduce the four point function choosing different pairs of operators, like 14 and 23. But since the fields are inside a correlation function, the choice of the contracting pairs should not affect the final result. This property is know as the *crossing symmetry* and it is graphically represented in fig. 1.1. The corresponding equation is

$$\frac{\mathcal{G}(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}} = \frac{\mathcal{G}(v,u)}{x_{14}^{2\Delta}x_{23}^{2\Delta}} \tag{1.39}$$

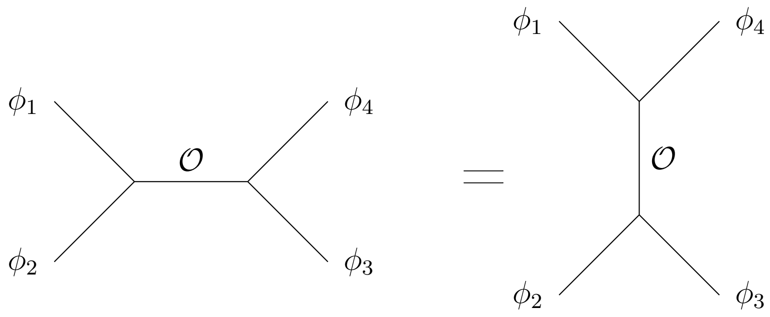


Figure 1.1: Crossing symmetry

Writing the functions $\mathcal{G}$ in terms of the OPE expansion we arrive at the *bootstrap equation*

$$\sum_{\Delta,\ell} \lambda^2_{\Delta,\ell} \left(\frac{v^{\Delta_\phi} G_{\Delta,\ell}(u,v) - u^{\Delta_\phi} G_{\Delta,\ell}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}} \right) = 1 \tag{1.40}$$

Equation (1.40) relates OPE coefficients $\lambda_{\Delta,\ell}$ among themselves; we can write a coefficient in terms of all the others (and they are infinite). In principle, these equations are enough to compute every $\lambda_{\Delta,\ell}$ of the theory. Practically, solve them analytically is computationally very hard. But there are numerical approaches that can give bounds on the spectrum of a given CFT [13].

One way to make progress on the analytic solution of (1.40) is trying to reduce the problem to a subsector of the theory that is closed under OPE. This reduction will shrink the number of constraints thus leading to a solvable truncation of the bootstrap equation. Generally, adding more symmetry help to spot subsectors with this property, see e.g. supersymmetry. If having more symmetries is not viable, we can think of breaking (some) of them by considering defects inside the theory. The latter option will add more structure to the theory, giving different constraints.

1.2 Supersymmetry

Supersymmetry was first introduced [20] as an extension of the Poincaré algebra to escape the no-go theorem [21], stating that the only possible symmetries of a QFT with unitary S-matrix are the Poincaré group and internal symmetries. Supersymmetry is then realized as a graded algebra formed by bosonic and fermionic generators, which respect super-commutation rules.

In supersymmetric theories we have the same amount of bosonic and fermionic degrees of freedom, giving strong constraints on the theory. One of the consequences of this peculiarity is the divergences cancellation in quantum corrections (non-renormalization theorems [22]). SUSY provides a nice example of unification at large scales, showing that Standard Model couplings converge to one value [23].

Usually, the high amount of symmetries then constrains the theory enough to allow for exact results. For more details on this huge topics, see e.g. [24, 25].

SUSY is realized introducing fermionic generators Q^I_α and $\bar{Q}^{\dot{\alpha},\,J}$, called supercharges, whose anti-commutators close on translations. More explicitly, they extend the Poincaré algebra in the following way

$$\left[M_{\mu\nu}, Q^I_\alpha\right] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \tag{1.41}$$

$$\left[M_{\mu\nu}, \bar{Q}^{\dot{\beta},\,J}\right] = i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta},\,J} \tag{1.42}$$

$$\left\{Q^I_\alpha, \bar{Q}^J_{\dot{\beta}}\right\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{IJ} \tag{1.43}$$

$$\left\{Q^I_\alpha, Q^J_\beta\right\} = \varepsilon_{\alpha\beta} Z^{IJ} \tag{1.44}$$

$$\left\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\right\} = \varepsilon_{\dot{\alpha}\dot{\beta}} \left(Z^{IJ}\right)^* \tag{1.45}$$

where $Z^{IJ} = -Z^{JI}$ is the central extension of the algebra. The relations (1.41) and (1.42) show that the fermionic generators are spinors that transform , respectively, in the $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ representations of the Lorentz group.

Focus on eq. (1.43): it is valid for rigid SUSY, namely the supercharges are independent of the coordinates. For local SUSY, it will be modified in the r.h.s. introducing a local parameter. In this case, the relation will describe a theory invariant under coordinate diffeomorphism; then, gauging supersymmetry will automatically give a supersymmetric theory of gravity.

The supercharges have an additional internal symmetry, called R-symmetry, that rotates the supercharges among them. Usually the R-symmetry group is identified as $SO\left(\mathcal{N}\right)$ or $SU\left(\mathcal{N}\right)$, depending on the reality conditions of the theory. Usually, we refer to $\mathcal{N}$ as the number of the supersymmetries we are considering.

Supermultiplets

The irreducible representations of the SUSY algebra are called supermultiplets. Since the supercharges Q_α and $\bar{Q}^\beta$ are spinors, they can lower or rise the spin by half unit and we can think of them as creation/annihilation operators on the Hilbert space. Choosing a state annihilated by all the Qs (Clifford vacuum, with helicity λ_0), the action of the $\bar{Q}$s will create a state with increased helicity by half. Depending on how many supersymmetries we consider ($\mathcal{N}$), we can have multiple actions of $\bar{Q}^J$ on the Clifford vacuum. In particular, for anti-symmetry, the number of intermediate states after p action of $\bar{Q}^J$ is the binomial $\begin{pmatrix}\mathcal{N}\\ p\end{pmatrix}$ with $p = 0, \ldots, \mathcal{N}$ and the total number of states in a given irreducible representation is

$$\sum_{p=0}^{\mathcal{N}} \begin{pmatrix}\mathcal{N}\\ p\end{pmatrix} = 2^{\mathcal{N}} = \left(2^{\mathcal{N}-1}\right)_{bosons} + \left(2^{\mathcal{N}-1}\right)_{fermions} \tag{1.46}$$

The final state will have helicity $\lambda_0 + \frac{\mathcal{N}}{2}$. Then, the fields content of theory with a different number of SUSY are organized in different supermultiplets:

- $\mathcal{N} = 1$: here we have 4 type of multiplets that are the matter multplet, whose degrees of freedom are $\left(0, \frac{1}{2}\right)$, the gauge multiplet $\left(\frac{1}{2}, 1\right)$, the spin 3/2-multiplet $\left(1, \frac{3}{2}\right)$ and the graviton multiplet $\left(\frac{3}{2}, 2\right)$, together with the respective CPT conjugates.

- $\mathcal{N} = 2$: hypermultiplet $\left(-\frac{1}{2}, 0, 0, \frac{1}{2}\right)$, vector multiplet $\left(0, \frac{1}{2}, \frac{1}{2}, 1\right)$, spin 3/2-multiplet $\left(\frac{1}{2}, 1, 1, \frac{3}{2}\right)$ and the graviton multiplet $\left(1, \frac{3}{2}, \frac{3}{2}, 2\right)$.

$\mathcal{N} = 4$ Super Yang-Mills

Probably the most famous example of theory with extended supersymmetry is $\mathcal{N} = 4$ SYM theory in four dimensions. The amount of symmetry, in this case, allows only for one vector multiplet

$$V = \left(A_\mu, \lambda_\alpha^I, \phi^J\right) \tag{1.47}$$

consisting in a gauge field, four Weyl spinors and six real scalars all living in the adjoint representation of the gauge group.

The action can be obtained by dimensional reduction from $\mathcal{N} = 1$ SYM in 10 dimensions and it is classically conformal. At the quantum level, the theory is UV finite and the beta function vanishes for every value of the coupling [16]. This result holds up to three loops [26]. From light-cone gauge arguments, it was argued that the vanishing beta function holds at all loop orders [27, 28]. Hence, conformal symmetry survives at the quantum level too.

Defining new classes of operators

Adding new fermionic generators gives us the possibility to define new classes of operators, closed under OPE. It's the case for the *chiral ring*, defined as the set of operators annihilated by a certain number of supercharges, usually chosen as half of the total supercharges i.e. $[Q, \mathcal{O}] = 0$, provided that the operator $\mathcal{O}$ is not Q-exact. This is a nice example of an OPE-closed subset that can reduce the bootstrap equation, but correlation functions of chiral operators are trivially vanishing, due to their independence on the position of the operators. Still, there are hopes left to exploit properties of this subclass, as we will see in section 3.1.

1.3 SCFT

We saw that we can realize an extension of the Poincaré algebra introducing spinorial generators. We also saw that the Poincaré group is the isometry subgroup of the conformal symmetry. The question is, then, whether we can or not make a supersymmetric extension of the conformal group. The answer turns out to be positive. In fact, introducing the so-called *conformal supercharges* S^I_α and $\bar{S}^J_{\dot\alpha}$ and requiring they close on special conformal transformation

$$\{S, \bar{S}\} \sim K \tag{1.48}$$

we notice that the new algebra closed in what we call the superconformal algebra. The complete classification of superconformal algebras was given in [29].

Since the main subject of this thesis are three-dimensional superconformal theories, and in particular ABJ(M), here we report, as an example, the full superconformal algebra for $d = 3$ in Euclidean signature[2].

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= \delta^{\sigma\mu} M^{\nu\rho} - \delta^{\sigma\nu} M^{\mu\rho} + \delta^{\rho\nu} M^{\mu\sigma} - \delta^{\rho\mu} M^{\nu\sigma} & [P^\mu, K^\nu] &= 2(\delta^{\mu\nu} D + M^{\mu\nu}) \\
[P^\mu, M^{\nu\rho}] &= \delta^{\mu\nu} P^\rho - \delta^{\mu\rho} P^\nu & [K^\mu, M^{\nu\rho}] &= \delta^{\mu\nu} K^\rho - \delta^{\mu\rho} K^\nu \\
[D, P^\mu] &= P^\mu & [D, K^\mu] &= -K^\mu
\end{aligned} \tag{1.49}$$

The fermionic generators Q^{IJ}_α, S^{IJ}_α satisfy the following anti-commutation rules

$$\begin{aligned}
&\{Q^{IJ}_\alpha, Q^{KL,\beta}\} = \varepsilon^{IJKL} (\gamma^\mu)_\alpha{}^\beta P_\mu \qquad \{S^{IJ}_\alpha, S^{\beta KL}\} = \varepsilon^{IJKL} (\gamma^\mu)_\alpha{}^\beta K_\mu \\
&\{Q^{IJ}_\alpha, S^{\beta KL}\} = \varepsilon^{IJKL} \left(\frac{1}{2} (\gamma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} + \delta^\beta_\alpha D \right) + \delta^\beta_\alpha \varepsilon^{KLMN} (\delta^J_M R_N{}^I - \delta^I_M R_N{}^J)
\end{aligned} \tag{1.50}$$

and similarly for $\bar{Q}_{\alpha IJ} = \frac{1}{2}\varepsilon_{IJKL} Q^{KL}_\alpha$ and $\bar{S}_{\alpha IJ} = \frac{1}{2}\varepsilon_{IJKL} S^{KL}_\alpha$. The mixed commutator are

$$\begin{aligned}
[K^\mu, Q^{IJ}_\alpha] &= (\gamma^\mu)_\alpha{}^\beta S^{IJ}_\beta & [P^\mu, S^{IJ}_\alpha] &= (\gamma^\mu)_\alpha{}^\beta Q^{IJ}_\beta \\
[M^{\mu\nu}, Q^{IJ}_\alpha] &= -\frac{1}{2} (\gamma^{\mu\nu})_\alpha{}^\beta Q^{IJ}_\beta & [M^{\mu\nu}, S^{IJ}_\alpha] &= -\frac{1}{2} (\gamma^{\mu\nu})_\alpha{}^\beta S^{IJ}_\beta \\
[D, Q^{IJ}_\alpha] &= \frac{1}{2} Q^{IJ}_\alpha & [D, S^{\alpha IJ}] &= -\frac{1}{2} S^{\alpha IJ} \\
[R_I{}^J, Q^{KL}_\alpha] &= \delta^K_I Q^{JL}_\alpha + \delta^L_I Q^{KJ}_\alpha - \frac{1}{2}\delta^J_I Q^{KL}_\alpha & [R_I{}^J, S^{\alpha KL}] &= \delta^K_I S^{\alpha JL} + \delta^L_I S^{\alpha KJ} - \frac{1}{2}\delta^J_I S^{\alpha KL}
\end{aligned} \tag{1.51}$$

From the commutation relations with the dilation generator, we can think of Q and S as ladder operators, increasing or decreasing operators scaling dimension by $1/2$. Thus, there will be a class of operators whose scaling dimension can't be lowered by the superconformal charges: those operators are then annihilated by the superconformal charges. They are called superprimary operators and from (1.48) we can easily see they are primary operators as well.

[2] Here we keep the R-symmetry group generic.

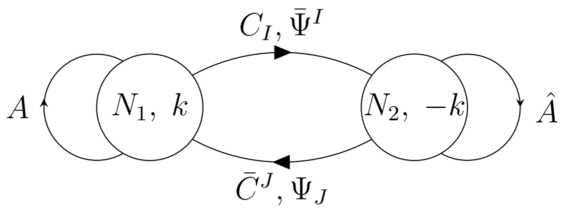


Figure 1.2: ABJ(M) quiver diagram

1.3.1 $\mathcal{N} = 6$ Super Chern-Simons Matter Theory (ABJ(M))

Three-dimensional field theories are interesting to study since they represent planar systems in condensed matter. In particular, the C–S action is used in describing the topological order in the fractional quantum Hall effect (for a nice review see [30] and reference therein). Moreover, three-dimensional SCFT play a pivotal role because they represent a nice framework to study strongly coupled planar systems in the holographic theory. Indeed, the Super C–S action describes the world-volume of eleven-dimensional M2-branes at low energies inside the M-theory [31]. Afterwards, a theory with $\mathcal{N} = 8$ supersymmetry was built in [32] and conjectured to represent a specific M2-brane theory for particular values of the C–S level.

In a recent paper [33], the authors discovered a three-dimensional theory with as gauge group a double Chern-Simons $U(N) \times U(N)$, whose supersymmetry is $\mathcal{N} = 6$, known in the literature as ABJM. Short after they generalized it for the different rank case $U(N_1) \times U(N_2)$ [34]. In the case $k = 1, 2$ the supersymmetry is enhanced to $\mathcal{N} = 8$. We will denote the theory ABJ(M), meaning that all the results we are presenting are both valid in ABJ and ABJM.

For $k > 2$, the global symmetry of ABJ(M) is represented by the Euclidean group $Osp(6|4)$, whose algebra satisfies the commutation relations in (1.49), (1.50) and (1.51) with the R-symmetry algebra being $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$. Its generators are traceless and they satisfy

$$[J_I{}^J, J_K{}^L] = \delta_I^L J_K{}^J - \delta_K^J J_I{}^L \tag{1.52}$$

Furthermore, the bosonic generators satisfy the following conjugation rules

$$(P^\mu)^\dagger = K^\mu \qquad (K^\mu)^\dagger = P^\mu \qquad D^\dagger = D \qquad (M^{\mu\nu})^\dagger = -M^{\mu\nu} \qquad (J_K{}^L)^\dagger = J_L{}^K \tag{1.53}$$

while the fermionic ones

$$(Q^{IJ}_\alpha)^\dagger = \frac{1}{2}\varepsilon_{IJKL}\, S^{KL\alpha} = \bar{S}^\alpha_{IJ} \qquad\qquad (S^{IJ}_\alpha)^\dagger = \frac{1}{2}\varepsilon_{IJKL}\, Q^{KL\alpha} = \bar{Q}^\alpha_{IJ} \tag{1.54}$$

and the action of the $\mathfrak{su}(4)$ R-symmetry generators on operators with R-symmetry indexes $\mathcal{O}_I$ ($\bar{\mathcal{O}}^I$) in the (anti-)fundamental representation reads

$$[J_I{}^J, \mathcal{O}_K] = \frac{1}{4}\delta_I^J \mathcal{O}_K - \delta_K^J \mathcal{O}_I \qquad [J_I{}^J, \bar{\mathcal{O}}^K] = \delta_I^K \bar{\mathcal{O}}^J - \frac{1}{4}\delta_I^J \bar{\mathcal{O}}^K \tag{1.55}$$

The details on the $\mathfrak{osp}(6|4)$ supermultiplets can be found in [35].

Since we have a two gauge groups, the theory can be represented in a quiver diagram, see fig. 1.2. We have two gauge fields, denoted as $(A)_i{}^j$ and $(\hat{A})_{\hat{i}}{}^{\hat{j}}$, in the adjoint representation of the respective group (i and $\hat{i}$ being the gauge indices). The matter fields are the complex scalars $(C_I)_i{}^{\hat{j}}$, $(\bar{C}^I)_{\hat{i}}{}^j$ and the fermions $(\Psi_I)_{\hat{i}}{}^j$, $(\bar{\Psi}^I)_i{}^{\hat{j}}$. They live in the (anti-)bifundamental representation of the gauge groups, meaning they transform as $(\mathbf{N_1}, \bar{\mathbf{N}}_\mathbf{2})$ for bifundamental and as $(\mathbf{N_2}, \bar{\mathbf{N}}_\mathbf{1})$ for the anti-bifundamental. The index $I = 1, \ldots, 4$ is the R-symmetry index of $SU(4)$. Operatively, you can think of matter fields as rectangular matrices and to construct gauge-invariant observables one has to take the product to form a square-matrix of one of the two gauge fields, i.e. construct an object transforming in one of the adjoint representations, and then take the trace on the gauge indices.

One nice property of ABJ(M) is that we can write an explicit action

$$S_{ABJM} = S_{gauge} + S_{ghost} + S_{matter} + S_{int} \tag{1.56}$$

where the pieces are

$$S_{CS} = \tfrac{-ik}{4\pi} \int d^3x \varepsilon^{\mu\nu\rho} \left[\mathrm{Tr}\left(A_\mu \partial_\nu A_\rho + \tfrac{2i}{3} A_\mu A_\nu A_\rho\right) - \mathrm{Tr}\left(\hat{A}_\mu \partial_\nu \hat{A}_\rho + \tfrac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho\right)\right] \tag{1.57}$$

$$S_{gf} = \frac{k}{4\pi} \int d^3x \, \mathrm{Tr}\left[\frac{1}{\xi}(\partial_\mu A^\mu)^2 + \partial_\mu \bar{c} D^\mu c - \frac{1}{\xi}\left(\partial_\mu \hat{A}^\mu\right)^2 - \partial_\mu \bar{\hat{c}} D^\mu \hat{c}\right] \tag{1.58}$$

$$S_{mat} = \int d^3x \, \mathrm{Tr}\left[D_\mu C_I D^\mu \bar{C}^I + i\bar{\psi}^I \gamma^\mu D_\mu \psi_I\right] + S_{int} \tag{1.59}$$

and the covariant derivative act in the following way on the fields

$$\begin{aligned} D_\mu C_I &= \partial_\mu C_I + i A_\mu C_I - i C_I \hat{A}_\mu, \qquad & D_\mu \bar{C}^I &= \partial_\mu \bar{C}^I + i\hat{A}_\mu \bar{C}^I - i\bar{C}^I A_\mu \\ D_\mu \bar{\psi}^I &= \partial_\mu \bar{\psi}^I + i A_\mu \bar{\psi}^I - i\bar{\psi}^I \hat{A}_\mu, \qquad & D_\mu \psi_I &= \partial_\mu \psi_I + i\hat{A}_\mu \psi_I - i\psi_I A_\mu \end{aligned} \tag{1.60}$$

The interaction part S_{int} is composed of a Yukawa-like potential of the type $C^2\psi^2$ and a sextic scalar interactions potential

$$\begin{aligned} S_{6-pt} = \frac{4\pi}{3k^2} \int d^3x \, \mathrm{Tr}\,[&C_I \bar{C}^I C_J \bar{C}^J C_K \bar{C}^K + C_K \bar{C}^I C_I \bar{C}^J C_J \bar{C}^K + \\ &+ 4 C_I \bar{C}^J C_K \bar{C}^I C_J \bar{C}^K - 6 C_I \bar{C}^J C_J \bar{C}^I C_K \bar{C}^K] \end{aligned} \tag{1.61}$$

$$\begin{aligned} S_{4-pt} = \frac{2\pi i}{k} \int d^3x \, \mathrm{Tr}\,[&C_I \bar{\psi}_J \psi^J \bar{C}^I - C_I \bar{C}^I \psi^J \bar{\psi}_J - 2 C_J \bar{\psi}_I \psi^J \bar{C}^I + 2 C_I \bar{C}^J \psi^I \bar{\psi}_J + \\ &+ \varepsilon_{IJKL} \psi^I \bar{C}^J \psi^K \bar{C}^L - \varepsilon^{IJKL} \bar{\psi}_I C_J \bar{\psi}_K C_L] \end{aligned} \tag{1.62}$$

where $\varepsilon_{1234} = \varepsilon^{1234} = 1$ and for the group generators we use the following relations

$$\mathrm{Tr}(T^A T^B) = \delta^{AB} , \qquad [T^A, T^B] = i f^{AB}{}_C T^C \tag{1.63}$$

The fields have dimensions $[A] = [\hat{A}] = [\Psi] = 1$, while $[C] = [\bar{C}] = \frac{1}{2}$, thus the coupling is dimensionless, making the theory classicaly conformal. The action is invariant under the supersymmetry transformation listed in (B.1).

However, there are evidences [34] for which ABJ exist as unitary superconformal theory only when

$$|N_1 - N_2| \leq |k| \tag{1.64}$$

We will see a similar issue when computing the anomalous dimension of operator insertions inside the 1/2-BPS Wilson Loop.

The presence of the action allows us to perform perturbative computations, by means of Feynman diagrams related to Feynman rules (listed in the appendix A.2), matching the predictions with those obtained by exact techniques. As we will see, this is the way we get through in the topological sector. Most of the time, exact results are difficult to obtain, especially when dealing with new systems. Perturbation theory is then a very effective tool to probe these new configurations and get insights on them, as we will see for operator insertions inside the fermionic Wilson Loop.

Gravity Dual

In the special case of ABJM for $k = 1, 2$, the conjectured dual description is a low-energy theory on N M2-branes at a $\mathbb{C}^4/\mathbb{Z}_k$ singularity. ABJ can be obtained by considering a system of $N_1 - N_2$ M2-branes sitting on as $\mathbb{C}^4/\mathbb{Z}_k$ singularity while N_2 branes are free to move.

For $k > 2$ instead, ABJ(M) holographic dual is a M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. The sphere S^7 can be viewed as a S^1-fibration over $\mathbb{CP}^3$,i.e. $S^7/\mathbb{Z}_k \simeq \mathbb{CP}^3 \times S^1/\mathbb{Z}_k$ and the radius of the one-sphere behaves as $R_{S^1} \sim (Nk^{-5})^{\frac{1}{6}}$. Thus, in the large N limit with $N \gg k^5$, namely when the radius of the one-sphere is large, the theory is strongly coupled and the eleven-dimensional description is still valid; in the opposite limit $N \ll k^5$, the one-sphere shrinks and the effective description is in terms of a ten-dimensional type IIA string theory in $AdS_4 \times \mathbb{CP}^3$.

Parity-like Symmetry

It is known that the C–S term breaks parity. Nevertheless, ABJM is invariant under a parity-like transformation, realized by the reflection together the exchange $k \to -k$. Having the same rank of the group, the quiver is then invariant. ABJ is also invariant under a modified parity-like transformation. Some features of this transformation for ABJ have been studied in previous works in the context of integrability [36, 37]. Parity-like (PL) transformation is a symmetry for ABJ(M) if we simultaneously make the exchanges $N_1 \leftrightarrow N_2$ and $k \to -k$. In this case the quiver diagram will change as in fig. 1.3. We conclude that to pass from a parity to another, one has to be careful to exchange the barred fields with the unbarred ones. Fig. 1.4 shows that after renaming the fields, what we obtain is just a reflection of the initial quiver fig. 1.2. These swaps are realized on fields demanding that

$$(A_\mu)^{PL} = -\hat{A}_\mu(x) \qquad (C_I)^{PL} = \bar{C}^I(x) \tag{1.65}$$

and viceversa. For the fermions, the transformations are defined up to a phase

$$\left(\bar{\Psi}_\alpha\right)^{PL} = \bar{\omega}\Psi^\alpha \qquad (\Psi^\alpha)^{PL} = \omega\bar{\Psi}\alpha \tag{1.66}$$

where $\bar{\omega}$, ω are constrained by the requirement $\bar{\omega}\omega = 1$. We will see how consistency on the parity-like transformation on the fermionic loop will fix these constants.

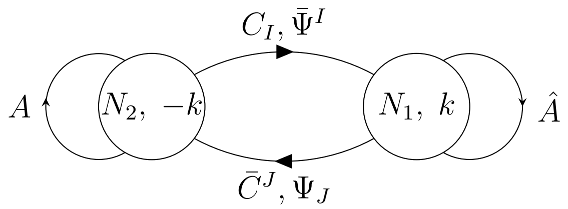


Figure 1.3: ABJ(M) quiver diagram after parity

In particular, these transformations will change the sign in the quiver-derivatives (1.60). The Dirac term in the kinetic part of the action is invariant because in three dimensions, there are two inequivalent choice of gamma matrices that are related by parity

$$\{\sigma^1, \sigma^2, \sigma^3\} \quad \overset{P}{\leftrightarrow} \quad \{\sigma^1, \sigma^2, -\sigma^3\} \tag{1.67}$$

so the product $\gamma^\mu D_\mu$ is parity-invariant.

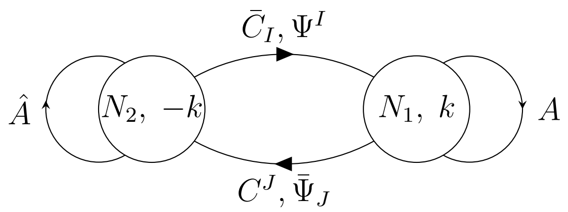


Figure 1.4: ABJ(M) quiver diagram after parity and exchanges

2

Conformal Defects and Wilson Loops

The beauty of a theory lies in its defects.

A. Söderberg

We have seen how powerful conformal symmetry is in constraining physical observables and providing consisteny relations for them. However, real world situation are far from being perfectly conformal. Finite size effects or the presence of boundaries (e.g. domain walls separating differently ordered regions) and impurities are everyday configurations whose contributions are not negligible. In QFT, we can describe these objects by considering the presence of a defect inside an ambient theory (bulk). The most famous example is the Kondo problem, describing magnetic impurities in metals; results related to it led to important progress in the study the renormalization group beyond the critical endpoints and in integrability [38]. They are mostly used to describe probes and to measure the response of the theory to the presence of the probe, e.g. fluctuations of the vacuum.

In QFT, defects are usually realized as extendend operators beside the action

$$S \to S + \int d^d x \mathcal{O}(x) \tag{2.1}$$

The bulk symmetry breaking enrich the structure of correlation functions, making them more intricate. For instance, near the defect, bulk operators can have a non-vanishing one-point function.

Line defects represent an important class since they find applications from condensed matter to high energy physics[1]. In particular, the study of one-dimensional

[1]For recent studies, see [39, 40, 41, 42, 43, 44] and reference therein.

defects in three dimensions plays a central role because it describes impurities in planar systems. Moreover, a stationary impurity on a surface is represented as a line defect along the time direction in a 2+1 manifold. Thus, we consider conformal one-dimensional defects, on which the line conformal symmetry is preserved.

In this Chapter, we present the general approach to defects, introducing an important operator arising near them, the *displacement operator* and arguing its relation with the Bremsstrahlung function. We then turn our attention to Wilson Loops, that represent a important realization of defects in gauge theories.

2.1 Generalities on defects

The systematic approach to defect is presented in [45]. Consider a d-dimensional euclidean ambient space and inside it a defect of dimension p. Moreover, the defect co-dimension is defined as $q = d-p$. Let's denote with $G(d)$ the ambient space-time symmetry group; the defect breaks it as

$$G(d) \to G(p) \times SO(q) \tag{2.2}$$

namely, we are left with the same reduced space.time symmetry group on the defect times rotations around it. Therefore, we can split the coordinates as the ones parallel $x_\parallel$ and perpendicular $x_\perp$ to the defects.

We are interested in conformal defect: we have then $G(d) = SO(1, d+1)$ that is broken down to

$$SO(1, d+1) \to SO(1, p+1) \times SO(q) \tag{2.3}$$

The field content of the theory in presence of a defect splits in two main classes: bulk and defect operators, respectively denoted by $\mathcal{O}_B$ and $\mathcal{O}_D$. Bulk operators are those living in the ambient space, while defect operators live only on the defect. In the bulk as well as in the defect, operators have consistent OPE. Defect operators can carry two types of spin, which are the representations of the two groups in (2.3).

We saw in the previous chapter that in CFT the observables are correlation functions. Let's see how they get modified in the presence of a defect.

The defect itself will possess a vacuum expectation value, i.e. $\langle D \rangle$, measuring the response of the theory at the presence of the defect. Let's consider insertions of m operators $\mathcal{O}_i$ in the bulk and n operators $\mathcal{O}_{D,j}$on the defect. Their correlation function is defined as

$$\langle\langle \mathcal{O}_1 \cdots \mathcal{O}_m \mathcal{O}_{D,1} \cdots \mathcal{O}_{D,n} \rangle\rangle_D = \frac{\langle \mathcal{O}_1 \cdots \mathcal{O}_m \left(\mathcal{O}_{D,1} \cdots \mathcal{O}_{D,n} D \right) \rangle}{\langle D \rangle} \tag{2.4}$$

where the part $(\mathcal{O}_{D,1}\cdots\mathcal{O}_{D,n}D)$ means that every defect insertion is connected by the defect, i.e.

$$\begin{aligned}&\big(\mathcal{O}_{D,1}(x_{\parallel,1})\cdots\mathcal{O}_{D,n}(x_{\parallel,n})D\big)=\\=&D_{-\infty,x_{\parallel,1}}\mathcal{O}_{D,1}(x_{\parallel,1})D_{x_{\parallel,1},x_{\parallel,2}}\mathcal{O}_{D,2}(x_{\parallel,2})\cdots D_{x_{\parallel,n-1},x_{\parallel,n}}\mathcal{O}_{D,n}(x_{\parallel,n})D_{x_{\parallel,n},+\infty}\end{aligned}\tag{2.5}$$

with the compact notation $D_{x_{\parallel,1},x_{\parallel,2}}$ meaning the defect joining the point $x_{\parallel,1}$ and $x_{\parallel,2}$. The set of defect correlation functions of operators on the defect are usually referred as *defect Conformal Field Theory* (dCFT).

2.2 Correlation functions in the presence of defects

Since on the defect the theory is still conformal, the correlation functions between defect operators are just as we saw in the previous chapter, just restricted on the parallel coordinates. The only aspect to pay attention to is how to choose the normalization of the two-point function, depending on whether there is another global symmetry or not. The same for bulk operators very far from the defect, in this case, there's no splitting as in (2.3) and the theory looks perfectly conformal.

The new kind of interactions come from the correlation between bulk and defect operators, ore when considering bulk operators near the defect.

Following [46], on the defect we have a complete basis of defect operator. When the bulk operator is approaching the defect, the can expand it on the defect basis

$$\mathcal{O}(x_\parallel,x_\perp)=\sum_{\hat{\Delta}}\frac{b_{\mathcal{O}_B\mathcal{O}_D}}{|x_\perp|^{\Delta_{\mathcal{O}}-\hat{\Delta}}}\mathcal{O}_{D,\hat{\Delta}}(x_\parallel)\tag{2.6}$$

and the sum starts from $\hat{\Delta}=0$, namely the identity operator. This expansion is called Boundary Operator Expansion (BOE).

The first effect of the BOE is that one-point function of bulk operator are no more vanishing near the defect, i.e. they acquire a non-trivial VEV since

$$\langle\mathcal{O}(x_\parallel,x_\perp)\rangle=\sum_{\hat{\Delta}}\frac{b_{\mathcal{O}_B\mathcal{O}_D}}{|x_\perp|^{\Delta_{\mathcal{O}}-\hat{\Delta}}}\,\langle\mathcal{O}_{D,\hat{\Delta}}(x_\parallel)\rangle\tag{2.7}$$

the only term contributing is the first coming from the identity (the term corresponding to $\hat{\Delta}=0$), so

$$\langle\mathcal{O}\rangle=\frac{b_{\mathcal{O}\mathbb{1}}}{|x_\perp|^{\Delta_{\mathcal{O}}}}\tag{2.8}$$

and can only be function of the distance from the defect. Eq. (2.8) is the manifestation of the broken translation near the defect. It is manifestly divergent as the bulk-operator get closer and closer to the defect.

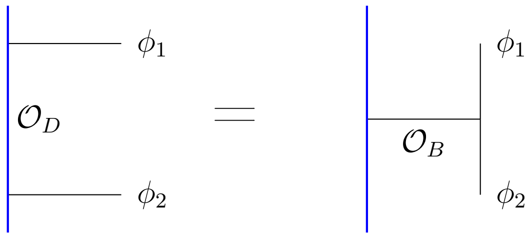


Figure 2.1: Crossing symmetry near the defect.

The BOE controls also the 2-point function between a bulk operator and a defect one

$$\langle \mathcal{O}(x_{\parallel,1}, x_\perp)\mathcal{O}_{D,n}(x_{\parallel,2})\rangle = \sum_{\hat{\Delta}_m} \frac{b_{\mathcal{O}_B \mathcal{O}_{D,m}}}{|x_\perp|^{\Delta_\mathcal{O} - \hat{\Delta}_m}} \langle \mathcal{O}_{D,m}(x_{\parallel,1})\mathcal{O}_{D,n}(x_{\parallel,2})\rangle \tag{2.9}$$

and from the defect two-point function we obtain

$$\langle \mathcal{O}(x_{\parallel,1}, x_\perp)\mathcal{O}_{D,n}(x_{\parallel,2})\rangle = \frac{b_{\mathcal{O}\mathcal{O}_D}}{|x_\perp|^{\Delta - \hat{\Delta}_n} |x_{\parallel,12}|^{2\Delta_n}} \tag{2.10}$$

2.2.1 Two-point function of bulk operators near the defect

Near the defect, the two-point function of two bulk operators is no more completely fixed by conformal invariance. Indeed, we can expand the operators using BOE

$$\langle \mathcal{O}_1 \mathcal{O}_2\rangle = \sum_{\hat{\Delta}_1,\, \hat{\Delta}_2} \frac{b_{\mathcal{O}_1 \mathcal{O}_{D,m}} b_{\mathcal{O}_2 \mathcal{O}_{D,n}}}{|x_{\perp,1}|^{\Delta_1 - \hat{\Delta}_m} |x_{\perp,2}|^{\Delta_2 - \hat{\Delta}_n}} \langle \mathcal{O}_{D,m}\mathcal{O}_{D,n}\rangle \tag{2.11}$$

and exploiting the two-point function we have

$$\langle \mathcal{O}_1 \mathcal{O}_2\rangle = \sum_{\hat{\Delta}} \frac{b_{\mathcal{O}_1 \mathcal{O}_{D,n}} b_{\mathcal{O}_2 \mathcal{O}_{D,n}} |x_{\parallel,12}|^{-2\hat{\Delta}_n}}{|x_{\perp,1}|^{\Delta_1 - \hat{\Delta}_n} |x_{\perp,2}|^{\Delta_2 - \hat{\Delta}_n}} \tag{2.12}$$

As for the OPE coefficient, the BOE coefficients depend on coordinates and derivatives (i.e. contain descendants contributions), and in general we can denote it as a function $f_{12}(\xi, \cos\phi)$ of two defect cross-ratios

$$\xi = \frac{x^2_{\parallel,12}}{|x_{\perp,1}|\;|x_{\perp,2}|} \qquad \cos\phi = \frac{\xi}{\xi+1} \tag{2.13}$$

The function f_{12} is constrained by a crossig symmetry similar to the one leading to the bootstrap equation and it's depicted in fig. 2.1, where its l.h.s. is the graphical representation of the expansion (2.11). The r.h.s. means to take the bulk OPE and then expanding using BOE. We can phrase the crossing symmetry as

$$\text{defect OPE} \circ BOE = BOE \circ \text{bulk OPE} \tag{2.14}$$

So take the bulk OPE, as (1.18)

$$\langle \mathcal{O}_1\mathcal{O}_2\rangle = \sum_{\Delta} C_{\Delta}(x,\partial)\langle \mathcal{O}_{\Delta}\rangle = \sum_{\Delta}\frac{C_{\Delta}(x,\partial)b_{\mathcal{O}_{\Delta}\hat{\mathbb{1}}}}{|x_{\perp}|^{\Delta}} \tag{2.15}$$

Equating (2.12) and (2.15) we obtain the *defect bootstrap equation*, giving constraints in both OPE and BOE coefficient [47]. Of course, on the defect there will be a conformal bootstrap equation relating the defect OPE coefficients.

Displacement operator

The presence of the defect will break the translational invariance. As usual, the breaking of symmetry gives rise to new operators. In this case, the breaking of the translation is encoded in the non-conservation of the stress-energy tensor

$$\partial_{\mu}T^{\mu i} = \delta^{p}(x - x_D)\mathbb{D}^{i} \tag{2.16}$$

where the index i represents coordinates transverse to the defect. The object $\mathbb{D}$ is called Displacement operator and it measures the response of the defect under small deformation of its shape. The delta function means that the stress-energy conservation is broken on the defect.

2.3 Wilson Loops

Wilson Loops were introduced [48] in the context of QCD trying to explain the confinement of quarks at low energies. Due to the nature of the strong interaction, we can not see single quarks at low energies. It is possible at high energies, thanks to the asymptotic freedom, where perturbation theory is viable. Moreover, Wilson Loops are considered as the order parameter between the confined and free phases. They are the most general gauge-invariant observables and they can be defined in any gauge theory.

The Wilson Loop represents the phase factor picked up by a charge moving in an external potential. It is realized as the holonomy of the gauge connection

$$W_C = \frac{1}{\dim_R} Tr_R \mathcal{P}exp\left(i\lambda \oint_C dx^{\mu}A_{\mu}(x)\right) \tag{2.17}$$

where R is the representation of the gauge group and λ the coupling of the theory. The $\mathcal{P}exp$ is the *Path-ordered exponential*, whose series expansion is

$$\mathbb{1} + i\lambda\int d\tau \dot{x}^{\mu}A_{\mu} - \lambda^2\int d\tau 1\int^{\tau_1} d\tau_2 \dot{x}_1^{\mu}\dot{x}_2^{\nu}A_{\mu}(x_1)A_{\nu}(x_2) + \dots \tag{2.18}$$

provided that $\tau_1 > \tau_2$. It's well know how to use the Wilson Loop to describe quark-antiquark potential [49, 50]. Moreover, Wilson Loops, together with correlation functions of operators inserted in them, provide a natural example of dCFT.

2.3.1 Supersymmetric Wilson Loops

Wilson Loops can be defined in supersymmetric gauge theories as well. The supersymmetry condition requires a modified version of the gauge connection, including the presence of other fields to make the object SUSY-invariant. Here, as an example, we follow the construction of the Maldacena-Wilson Loop in $\mathcal{N} = 4$ SYM [51]. It is defined as

$$W_C = \frac{1}{dim_R}\,\mathrm{Tr}_R\,\mathcal{P}\exp\left[\oint_C d\tau\,\left(i\dot{x}^\mu A_\mu + |\dot{x}|\,\theta_I\phi^I\right)\right] \tag{2.19}$$

where τ is the parameter of the curve $x^\mu(\tau)$ and $\theta_I,\ I = 1,\ldots,6$ is a unit vector on the 5-sphere, specifying which scalars enter in the new connection. The amount of supersymmetry preserved by the loop (2.19) depends on the shape of the contour [52]:

- for the most general contour lying in $\mathbb{R}^4$, the Wilson Loop preserves $1/16$ of the supercharges (1/6-BPS);
- if it lies in a three-dimensional subspace, e.g. a $\mathbb{R}^3$ time slice at $x^0 = 0$, the amount of supercharges preserved are doubled, i.e. $1/8$ of the total number (1/8-BPS);
- if the contour lies on a plane, then the supersymmetry is enhanced to $1/4$ supercharges preserved (1/4-BPS);
- the highest number of supercharged preserved is for line contour, thus 1/2-BPS.

In [53, 54] they studied Wilson Loop on S^3, called DGRT-loop. Their construction is similar to the one in flat space, but the expectation values are usually more involved functions of the couplings and the preserved supersymmetries are usually combinations of Poincaré and conformal supercharges.

2.4 Wilson Loops In ABJ(M)

In this section, we present the main character of this thesis, the 1/2-BPS Wilson Loop inside ABJ(M). The gauge structure of ABJ(M) gives rise to a variety of loops: the straightforward generalization of (2.19) in this case turns out to be less supersymmetric than its counterpart. To find a maximal supersymmetric object, the gauge quiver has to be augmented to a supergroup, forcing the superconnection to be a supermatrix including fermions, whilst relaxing the supersymmetry condition to a supergauge transformation. In what follows, we will restrict ourselves to the maximum supersymmetric case, the infinite straight line (or the circle).

2.4.1 1/6-BPS "Bosonic" Loops

Since in ABJ(M) we have two different gauge groups, the simplest thing to do is to generalize the Maldacena-Wilson Loop (2.19) for each gauge field. In ABJ(M), scalar fields live in the bifundamental representation of the gauge group. But the product of two scalars fields live in the adjoint representation of one of the two gauge groups, depending on the product order, namely

$$C_I \bar{C}^J \to \text{adjoint of } U(N_1)_k \tag{2.20}$$

$$\bar{C}^I C_J \to \text{adjoint of } U(N_2)_{-k} \tag{2.21}$$

such that we can from two types of loops

$$W_C = \frac{1}{dim_R} \operatorname{Tr}_R \mathcal{P} \exp\left[\int_C d\tau \left(i\dot{x}^\mu A_\mu + \frac{2\pi|\dot{x}|}{k} M^I{}_J C_I \bar{C}^J\right)\right] \tag{2.22}$$

and

$$\widehat{W}_C = \frac{1}{dim_R} \operatorname{Tr}_R \mathcal{P} \exp\left[\int_C d\tau \left(i\dot{x}^\mu \hat{A}_\mu + \frac{2\pi|\dot{x}|}{k} \widehat{M}_I{}^J \bar{C}_I C^J\right)\right] \tag{2.23}$$

and requiring them to be supersymmetric fixes the shape of the scalar coupling matrices [55]

$$M^I{}_J = \widehat{M}_I{}^J = diag(-1, 1, -1, 1) \tag{2.24}$$

while highlighting that these operators preserve only 1/6 of the supercharges.

In the dual theory, the most supersymmetric object is a classical string preserving half of the supercharges. Therefore, there should be an analog object in the gauge theory.

2.4.2 1/2-BPS "Fermionic" Loops

In order to find the loop that preserves half of the supercharges, we have to add fermions, from this the name *fermionic loop*. The only way to fit the gauge structure and the requirement of scaling dimension to be one is to pack everything in a superconnection written in a supermatrix form [56]:

$$\mathcal{L} = \begin{pmatrix} A_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| \, M^I_J C_I \bar{C}^J & i\sqrt{\frac{2\pi}{k}} \, |\dot{x}| \, \eta \bar{\Psi} \\ -i\sqrt{\frac{2\pi}{k}} \, |\dot{x}| \, \Psi \bar{\eta} & \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| \, M_I{}^J \bar{C}^J C_I \end{pmatrix} \tag{2.25}$$

where the fermionic couplings $\eta, \bar{\eta}$ are even spinors. The susy condition has to be relaxed to a weaker form[2]

$$\delta_{SUSY} \mathcal{L} = \mathfrak{D}_\tau G = \partial_\tau G + i\,[\mathcal{L}, G\} \tag{2.26}$$

[2]Imposing $\delta_{SUSY}\mathcal{L} = 0$ will lead us to the bosonic loops, i.e. $\eta = \bar{\eta} = 0$.

where $\mathfrak{D}$ is the covariant derivative on the loop:

$$\mathfrak{D} = \partial + i\,[\mathcal{L}, \cdot\}\tag{2.27}$$

and $[\cdot, \cdot\}$ is the supercommutator as defined in (D.5). The 1/2-BPS Wilson Loop then is defined as

$$W_C = s\,\mathrm{Tr}\,\mathcal{P}\exp\left(-i\oint_C ds\,\mathcal{L}\right)\tag{2.28}$$

Note that when dealing with supergroups, the supertrace is the one invariant under similitude transformation. The minus sign in front of the integral is dictated by gauge covariance. However, for closed loops, there are boundary conditions to take care of. In the circle case, we have anti-periodic boundary conditions, making the loop gauge-invariant only when taking its trace. On the other hand, for infinite straight circuits, i.e. the infinite line, we can choose the fields to vanish at the infinity, giving us the freedom to choose either the trace or the supertrace, and the result will still be gauge invariant. In the following, since line and circle are conformally equivalent, we will choose the trace.

SUSY condition (2.26) fixes the form of the scalar coupling matrix to be [57]

$$M_I{}^J = \ell\begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\tag{2.29}$$

where ℓ can be ± 1, and gives constraints to the fermionic couplings

$$\delta^\beta_\alpha = \frac{1}{2i}(\eta^\beta\bar{\eta}_\alpha - \eta_\alpha\bar{\eta}^\beta) \qquad (\dot{x}\cdot\gamma)_\alpha{}^\beta = \frac{\ell}{2i}|\dot{x}|(\eta^\beta\bar{\eta}_\alpha + \eta_\alpha\bar{\eta}^\beta)\tag{2.30}$$

The choice of the free parameter ℓ seems to be related to the parity-like symmetry of ABJ(M).

Moreover, the fermionic loop share with the bosonic ones all the conserved supercharges. Thus, the difference between the two types of loops can be related to an action of a linear combination of the common supercharges on an object V such that [56]

$$W_{1/2} - \left(W_{1/6} + \widehat{W}_{1/6}\right) = \mathcal{Q}\,V\tag{2.31}$$

Thus, $\mathcal{Q}$-exact terms do not contribute to correlation functions. Then, the expectation value of the loops are equal

$$\langle W_{1/2}\rangle = \langle W_{1/6}\rangle + \langle\widehat{W}_{1/6}\rangle\tag{2.32}$$

2.4.3 More general contours

We saw for the Maldacena-Wilson loop that the choice of the contour is crucial in determining the number of supersymmetry preserved by the loop. Here we want to present some other configurations that have been studied.

Latitude

Latitude Wilson Loops in ABJ(M) are the cousins of DGRT-loop in four dimensions. They are defined on the sphere S^2, characterized by non-constant couplings, depending on an internal angle that specifies the *latitude* on the sphere [57]. For the special value of $\alpha = 0$, namely the great circle or equator of the sphere, we recover the usual Wilson Loop defined on the circle. For $\alpha \neq 0$, these new loops preserve a reduced number of supercharges, usually half of the amount preserved on the respective great circle, so that we have the bosonic latitude to be 1/12-BPS, whilst the fermionic one is 1/4-BPS. A cohomological relation between fermionic and bosonic loops as (3.26) holds for the latitudes as well [58].

Latitude Wilson Loops has been studied intensively in three dimension [59, 60], four dimensions [61, 62] and even for more complicated geometry, like the squashed sphere[63, 64].

Cusp

Another interesting contour configuration is the cusp. It is realized as two infinite lines connecting in the origin, forming an angle between them. The cusp has two more parameters, namely the angle φ between the two lines and the internal angle θ specifying the difference between the couplings on the different lines. This configuration globally breaks all the supersymmetries, while preserving half of them on the separate lines. The small angles limit has to reproduce the infinite straight line. In the next section we see why the cusp Wilson Loop is related to the Bremstrahlung function.

2.5 The Bremsstrahlung function

Wilson Loops usually describe the motion of heavy charged particles in an external fields. It is very well known that an accelerated charge emits radiation, and the process takes the name of *Bremsstrahlung*. The emitted radiation is controlled by the Bremmstrahlung function, and it is the main physical observable in the context of conformal gauge theories. It can be defined in the following way [65]

$$\Delta E = 2\pi B \int dt \, (\dot{v})^2 \tag{2.33}$$

for a slow moving heavy quark. In particular, for $\mathcal{N} = 4$ SYM the Bremmstrahlung can be computed exactly, combining perturbative and non-perturbative methods [65]. The Bremsstrahlung function usually appears in the following situations:

- it is the coefficient of the small angle limit of the cusp anomalous dimension

$$\Gamma_{cusp}(g, \phi, \theta) = -B(g)\left(\phi^2 - \theta^2\right) \tag{2.34}$$

where ϕ is the angle between the two lines forming the cusp and θ is the internal couplings angle [66, 67, 68, 69];

- it can be computed as the derivative of the logarithm of the latitude WL vev w.r.t. the latitude parameter ν, at $\nu = 1$ [60, 58]

$$B_{1/2}(\lambda) = \frac{1}{4\pi^2} \partial_\nu \log \langle W_F(\nu) \rangle \bigg|_{\nu=0} \tag{2.35}$$

- it is the coefficient of the two-point correlation function of the Displacement operator in the dCFT living on the Wilson Loop [70, 71, 72]

$$\langle \mathbb{D}(x_1) \mathbb{D}(x_2) \rangle \sim \frac{B_{1/2}(\lambda)}{x_{12}^4} \tag{2.36}$$

since there are evidences that this quantity is related to the one point-function of the bulk stress-energy tensor [45]. Indeed, the displacement operator measure the response of the Wilson Loop under deformation of the contour, that has to be under emitted energy.

On the last point, since the displacement operator presence is due to a broken symmetry, we expect it is a protected operator. Thus, its correlation functions should be free from divergences, thus allowing us to compute the Bremmstrahlung function and its correction by perturbation theory. In the last chapter of this thesis we will check the finiteness at 1-loop for the fermionic Wilson Loop.

The Bremsstrahlung function is known exactly in four dimensions for $\mathcal{N} = 2$ theories [73] and for $\mathcal{N} = 4$ SYM [65]. In ABJ(M), although we have many results both at weak at strong coupling, an exact derivation is still missing. Recently, in [74], it has been conjectured an exact form for the interpolating function of ABJ(M), governing the behavior of the Bremsstrahlung function as well.

II

One-dimensional sectors in ABJ(M)

3

The topological sector

You can recognize truth by its beauty and simplicity.

R. P. Feynman

Combining supersymmetric localization [75, 76] with bootstrap techniques has made analytic computations in SCFTs possible, as shown for example in [77, 78, 79, 80, 81, 82, 83]. In these advances, a pivotal role has been played by topological sectors, consisting of a completely solvable set of correlation functions in a given SCFT. Their existence allows the extraction of useful information regarding the quantum theory, like OPE coefficients, numerical bounds on the spectrum exploiting bootstrap techniques, coefficients of Witten diagrams in the AdS duals, or the computation of exact quantities interpolating between strong and weak couplings regimes. A prototypical example of the topological sector appears in $\mathcal{N} = 4$ SYM in four dimensions [54, 84, 85]. In three dimensions, general properties of the superconformal algebra suggest that SCFTs with $\mathcal{N} \geq 4$ always contain a topological sector [78, 86].

In this light, the topological sector can represent a solvable truncation of the bootstrap equation [3]. In particular, the topological sector has played a notable role in performing a precision study of maximally supersymmetric ($\mathcal{N} = 8$) SCFTs through conformal bootstrap, allowing to compute exactly some OPE data and constraining regions in the parameter space [78, 82, 83]. At the same time, it has been instrumental in fixing contributions to the scattering amplitudes of supergravitons in M-theory in eleven dimensions [87].

In ABJ(M), the Chern-Simons term prevents us to build a one-dimensional action for the topological sector: we found that twisting conformal symmetry with R-symmetry will produce a cohomological relation satisfied by topological operators. The shape of the symmetries allows us to twist only a one-dimensional sector of the theory.

In this chapter we describe in detail the twisting procedure to build the topological sector, finding the selections rules on the Dynkin labels, and identifying the supermultiplets involved. We provide a field realization of the superprimary operator, generalizing it to higher-dimensional operators. We compute the quantum correction to the two-point function, by means of Feynman diagrams regulated by dimensional reduction scheme. We then present the evaluation of the integrated two-point function and the central charge c_T at weak coupling from the mass-deformed matrix-model, comparing the results to the one obtained by perturbation theory.

3.1 Twist

In general, a twist is an identification of two isomorphic groups. The idea is to define a sub-sector of operators under this new symmetry group (made out from a combination of the original groups) for which correlation functions are easier to study. We start with an example and then generalize it to our case.

3.1.1 Chiral ring twist

We know that in CFT, and in SCFT too, a central role is played by correlation function. In particular, theories are fully determined once we specify the conformal data, encoded in the coefficient of the 2- and 3-pt functions. Although giving the complete set of conformal data is a somehow hard task, being the set of correlators infinite, it legit to try to restrict the problem to sub-sectors of the theory, finding classes of operators that transform in a particular way under the symmetries of the theory.

In SCFT, we can exploit supersymmetry to find a class of operators called *chiral ring*: it is formed by operators annihilated by a certain number of supercharges

$$[Q_\alpha, \mathcal{O}] = 0 \tag{3.1}$$

Operators in the chiral ring have position-independent correlation functions. Moreover, they are identically vanishing, due to the fact that the superconformal algebra forces the R-symmetry charge to be proportional to the scaling dimension. Since the scaling dimension is always positive for unitary theories, the total R-symmetry charge will not be zero, forcing the correlation functions to vanish identically.

But this is not a trivial sectors as at first sight. Troubles come from not having a singlet under the R-symmetry [78]. Then the key idea is to find a combination of translation plus R-symmetry rotations that are exact under a new supercharge

$\mathcal{Q}$, $\mathcal{Q}^2 = 0$. The new supercharge will define a class of operators at the origin, and the new supertranslation will keep this class unaffected. In particular, we will find, together with the exact generators $\hat{P}$, some $\mathcal{Q}$-closed generators $\tilde{P}$, such that

$$\hat{\mathcal{O}}(\tilde{x},\hat{x}) = e^{i\tilde{x}_i\tilde{P}^i + i\hat{x}_j\hat{P}^j}\hat{\mathcal{O}}(0)e^{-i\tilde{x}_i\tilde{P}^i - i\hat{x}_j\hat{P}^j} \tag{3.2}$$

is in the same class of $\hat{\mathcal{O}}$, provided that $\left[\mathcal{Q},\hat{\mathcal{O}}(0)\right] = 0$. Correlation functions then

$$\langle\hat{\mathcal{O}}_1(\tilde{x}_1,\hat{x}_1)\cdots\hat{\mathcal{O}}_n(\tilde{x},\hat{x}_n)\rangle = f\left(\tilde{x}_1,\ldots,\tilde{x}_n\right) \tag{3.3}$$

will be function only of the $\mathcal{Q}$-closed coordinates; in this sense can be viewed as a lower dimensional theory.

This procedure seems a bit abstract now, but it will be clear in all details when applied to the ABJM case.

3.1.2 Twisting the line in ABJ(M)

We want to restrict to the line along the third dimension, parametrized by $x^\mu = 0, 0, s,\ s \in (-\infty,\infty)$ being its proper time. The conformal group on the line is $\mathfrak{su}(1,1)$, with generators

$$P = iP_3 \qquad K = iK_3 \qquad D \tag{3.4}$$

satisfying the following algebra

$$[D,P] = P \qquad [D,K] = -K \qquad [P,K] = -2D \tag{3.5}$$

The set of the fermionic supercharges generating the z-line are

$$Q_1^{12},\ Q_1^{13},\ Q_1^{14},\ Q_2^{23},\ Q_2^{24},\ Q_2^{34} \tag{3.6}$$

together with the superconformal charges

$$S_1^{12},\ S_1^{13},\ S_1^{14},\ S_2^{23},\ S_2^{24},\ S_2^{34} \tag{3.7}$$

It's useful to notice that these supercharges can be reorganized in a $SU(3)$ invariant form as

$$\begin{array}{ll} Q^{k-1} = Q_1^{1k} & \bar{Q}_{k-1} = \frac{i}{2}\varepsilon_{klm}Q_2^{lm} \\ S^{k-1} = iS_1^{1k} & \bar{S}_{k-1} = \frac{1}{2}\varepsilon_{klm}S_2^{lm} \end{array} \qquad k,\ l,\ m = 2,\ 3,\ 4 \tag{3.8}$$

and renaming the $SU(3)$ index as $a = k-1$ such that a runs from 1 to 3. In the appendix B.2 we list all the commutation relations defining this one-dimensional superconformal algebra. The number of the supercharges to generate the line it's half of the total ABJM supercharges, thus the line is 1/2-BPS.

It becomes natural then to split the R-symmetry in a $\mathfrak{su}(3)$ subalgebra generated by

$$R_a{}^b = \begin{pmatrix} J_2{}^2 + \frac{1}{3}J_1{}^1 & J_2{}^3 & J_2{}^4 \\ J_3{}^2 & J_3{}^3 + \frac{1}{3}J_1{}^1 & J_3{}^4 \\ J_4{}^2 & J_4{}^3 & -J_3{}^3 - J_2{}^2 - \frac{2}{3}J_1{}^1 \end{pmatrix} \tag{3.9}$$

satisfying the algebraic relation

$$[R_a{}^b, R_c{}^d] = \delta_a^d R_c{}^b - \delta_c^b R_a{}^d \tag{3.10}$$

There is still a residual $\mathfrak{u}(1)$ generator M left, defined as

$$M \equiv 3iM_{12} - 2J_1{}^1 \tag{3.11}$$

The superconformal algebra preserved by this line is a $\mathfrak{su}(1,1|3) \oplus \mathfrak{u}(1)_b$ inside the original $\mathfrak{osp}(6|4)$ of ABJ(M). We have already seen the commutation relations for $\mathfrak{osp}(6|4)$ in Section 1.3.1. The conventions for the $\mathfrak{su}(1,1|3)$ superalgebra are listed in Appendix B.2. In the latter, we also specify our choice for the embedding of the preserved superalgebra inside $\mathfrak{osp}(6|4)$. From the shape of the preserved subalgebra, it is convenient to reorganize the scalars $C_I, \bar{C}^I$ and the fermions $\psi_I, \bar{\psi}^I$, $I = 1, 2, 3, 4$ in irreducible representations of $SU(3)$; we split them as

$$C_I = (Z, Y_a) \qquad \bar{C}^I = (\bar{Z}, \bar{Y}^a) \qquad \psi_I = (\psi, \chi_a) \qquad \bar{\psi}^I = (\bar{\psi}, \bar{\chi}^a) \qquad a = 1, 2, 3 \tag{3.12}$$

where $Y_a(\bar{Y}^a), \chi_a(\bar{\chi}^a)$ belong to the $\mathbf{3}(\bar{\mathbf{3}})$ of $SU(3)$, while $Z, \bar{Z}, \psi, \bar{\psi}$ are $SU(3)$-singlets. Gauge fields split according to the new spacetime symmetry as

$$\begin{aligned} A_\mu =& (A \equiv A_1 - iA_2,\ \bar{A} \equiv A_1 + iA_2,\ A_3) \\ \hat{A}_\mu =& (\hat{A} \equiv \hat{A}_1 - i\hat{A}_2,\ \hat{\bar{A}} \equiv \hat{A}_1 + i\hat{A}_2,\ \hat{A}_3) \end{aligned} \tag{3.13}$$

together with the corresponding covariant derivatives (see their definition in (1.60))

$$D_\mu = (D \equiv D_1 - iD_2,\ \bar{D} \equiv D_1 + iD_2,\ D_3) \tag{3.14}$$

The R-symmetry group is still bigger than the conformal line group. In order to perform the twist, we need the groups to be isomorphic. We can choose a $\mathfrak{su}(1,1) \simeq \mathfrak{su}(2)$ inside $\mathfrak{su}(3)$

$$\mathfrak{su}(1,1) \simeq \left\{ \mathcal{R}_+ = iR_3{}^1, \quad \mathcal{R}_- = iR_1{}^3, \quad \mathcal{R}_0 = \frac{1}{2}\left(R_1{}^1 - R_3{}^3\right) \right\} \tag{3.15}$$

satisfying the algebra

$$[\mathcal{R}_0, \mathcal{R}_\pm] = \pm\mathcal{R}_\pm \qquad [\mathcal{R}_+, \mathcal{R}_-] = -2\mathcal{R}_0 \tag{3.16}$$

coming along with a $\mathfrak{u}(1)$ generator $\frac{1}{2}\left(R_1{}^1 + R_3{}^3\right)$ that commutes with every generators in (3.15).

The twisting is defined as the diagonal sum of the generators of the two subalgebras

$$\widehat{\mathfrak{su}}(1,1) = \left\{ \hat{L}_+ = P + \mathcal{R}_+, \quad \hat{L}_- = K + \mathcal{R}_-, \quad \hat{L}_0 = D + \mathcal{R}_0 \right\} \tag{3.17}$$

and it is easy to see that they respect the algebra

$$\left[\hat{L}_0, \hat{L}_\pm\right] = \pm \hat{L}_\pm \qquad \left[\hat{L}_+, \hat{L}_-\right] = -2\hat{L}_0 \tag{3.18}$$

Under this new group, the supercharges Q^3, S^1 together with their hermitian conjugates become scalars, i.e.

$$\left[\hat{L}_0, Q^3\right] = 0 = \left[\hat{L}_0, S^1\right] \tag{3.19}$$

We can use them to define two new nihilpotent supercharges

$$\mathcal{Q}_1 = Q^3 + iS^1 \qquad \mathcal{Q}_2 = \bar{S}_3 + i\bar{Q}_1 \qquad \mathcal{Q}_1^2 = \mathcal{Q}_2^2 = 0 \tag{3.20}$$

such that the generators (3.17) are $\mathcal{Q}$-exact in both supercharges

$$\begin{aligned} \hat{L}_+ &= \left\{\mathcal{Q}_1, \bar{Q}_3\right\} = -i\left\{\mathcal{Q}_2, Q^1\right\} \\ \hat{L}_- &= \left\{\mathcal{Q}_2, \bar{S}_1\right\} = -i\left\{\mathcal{Q}_1, S^3\right\} \\ \hat{L}_0 &= \frac{1}{2}\left\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\right\} = \frac{1}{2}\left\{\mathcal{Q}_2, \mathcal{Q}_2^\dagger\right\} \end{aligned} \tag{3.21}$$

The central extension of the superalgebra spanned by $\left\{\hat{L}_\pm, \hat{L}_0, \mathcal{Q}_{1,2}\right\}$ is given by

$$\mathcal{Z} = \frac{1}{4}\left\{\mathcal{Q}_1, \mathcal{Q}_2\right\} = \frac{1}{3}M - \frac{1}{2}\left(R_1{}^1 + R_3{}^3\right) \tag{3.22}$$

where M is the super-rotational generator defined in (3.11).

We can use these two new supercharge to define a new class of operators, those annihilated by one of them, or a combination. We will denote with $\mathcal{Q}$ the supercharge used for the cohomology. The new class of operators is defined as

$$[\mathcal{Q}, \mathcal{O}(s)\} = 0 \text{ provided that } \mathcal{O}(s) \neq [\mathcal{Q}, \mathcal{O}'(s)\} \tag{3.23}$$

where s is the coordinate along the third direction. Since the twisted translation operator $\hat{L}_+$ is $\mathcal{Q}$-exact, we can focus on operators placed in the origin. Indeed, the translation

$$\mathcal{O}(s) = e^{is\hat{L}_+}\mathcal{O}(0)e^{-is\hat{L}_+} \tag{3.24}$$

will not change the cohomology class.

Let's see the operator content of this class. The condition (3.23) gives two strong constraint on the quantum numbers. In the appendix B.2.1, we describe the choice of quantum numbers; just to recap, in our case it is convenient to choose labels as $|\Delta, m, j_1, j_2\rangle$, where Δ is the dilatation eigenvalue, m the eigenvalue of M and $j_{1,2}$ are the $\mathfrak{su}(3)$ Dynkin labels.

Since $\hat{L}_0$ and $\mathcal{Z}$ commute, condition (3.23) only allows the eigenvalues to be zero. Written in term of our labels, the two conditions read

$$\hat{\ell}_0 = \Delta - \frac{j_1 + j_2}{2} = 0 \qquad z = \frac{1}{3}\left(m - \frac{j_2 - j_1}{2}\right) = 0 \tag{3.25}$$

then operators in our cohomology have quantum numbers related by

$$\Delta = \frac{j_1 + j_2}{2} \qquad m = \frac{j_2 - j_1}{2} \tag{3.26}$$

The long multiplet $\mathcal{A}^{\Delta}_{m;j_1,j_2}$ satisfies (3.26) only at the unitary threshold. Therefore, the superconformal primaries of the $\mathcal{A}$ multiplets at the threshold certainly belong to the cohomology of $\mathcal{Q}$. However, at the threshold, the long multiplet recombines into short multiplets according to the decomposition (B.24). Looking closer, the topological operators are actually the highest weight operators of the short multiplets $\mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1,j_2}$ in (B.24)[1].

For the short multiplet case, the shortening condition (B.30) together with (3.26) are always satisfied by the superprimaries of $\mathcal{B}^{\frac{1}{6},0}_{\frac{j_2-j_1}{2};j_1,j_2}$ and $\mathcal{B}^{0,\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1,j_2}$, for generic values of j_1 and j_2.

Field realization

The conditions (3.26) give us a straightforward method to build the operators in the cohomology starting from the ABJM fields. The lowest dimensional local, gauge invariant operator is build as a product of two scalar field. If we take, in the origin, the couple Y_1 and $\bar{Y}^3$, our selection rules are matched. Define

$$\mathcal{O}(0) = \mathrm{Tr}\left[Y_1(0)\bar{Y}^3(0)\right] \tag{3.27}$$

mathcing (3.26) with eigenvalues $[1, 0, 1, 1]$. Using (3.24) to translate it to a general point s on the line, we can rewrite the result in a compact form, by introducing position dependent R-symmetry polarization vectors

$$\mathcal{O}(s) = \mathrm{Tr}\left[Y_a(s)\bar{Y}^b(s)\right]\bar{u}^a(s)v_b(s), \qquad \bar{u}^a = (1, 0, s)\,,\; v_b = (-s, 0, 1) \tag{3.28}$$

[1]No other topological operator arises from the descendants of $\mathcal{B}$ multiplets in decomposition (B.24). More details can be found in [78, 35].

Higher dimensional operators and OPE

In general, we can build an operator inside the cohomology with dimension $\Delta = n$ just taking the product of$n$ couple of $Y, \bar{Y}$:

$$\mathcal{O}_n(0) = \mathrm{Tr}\left[Y_1(0)\bar{Y}^3(0)\ldots Y_1(0)\bar{Y}^3(0)\right] \tag{3.29}$$

and using (3.24), expanding the exponential at the right order, we will get the same as (3.28) with n $\bar{u}$s and vs. Moreover, multi-trace operators build from traces of the form (3.29) will still lie in the same cohomology class.

We can generalize this procedure in two cases: single-trace $\tilde{O}$ and a multi-trace $\widehat{O}$

$$\tilde{O}_J(s) = \mathrm{Tr}\left[\left(\bar{u}^a(s)v_b(s)Y_a(s)\bar{Y}^b(s)\right)^J\right] \qquad \widehat{O}_J(s) = \left(\mathrm{Tr}\left[\bar{u}^a(s)v_b(s)Y_a(s)\bar{Y}^b(s)\right]\right)^J \tag{3.30}$$

In $\tilde{O}$ we take the trace after the product oj J couples of $Y\bar{Y}$, while $\widehat{O}$ is the product of J different traces. It is easy to check they are still topological operators. When $J = 1$, they both reduce to (3.28), so we consider only $J > 1$. It is very instructive to take a look the following three-point function

$$\langle \mathcal{O}(s_1)\mathcal{O}(s_2)\tilde{O}_J(s_3)\rangle \tag{3.31}$$

For $J = 1$, it reduce to a three-point function of three topological operator, and we know it vanishes. For $J = 2$, it doesn't vanish, and for $J > 2$ it vanishes again. The same happens for the multi-trace operator. In general, if we consider the arbitrary three-point function

$$\langle \tilde{O}_{J_1}(s_1)\tilde{O}_{J_2}(s_2)\tilde{O}_{J_3}(s_3)\rangle \qquad J_3 > J_1, J_2 \tag{3.32}$$

we can see that can only be non vanishing if[2] $J_1 + J_2 = J_3$. Indeed, if we expand $\tilde{\mathcal{O}}_{\Delta_1}\tilde{\mathcal{O}}_{\Delta_2}$ with OPE

$$\tilde{\mathcal{O}}_{\Delta_1}\tilde{\mathcal{O}}_{\Delta_2} \sim \sum_k \frac{1}{s^k}\left(\tilde{\mathcal{O}}_{\Delta} + \widehat{\mathcal{O}}_{\Delta}\right) \tag{3.33}$$

we see that the only operators that can enter in the OPE for topological operators are those with $k = 0$ and for the constraint (1.19) we have

$$\Delta = \Delta_1 + \Delta_2 \tag{3.34}$$

This is an example of truncation of the OPE series since only two operators enter in the expansion of $\mathcal{O}\mathcal{O}$. It would be interesting to understand how this truncation affects the bootstrap equation.

[2] A similar condition has been found using general superspace arguments [35]

3.2 Correlation functions and quantum corrections

So far we have defined a new class of operators exploiting the twisting procedure. To further investigate the properties of this class, we have to study the correlation functions between operators in the class. In this section, we will do it using perturbation theory. We will then compare the results with the ones obtained via localization.

The perturbative computation are done expanding the Euclidean path integral in powers of the coupling constant k^{-1}

$$\langle \mathcal{O}_1(s_1)\cdots\mathcal{O}_n(s_n)\rangle = \int [\mathcal{D}\Phi]\,\mathcal{O}_1(s_1)\cdots\mathcal{O}_n(s_n)\; e^{-S_{ABJ(M)}} \tag{3.35}$$

where $S_{ABJ(M)}$ is the action of ABJ(M), whose explicit expression is given in 1.56, and then performing all possible fields contractions, using Feyman rules listed in appendix A.2.

From the tree level we see already it is topological. Using Feynman rule (A.16) we have

$$\langle \bar{u}^a v_b \,\mathrm{Tr}\left[Y_a\bar{Y}^b\right]\mathrm{Tr}\left[Y_1\bar{Y}^3\right]\rangle^{(0)} = -\frac{N_1N_2}{(4\pi)^2} \tag{3.36}$$

The space-time dependence cancellation comes from the fact that the contraction of the polarization vectors is identical to the world-line dependence at the propagator denominator.

The three-point function is always vanishing, while the four-point function

$$\langle \mathcal{O}(z)\mathcal{O}(t)\mathcal{O}(s)\mathcal{O}(0)\rangle^{(0)} = 2\frac{N_1N_2}{(4\pi)^4} \tag{3.37}$$

In computing the one-loop correction, the only non-a-priori vanishing diagrams are those in figure (3.1). Those contribution will eventually all be proportional to the integral

$$\int d^3x d^3y \frac{\varepsilon_{\mu\nu\rho}\, x^\mu y^\nu\,(x-y)^\rho}{|s-x||s-y|(x-y)^3x^3y^3} \tag{3.38}$$

that vanishes since the tensor contraction

$$\varepsilon_{\mu\nu\rho}\, x^\mu y^\nu (x-y)^\rho = 0 \tag{3.39}$$

So the one-loop correction is always vanishing for 2-, 3- and 4-point correlation functions of the topological operators.

The first non-vanishing correction comes at two loops. We have performed these computations only for the two-point function and the diagrams involved are drawn in fig. 3.2. The corresponding integrals are evaluated in momentum space, using the DRED scheme [88, 89] in $d = 3-2\varepsilon$, introducing a mass scale μ to correct the coupling constant.

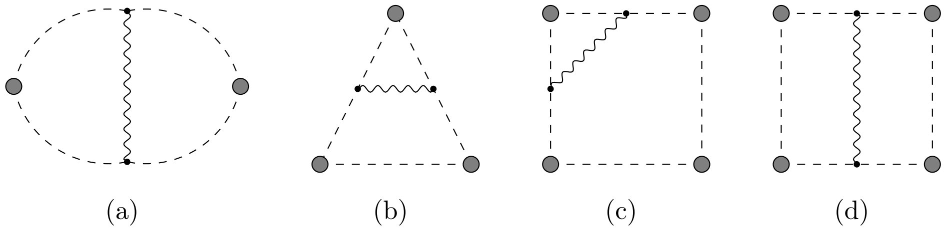


Figure 3.1: Topologies of one-loop diagrams contributing to the correlators.

The integral can be evaluated both using uniqueness method [90] or exploiting master integrals [91][3]. Here we list the results obtained with the second approach, while in [5] we have presented the first method. The results for each diagram are

$$\begin{aligned}(3.2(a)) = \frac{N_1N_2}{128\pi^2}\frac{|\mu s|^{8\varepsilon}}{k^2}\Bigg[&\frac{N_1^2+4N_1N_2+N_2^2-6}{\varepsilon}+(3\gamma)\left(N_1^2+4N_1N_2+N_2^2-6\right)\\ &-\log\left(256\pi^3\right)\left(N_1^2+4N_1N_2+N_2^2\right)+\pi^2N_1^2-6N_1^2+\pi^2(4N_1N_2)\\ &-(88N_1)N_2-6N_2^2+\pi^2N_2^2-6\pi^2+100+48\log(2)+18\log(\pi)+O\left(\varepsilon^1\right)\Bigg]\end{aligned} \tag{3.40}$$

$$(3.2(b)) = \frac{N_1N_2|\mu s|^{8\varepsilon}}{k^2}\frac{(\pi^2-12)\left(N_1N_2-1\right)}{16\pi^2}+O\left(\varepsilon^1\right) \tag{3.41}$$

$$(3.2(c)) = -\frac{N_1N_2|\mu s|^{8\varepsilon}}{k^2}\frac{N_1^2-(4N_1)N_2+N_2^2+2}{128\pi^2\varepsilon} \tag{3.42}$$

$$-\frac{\left(3\gamma-2+\log\left(\frac{1}{256\pi^3}\right)\right)\left(N_1^2-(4N_1)N_2+N_2^2+2\right)}{128\pi^2}+O\left(\varepsilon^1\right) \tag{3.43}$$

$$(3.2(e)) = \frac{N_1N_2|\mu s|^{8\varepsilon}}{k^2}\frac{(5\pi^2-48)\left(N_1N_2-1\right)}{96\pi^2}+O\left(\varepsilon^1\right) \tag{3.44}$$

$$(3.2(f)) = -\frac{N_1N_2|\mu s|^{8\varepsilon}}{k^2}\frac{N_1N_2-1}{16\pi^2\varepsilon}-\frac{\left(3\gamma-2+\log\left(\frac{1}{256\pi^3}\right)\right)\left(N_1N_2-1\right)}{16\pi^2}+O\left(\varepsilon^1\right) \tag{3.45}$$

$$(3.2(g)) = \frac{N_1N_2|\mu s|^{8\varepsilon}}{k^2}\frac{(\pi^2-12)\left(N_1^2-(4N_1)N_2+N_2^2+2\right)}{128\pi^2}+O\left(\varepsilon^1\right) \tag{3.46}$$

$$(3.2(j)) = -\frac{N_1N_2|\mu s|^{8\varepsilon}}{k^2}\frac{(\pi^2-12)\left(N_1N_2-1\right)}{48\pi^2}+O\left(\varepsilon^1\right) \tag{3.47}$$

$$(3.2(k)) = -\frac{N_1N_2|\mu s|^{8\varepsilon}}{k^2}\frac{(\pi^2-12)\left(N_1^2+N_2^2-2\right)}{192\pi^2}+O\left(\varepsilon^1\right) \tag{3.48}$$

$$(3.2(l)) = \frac{N_1N_2|\mu s|^{8\varepsilon}}{k^2}\frac{1}{64}(N_1-N_2)^2+O\left(\varepsilon^1\right) \tag{3.49}$$

Summing all the partial contributions the divergences nicely cancel out and with end up with a finite correction

$$\langle\mathcal{O}(s)\mathcal{O}(0)\rangle^{(2)} = -\frac{N_1N_2}{96\,k^2}\left(N_1^2+N_2^2-2\right)+O\left(\varepsilon^1\right) \tag{3.50}$$

[3]The master integrals method presented in [91] are for large N computations in ABJM. We have modified the expressions and found the final result coincides with the uniqueness computations.

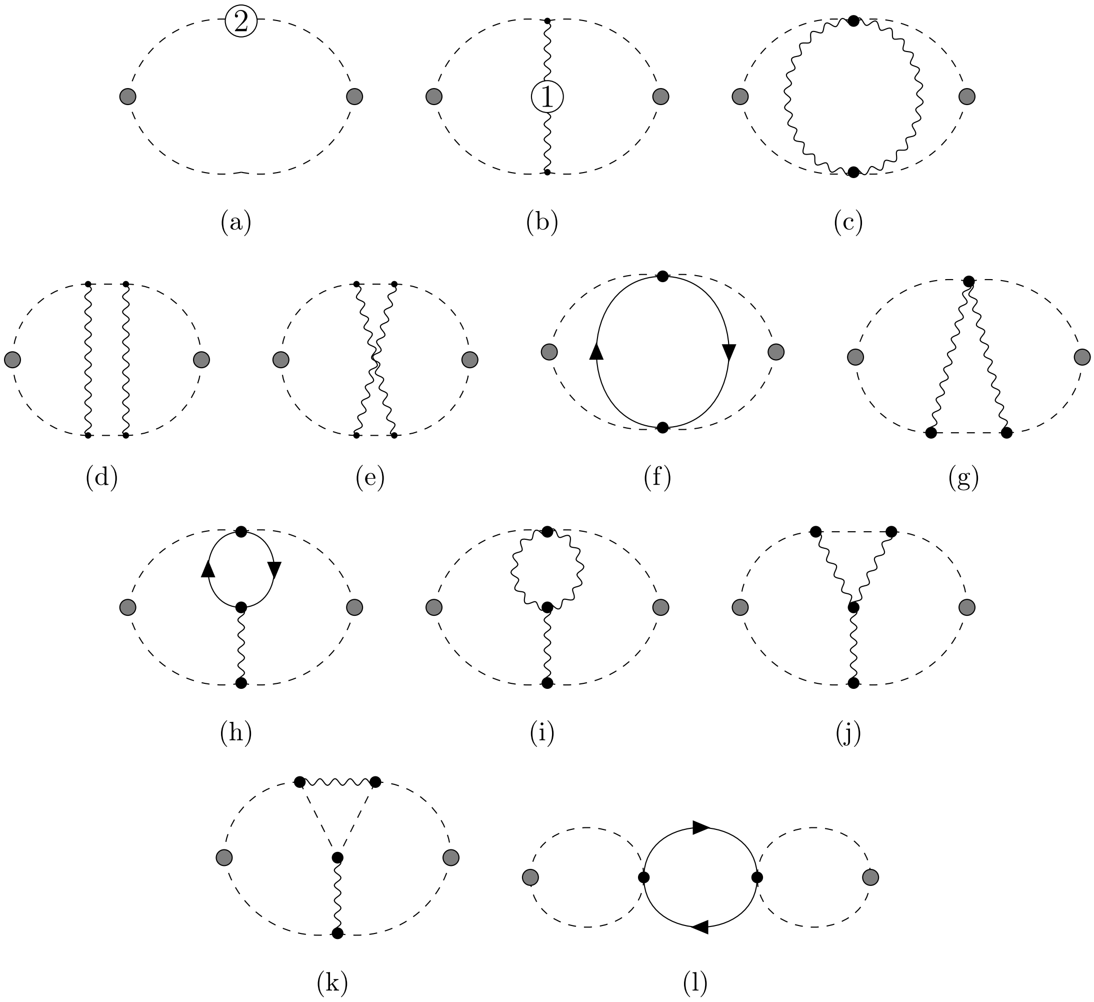


Figure 3.2: Two-loop diagrams for the two-point function. In (a) the white circle is the two-loop correction to the scalar propagator, while in (b) the circle is the one-loop correction to the gauge field propagator. Diagrams (h), (i), (j) and (k) sum up to provide the vertex correction.

Since all the divergences cancel out, we can safely take the limit $\varepsilon \to 0$ without worrying to possible scale dependent logarithm that could spoil the topologicity: they will always be produced at order ε and they are negligible in the limit we are interested in. The full 2-pt function read

$$\langle \mathcal{O}(s)\mathcal{O}(0)\rangle = -\frac{N_1 N_2}{(4\pi)^2}\left(1 - \frac{\pi^2}{6k^2}\left(N_1^2 + N_2^2 - 2\right)\right) + o(k^{-3}) \tag{3.51}$$

showing that the topological nature of the operators is preserved even at quantum level.

3.2.1 Central charge of ABJ(M) at weak coupling

Deforming the original SCFT by mass parameters m^a and localizing it on S^3 leads to a deformed MM which can be computed exactly in the large N limit []. On the other hand, this is equivalent to add to the one-dimensional Gaussian model mass terms for the fundamental (bosonic and fermionic) fields $\mathcal{J}^a$, of the form $-4\pi r^2 m^a \int_{-\pi}^{\pi} d\tau\, \mathcal{J}^a(\tau)$ [92]. Therefore, taking derivatives of the MM on S^3 respect to the mass parameters m^a provides integrated correlation functions of topologically twisted operators living on the great circle. Precisely, the crucial identity reads [82, 93]

$$\left\langle \int_{-\pi}^{\pi} d\tau_1 \ldots \int_{-\pi}^{\pi} d\tau_n\, \mathcal{J}^{a_1}(\tau_1)\ldots \mathcal{J}^{a_n}(\tau_n)\right\rangle = \frac{1}{(4\pi r^2)^n}\,\frac{1}{\mathcal{Z}}\,\frac{\partial^n}{\partial m^{a_1}\ldots\partial m^{a_n}}\mathcal{Z}[S^3, m^a]\Big|_{m^a=0} \tag{3.52}$$

where $\mathcal{Z}[S^3, m^a]$ is the partition function of the deformed theory on S^3 and r is the radius of the sphere. Since the topological correlators are position independent, the integrals on the l.h.s. can be trivially performed leading to a constant factor $(2\pi)^n$ times the correlator. Therefore, (3.52) provides an exact prescription for computing correlators in the one-dimensional topological sector in terms of the derivatives of the deformed MM of the three-dimensional theory. Read in the opposite direction, it allows to reconstruct the exact partition function of the three-dimensional theory on the sphere once we have solved the one-dimensional topological theory, i.e. we know exactly all its correlators.

On the other hand, as proved in [94], c_T can be independently computed from the mass deformed Matrix Model on S^3 as[4]

$$c_T = -\frac{64}{\pi^2}\frac{d^2}{dm^2}\log\mathcal{Z}[S^3, m]\Big|_{m=0} \tag{3.53}$$

Therefore, the consistency of the two independent results for c_T – the one obtained from the topological correlator and the one from (3.53) – represents an alternative way to prove the validity of (3.52), at least for $n = 2$. For the $\mathcal{N} = 8$ theories this has been discussed in details in [82].

In [93], the above ABJ(M) mass deformed matrix model has been used to fix some coefficient in Witten Diagrams computations of four-point functions at strong coupling. Derivatives with respect to the two massive deformation parameters give integrated correlation functions of operators sitting in the stress-energy supermultiplet, whose explicit form is

$$\mathcal{O}_I{}^J(x) = \mathrm{Tr}\left[C_I(x)\bar{C}^J(x)\right] - \frac{1}{4}\delta_I{}^J\,\mathrm{Tr}\left[C_K(x)\bar{C}^K(x)\right] \tag{3.54}$$

[4]We recall that an alternative prescription, which holds for any $\mathcal{N} \geq 2$ SCFT, amounts to placing the theory on the squashed sphere S_b^3, where b is the squashing parameter. It then follows that the central charge is given by the second derivative of the free energy in the squashed background w.r.t. to b [95].

Exploiting Ward Identities, we can relate the two-point function of (3.54) with the one of the stress-energy tensor, obtaining

$$\langle \mathcal{O}_I{}^J \mathcal{O}_K{}^L \rangle = \frac{c_T}{16} \left(\delta_I{}^L \delta_K{}^J - \frac{1}{4} \delta_I{}^J \delta_K{}^L \right) \frac{1}{16\pi^2 x_{12}^2} \tag{3.55}$$

where c_T is the central cherge of the theory.

If we write the topological operator (3.28) in $SU(4)$ formalism, we have

$$\mathcal{O}(s) = \mathrm{Tr}\left[C_I(0,0,s) \bar{C}^J(0,0,s) \right] U^I(0,\bar{0},s) V_J(0,0,s) \tag{3.56}$$

where

$$\bar{U}^I = (0,1,0,s) \qquad V_J = (0,-s,0,1) \tag{3.57}$$

we can relate the central charge to the two-point function of the topological operator, just by projecting (3.55) on the line $(0,0,s)$ and multiplying it by the polarization vectors $\bar{U}^I(s)V_J(s)\bar{U}^K(0)V_L(0)$, such that

$$c_T = -64(2\pi)^2 \langle \mathcal{O}(s)\mathcal{O}(0) \rangle \tag{3.58}$$

Inserting our result (3.51), we can easily obtain the correction to the ABJ(M) central charge

$$c_T = 16 N_1 N_2 \left(1 - \frac{\pi^2}{6k^2} \left(N_1^2 + N_2^2 - 2 \right) + o(k^{-3}) \right) \tag{3.59}$$

The above formula is valid for finite values of the gauge group ranks. Note that it is invariant under the ABJ parity-like symmetry. At tree-level, it reproduces the central charge correct value for a free theory of $4(N_1 N_2)$ chiral multiplets. In the specific case of ABJM with $N_1 = N_2 = 2$, the result matches with the two-loop approximation found in [77].

Since the topological operators $\mathcal{O}(\tau)$ are related to (3.54) and localized on the great circle $S^1 \subset S^3$, exploiting (3.55) we can compute c_T from their two-point function $\langle \mathcal{O}(\tau)\mathcal{O}(0) \rangle$ integrated on S^1. On the other hand, equation (3.53) is valid also for the ABJ(M) theory in the form

$$c_T = -\frac{64}{\pi^2} \frac{\partial^2}{\partial m_\pm^2} \log \mathcal{Z}[S^3, m_\pm] \Big|_{m_\pm = 0} \tag{3.60}$$

and provides an alternative way to compute the central charge. Now, if the two results – the one from the topological correlator and the one from the derivatives of the three-dimensional partition function – match, we can conclude that (3.52) is valid also in the ABJ(M) case.

In the next section we will check the validity of the following identity[5]

$$\left\langle \int_{-\pi}^{\pi} d\tau_1 \mathcal{O}(\tau_1) \int_{-\pi}^{\pi} d\tau_2 \mathcal{O}(\tau_2) \right\rangle = \frac{1}{\pi^2} \frac{\partial^2}{\partial m_\pm^2} \log \mathcal{Z}[S^3, m^\pm] \Big|_{m_\pm = 0} \tag{3.61}$$

by matching the weak coupling expansion of the derivatives of the mass deformed ABJ(M) Matrix Model on the r.h.s. against a genuine two-loop calculation of the two-point correlator $\langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\rangle$.

3.3 Result from the mass deformed Matrix Model

Consider the mass-deformed Matrix Model of the ABJ(M) theory [96]

$$Z = \frac{1}{(N!)^2} \int d\lambda\, d\mu\, \frac{e^{i\pi k \sum_i \left(\lambda_i^2 - \mu_i^2\right)} \prod_{i<j} 16 \sinh^2\left[\pi\left(\lambda_i - \lambda_j\right)\right] \sinh^2\left[\pi\left(\mu_i - \mu_j\right)\right]}{\prod_{i,j} 4\cosh\left[\pi(\lambda_i - \mu_j) + \frac{\pi m_+}{2}\right] \cosh\left[\pi(\lambda_i - \mu_j) + \frac{\pi m_-}{2}\right]} \tag{3.62}$$

We can choose to take the derivatives either with respect to m_+ or m_- (we will end with the same result) and then set them to zero

$$\frac{\partial^2}{\partial m_-^2} \log Z[S^3, m_\pm] \Big|_{m_\pm = 0} = \frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2 \tag{3.63}$$

where Z is the undeformed MM, whereas its derivatives are given by

$$Z' = -\frac{1}{(N!)^2} \int d\lambda\, d\mu\, e^{i\pi k \sum_i \left(\lambda_i^2 - \mu_i^2\right)} Z_{1-\text{loop}}(\lambda_i,\, \mu_j) \sum_{i,j} \tanh \pi(\lambda_i - \mu_j) \tag{3.64}$$

$$\begin{aligned} Z'' = \frac{1}{(N!)^2} \int d\lambda\, d\mu\, e^{i\pi k \sum_i \left(\lambda_i^2 - \mu_i^2\right)} Z_{1-\text{loop}}(\lambda_i,\, \mu_j) \\ \times \frac{\pi^2}{4} \left(\left(\sum_{i,j} \tanh(\pi(\lambda_i - \mu_j)) \right)^2 - \sum_{i,j} \frac{1}{\cosh^2(\pi\left(\lambda_i - \mu_j\right))} \right) \end{aligned} \tag{3.65}$$

with

$$Z_{1-\text{loop}}(\lambda_i,\, \mu_j) = \frac{\prod_{i<j} 16 \sinh^2\left[\pi\left(\lambda_i - \lambda_j\right)\right] \sinh^2\left[\pi\left(\mu_i - \mu_j\right)\right]}{\prod_{i,j} 4\cosh(\pi\left(\lambda_i - \mu_j\right)) \cosh(\pi\left(\lambda_i - \mu_j\right))} \tag{3.66}$$

The integrand in (3.64) is odd in the exchange $\lambda \leftrightarrow \mu$ then its integral must vanish. To compute the Z'' contribution, we first make the change of variables

$$x_i = \pi\sqrt{k}\lambda_i\,, \qquad y_j = \pi\sqrt{k}\mu_j\,, \qquad g_s = \frac{1}{\sqrt{k}} \tag{3.67}$$

[5]For notational convenience, in the rest of the paper we choose the radius of the sphere to be $r = 1/2$. Identity (3.61) is then consistent with (3.52) of the $\mathcal{N} = 4$ theory specialised to $n = 2$.

to get

$$Z=\int dX\,dY\,e^{\frac{i}{\pi}\sum_i\left(x_i^2-y_i^2\right)}f(x,y) \tag{3.68}$$

and

$$Z''=\int dX\,dY\,e^{\frac{i}{\pi}\sum_i\left(x_i^2-y_i^2\right)}f(x,y)\ \tfrac{\pi^2}{4}\left(\left(\sum_{i,j}\tanh(g_s(x_i-y_j))\right)^2-\sum_{i,j}\cosh^{-2}(g_s\,(x_i-y_j))\right) \tag{3.69}$$

where dX, dY are the Haar measures and

$$f(x,y)=\prod_{i<j}\frac{\sinh^2(g_s(x_i-x_j))}{g_s^2(x_i-x_j)^2}\frac{\sinh^2(g_s(y_i-y_j))}{g_s^2(y_i-y_j)^2}\frac{1}{\prod_{i,j}\cosh^2(g_s(x_i-y_j))} \tag{3.70}$$

where we have used the notation $g_s=k^{-\frac{1}{2}}$. It is convenient to normalize Z'' and Z with the free partition function Z_0

$$Z''\to\frac{Z''}{Z_0}=\mathcal{Z}''\qquad Z\to\frac{Z}{Z_0}=\mathcal{Z}\qquad Z_0\equiv\int dXdY\,e^{\frac{i}{\pi}\sum_i\left(x_i^2-y_i^2\right)} \tag{3.71}$$

Expanding the integrands in (3.68) and (3.69) up to $g_s^4\sim\frac{1}{k^2}$, i.e. at two loops, and evaluating the normalized gaussian matrix integrals, we obtain

$$\begin{aligned}
\mathcal{Z}''=-\frac{\pi^2}{4}N_1N_2\Big[1&+g_s^2\,\frac{i\pi}{6}\,(N_2-N_1)\,\left(1-(N_2-N_1)^2\right)\\
&-g_s^4\,\frac{\pi^2}{72}\Big(-24+16N_2^2-12N_1(N_2-N_1)+N_2^4+6N_2^2N_1^2\\
&\qquad+2N_2N_1^3-N_1^4+(N_2-N_1)^6\Big)+O(g_s^6)\Big]\\
\frac{1}{\mathcal{Z}}=1&-g_s^2\,\frac{i\pi}{6}\,(N_2-N_1)\left(1-(N_2-N_1)^2\right)\\
&-g_s^4\,\frac{\pi^2}{72}\Big(-2(N_2^2-N_1^2)+8N_2N_1-5N_2^4+2N_2N_1(N_2-N_1)(8N_2-7N_1)\\
&\qquad-3N_1^4+(N_2-N_1)^6\Big)+O(g_s^6)
\end{aligned} \tag{3.72}$$

Putting them back in (3.63), the final result reads

$$\begin{aligned}
\frac{1}{\pi^2}\frac{\partial^2}{\partial m_-^2}\log Z[S^3,m_\pm]\Big|_{m_\pm=0}&=\frac{1}{\pi^2}\frac{\mathcal{Z}''}{\mathcal{Z}}=\\
&=-\frac{N_1N_2}{4}\left(1-\frac{\pi^2}{6k^2}(N_1^2+N_2^2-2)\ +O\left(\frac{1}{k^3}\right)\right)
\end{aligned} \tag{3.73}$$

It is then easy to see that this expression coincides with the perturbative result 3.51 at every loop level. We have thus checked identity (3.61) at perturbative level.

The central charge in (3.59) indeed satisfies the identity

$$c_T=-\frac{64}{\pi^2}\frac{\partial^2}{\partial m_-^2}\log Z[S^3,m_\pm]\Big|_{m_\pm=0} \tag{3.74}$$

matching the general finding of [94].

4

Dynamics on the ABJ(M) Fermionic Wilson Line

What is research but a blind date with knowledge?

Will Harvey

We saw in a previous chapter that Wilson Loops are operators that can be defined in any gauge theories, and they have a pivotal role in studying it. Let's see how we can define them in the context of ABJ(M) theories. In this case, we will see we have differences compared to $\mathcal{N} = 4$ SYM in four dimensions.

Outline: dynamics and geometry of the fermionic Wilson Loop

The topological line can be seen as a trivial defect inside full ABJ(M) theory. One should expect to be able to reproduce the same sector even in the presence of interacting defects. Apparently, this is not the case for the 1/2-BPS Wilson Loop. Indeed, if one try to define the dCFT on the Wilson Loop by means of operator insertions, i.e.

$$\langle\langle \mathcal{O}_1 \mathcal{O}_2 \rangle\rangle_{\mathcal{W}} = \frac{\langle \mathcal{W} \mathcal{O}_1 \mathcal{W} \mathcal{O}_2 \mathcal{W} \rangle}{\langle \mathcal{W} \rangle} \tag{4.1}$$

then we have more operators to insert w.r.t. the free theory. For example, insertions of single scalars are allowed while for the free theory are excluded by gauge invariance. From this point of view, it seems natural to take pieces of the topological opertor and to generalize them in a supermatrix form suitable for WL insertions:

$$\bar{u}^a Y^a \to \bar{u}^a \begin{pmatrix} 0 & Y_a \\ 0 & 0 \end{pmatrix} \qquad v_b \bar{Y}^b \to v_b \begin{pmatrix} 0 & 0 \\ \bar{Y}^b & 0 \end{pmatrix} \tag{4.2}$$

Remark: we have to be careful in define the previous correspondence, because Y, $\bar{Y}$ are bosons, while $\mathbb{Y}$, $\bar{\mathbb{Y}}$ represent fermions.

Naively, if we try to compute the two-point function $\langle \bar{u}^a \mathbb{Y}_a v_b \bar{\mathbb{Y}}^b \rangle_{\mathcal{W}}$, we find the tree-level to be topological, while the one-loop correction diverges. This divergence doesn't occur when computing the correction to the Bremsstrahlung function: this effect is driven by the scalar coupling matrix $M_I{}^J = diag\,(-1, 1, 1, 1)$ inside $\mathcal{L}$. The change of sign in the $SU(3)$ part spoils supersymmetry cancellations in the perturbative corrections. This is the supermultiplet recombination signature and, from the divergence, it is possible to extract its anomalous dimension. We want to remark here that the recombination is a purely dynamical phenomenon, in the sense that it's activated by the interactions with the WL. But the presence of the anomalous dimension signal that operators $\mathbb{Y}$, $\bar{\mathbb{Y}}$ are no more protected; nevertheless, they are still classically annihilated by at least one supercharge. It's an open problem to find a mechanism that shows this remaining supercharge is broken at the quantum level.

All these problems leave us with a fundamental question: are we missing something deep (i.e. geometric properties) in the understanding of the fermionic loop? The supermatrix form is still not well understood as well as how the supercharge act on the WL. Luckily, from the superalgebra, we have some clues to proceed. We found Supercharges have a nice supermatrix representation that acts nicely on the component fields. Moreover, on the WL they gain an additional component, taking care of the WL SUSY condition (called "covariant" component). We tested them reproducing the Displacement supermultiplet and (for self-consistency) we compared it with what we have obtained from contour deformation.

The covariant supercharges also take into account how the short supermultiplet in the free case becomes long in the interacting case. Again, this is not the full story, since we miss half of the new components that should be visible. In addition, from group theory arguments we know we should obtain an operator with particular quantum numbers, but that can not be built from the ABJ(M) fields.

Another open problem is the one related to the conformal anomaly, visible already at one-loop in perturbation theory. It is related to the choice of the WL contour. The circle takes no correction while for the line we have a divergence, related to a cutoff not compatible with gauge invariance. For this reason, all the perturbative computations are performed on the circle, and we postpone the discussion of the conformal anomaly to future works.

What we learned from this work is that the dCFT living on interacting defects is not only determined by the residual superconformal algebra; the dynamics coming from interactions with the WL deeply change what we expect from geometry.

4.1 Parity-like symmetry

When we presented the fermionic loop in section 2.4.2, we found the scalar coupling matrix to be constrained up to a sign. We inferred that this freedom is fixed requiring invariance under the parity-like symmetry, that in this case will change $\ell \to -\ell$.

We parametrize the loop as follows: since the parity does not change sign in the proper time, i.e. under parity $s \to s$, we explicitly implement the sign change in the third coordinate with the parameter ℓ

$$x^\mu(s) = (0,0,\ell s) \to \dot{x}^\mu(s) = (0,0,\ell) \qquad \Rightarrow \qquad (A_\mu \dot{x}^\mu)^{PL} = \hat{A}_\mu \dot{x}^\mu \tag{4.3}$$

where we denote with the upper-case PL the change under ABJ(M) parity-like transformation. Making explicit the factor ℓ in the scalar coupling, i.e. $M \to \ell M$, we have

$$\left(-\frac{2\pi i\ell}{k}|\dot{x}|M_J{}^I C_I \bar{C}^J\right)^{PL} = -\frac{2\pi i\ell}{k}|\dot{x}|M_I{}^J \bar{C}^I C_J \tag{4.4}$$

since the change of ℓ is compensated by the change of k. If we restore the invariant combination ℓk^{-1} inside the dqare root in the fermionic coefficient, then those parts change as

$$\begin{aligned} \left(-i\sqrt{\frac{2\pi\ell}{k}}|\dot{x}|\Psi\bar{\eta}\right)^{PL} &= i\sqrt{\frac{2\pi\ell}{k}}|\dot{x}|\omega\,(\bar{\eta})^{PL}\,\bar{\Psi} \\ \left(i\sqrt{\frac{2\pi\ell}{k}}|\dot{x}|\eta\bar{\Psi}\right)^{PL} &= -i\sqrt{\frac{2\pi\ell}{k}}|\dot{x}|\bar{\omega}\Psi\,(\eta)^{PL} \end{aligned} \tag{4.5}$$

since η, $\bar{\eta}$ (and so their parity-transformed partners) are even spinors. If we pack everything in the superconnection, we get

$$\mathcal{L}^{PL} = \begin{pmatrix} \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i\ell}{k}\,|\dot{x}|\,M_I{}^J \bar{C}^I C^J & -i\sqrt{\frac{2\pi\ell}{k}}|\dot{x}|\bar{\omega}\Psi\,(\eta)^{PL} \\ i\sqrt{\frac{2\pi\ell}{k}}|\dot{x}|\omega\,(\bar{\eta})^{PL}\,\bar{\Psi} & A_\mu \dot{x}^\mu - \frac{2\pi i\ell}{k}\,|\dot{x}|\,M^I{}_J C_I \bar{C}^J \end{pmatrix} \tag{4.6}$$

but this is a supermatrix of $U(N_2|N_1)$. To restore the original structure, the superconnection has to transform as

$$\mathcal{L} \to \mathcal{L}' = \mathfrak{P}\mathcal{L}^{PL}\mathfrak{P}^{-1} \qquad \mathfrak{P} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{4.7}$$

Imposing $\mathcal{L}' = \mathcal{L}$, we found the transformation for the fermion couplings to be

$$(\bar{\eta})^{PL} = \omega^{-1}\eta \qquad (\eta)^{PL} = \bar{\omega}^{-1}\bar{\eta} \tag{4.8}$$

but they are not sufficient to fix the coefficients ω, $\bar{\omega}$. For this aim, we consider two operators, coming from the variation of $\mathcal{L}$ w.r.t. the two R-symmetry broken generators ${J_1}^a$ and ${J_a}^1$. They are

$$\mathbb{O}^a := i\delta_{J_1^a}\mathcal{L} = -2\sqrt{\frac{\pi\ell}{k}}\begin{pmatrix} 2\sqrt{\frac{\pi\ell}{k}}Z\bar{Y}^a & \bar{\chi}_1^a \\ 0 & 2\sqrt{\frac{\pi\ell}{k}}\bar{Y}^a Z \end{pmatrix} \tag{4.9}$$

$$\bar{\mathbb{O}}_a := i\delta_{J_a{}^1}\mathcal{L} = -2\sqrt{\frac{\pi\ell}{k}}\begin{pmatrix} -2\sqrt{\frac{\pi\ell}{k}}Y_a\bar{Z} & 0 \\ i\chi_a^1 & -2\sqrt{\frac{\pi\ell}{k}}\bar{Z}Y_a \end{pmatrix} \tag{4.10}$$

We focus, for the moment, only on $\mathbb{O}^a$, and we compute the parity-like associated operator using, for the fermion, the rule (4.5)

$$(\mathbb{O}^a)^{PT} = -2\sqrt{\frac{\pi\ell}{k}}\begin{pmatrix} 2\sqrt{\frac{\pi\ell}{k}}Y_a\bar{Z} & 0 \\ \bar{\omega}\chi_a^1 & 2\sqrt{\frac{\pi\ell}{k}}\bar{Z}Y_a \end{pmatrix} \tag{4.11}$$

and this is similar to $\bar{\mathbb{O}}_a$ only if $\bar{\omega} = i$, implying $\omega = -i$. We obtain

$$(\mathbb{O}_a)^{PL} = -\bar{\mathbb{O}}^a \tag{4.12}$$

and we will find a reason for the extra minus sign when we build the covariant supercharges. Conditions (4.8) become

$$(\bar{\eta})^{PL} = i\eta \qquad (\eta)^{PL} = -i\bar{\eta} \tag{4.13}$$

as well as the fermions fields general rule

$$\left(\bar{\Psi}_\alpha\right)^{PL} = i\Psi^\alpha \qquad (\Psi^\alpha)^{PL} = -i\bar{\Psi}_\alpha \tag{4.14}$$

Eventually, the parity-like invariant superconnection takes the following form

$$\mathcal{L} = \begin{pmatrix} A_\mu\dot{x}^\mu - \frac{2\pi i\ell}{k}\,|\dot{x}|\,M^I_J C_I\bar{C}^J & i\sqrt{\frac{2\pi\ell}{k}}\,|\dot{x}|\,\eta\bar{\Psi} \\ -i\sqrt{\frac{2\pi\ell}{k}}\,|\dot{x}|\,\Psi\bar{\eta} & \hat{A}_\mu\dot{x}^\mu - \frac{2\pi i\ell}{k}\,|\dot{x}|\,{M_I}^J\bar{C}^J C_I \end{pmatrix} \tag{4.15}$$

We found then the correct form for a parametrization and parity-like invariant superconnection. We will see how the parameter ℓ enters the perturbation theory and discuss the importance of its presence.

4.2 1-loop corrected VEV

Although we have a conformal transformation that maps the line into the circle, when considering Wilson Loops they are not equivalent. A similar effect happens in $\mathcal{N} = 4$ SYM [97], where the two-loop corrections are different but finite. In particular, the non-compact shape of the infinite line forces us to put a regularization that breaks gauge invariance. This effect is manifest when computing the one-loop correction to the Wilson Line VEV. In this section we perform the computations setting $\ell = 1$ and using the rescaled ABJ(M) action (A.5) and (A.8).

Circle

The VEV for the fermionic circle was already computed in [98]. The Wilson Loop in this case is

$$\mathcal{W} = \mathrm{Tr}\,\mathcal{P}\exp\left(-i\int_{-\pi}^{\pi} d\tau \mathcal{L}\right) \tag{4.16}$$

At tree level, the VEV is the trace of the identity of the supergroup $U(N_1|N_2)$

$$\langle \mathcal{W} \rangle^{(0)} = N_1 + N_2 \tag{4.17}$$

To compute the one-loop correction, we have to look at the series expansion of the $\mathcal{P}exp$:

$$\mathcal{P}\exp\left(-i\int_{-\pi}^{\pi} d\tau \mathcal{L}\right) = \mathbb{1} - i\int_{-\pi}^{\pi} d\tau \mathcal{L} - \int_{-\pi}^{\pi} d\tau_1 \int_{-\pi}^{\tau_1} d\tau_2 \mathcal{L}_1\mathcal{L}_2 + \dots \tag{4.18}$$

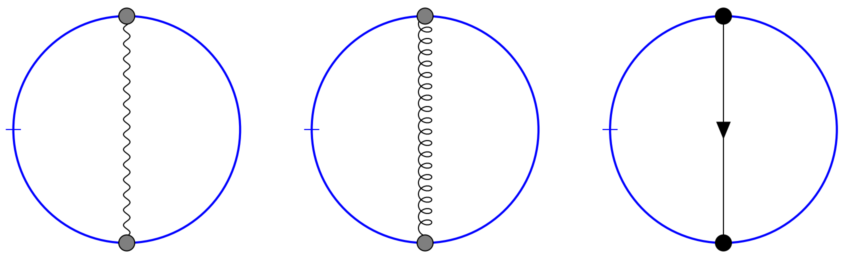
From the term with only one $\mathcal{L}$ we can't have contributions. From the other one

$$\mathcal{L}_1\mathcal{L}_2 = \begin{pmatrix} \frac{1}{k}A_{1,\mu}\dot{x}_1^\mu A_{2,\nu}\dot{x}_2^\nu + \frac{2\pi}{k}(\eta\bar{\psi})_1(\bar{\eta}\psi)_2 & \cdots \\ \cdots & \frac{1}{k}\hat{A}_{1,\mu}\dot{x}_1^\mu \hat{A}_{2,\nu}\dot{x}_2^\nu + \frac{2\pi}{k}(\psi\bar{\eta})_1(\eta\bar{\psi})_2 \end{pmatrix} + o\left(k^{-2}\right) \tag{4.19}$$

where we left "$\cdots$" in the anti-diagonal since it won't matter after taking the trace:

$$\mathrm{Tr}[\mathcal{L}_1\mathcal{L}_2] = \frac{1}{k}\left(A_{1,\mu}\dot{x}_1^\mu A_{2,\nu}\dot{x}_2^\nu + \hat{A}_{1,\mu}\dot{x}_1^\mu \hat{A}_{2,\nu}\dot{x}_2^\nu\right) + \frac{2\pi}{k}\left((\eta\bar{\psi})_1(\bar{\eta}\psi)_2 + (\psi\bar{\eta})_1(\eta\bar{\psi})_2\right) \tag{4.20}$$

and the trace on gauge indices in the R.H.S. is implied. The corresponding diagrams are shown if figure 4.1.



(a) (b) (c)

Figure 4.1: Diagrams appearing in the one-loop correction of the Wilson Circle VEV.

Diagrams (4.1(a)) and (4.1(b)) vanish due to parity. The one involving fermions includes the two contributions inside the second brackets of (4.20). Using propagators (A.27) we have

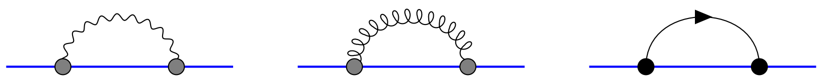
$$-\tfrac{4\pi}{k}\int_{-\pi}^{\pi} d\tau_1 \int_{-\pi}^{\tau_1} d\tau_2\, \mathrm{Tr}\left[(\psi\bar{\eta})_1(\eta\bar{\psi})_2\right] = \frac{2^{2\varepsilon}N_1N_2\Gamma\left(\frac{3}{2}-\varepsilon\right)}{k\pi^{\frac{1}{2}-\varepsilon}}\int_{-\pi}^{\pi} d\tau_1 \int_{-\pi}^{\tau_1} d\tau_2 \left(\sin\tfrac{\tau_1-\tau_2}{2}\right)^{2\varepsilon-2} \tag{4.21}$$

and this vanishes, so

$$\langle \mathcal{W} \rangle^{(1)} = 0 \tag{4.22}$$

Line

The infinite straight line case is almost the same of the circular one, although we have the integration extremes changed in $\pm\infty$. The tree level is the same as (4.17), since the contour identity is independent of the contour. The term with only one superconnection will not contribute for the same arguments of above. At second order we have the same structure as in (4.19) and the corresponding diagrams are shown in figure (4.2)



(a) (b) (c)

Figure 4.2: Diagrams from the second order expansion of the Wilson Loop.

Diagrams (4.2(a)) and (4.2(b)) vanish since

$$\langle A_3(x_1)A_3(x_2)\rangle = \langle \hat{A}_3(x_1)\hat{A}_3(x_2)\rangle = 0 \tag{4.23}$$

For the third diagram we have to solve the following integral

$$\int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{t_1} dt_2 (t_1 - t_2)^{2\varepsilon-2} \tag{4.24}$$

but it will lead to a divergent result. We regularize it putting a cut-off L on the length of the line

$$\int_{-L}^{L} dt_1 \int_{-L}^{t_1} dt_2 (t_1 - t_2)^{2\varepsilon-2} = -\frac{(2L)^{2\varepsilon}}{4\varepsilon\left(\frac{1}{2} - \varepsilon\right)} \tag{4.25}$$

We notice that removing the cut-off, i.e. taking the limit $L \to \infty$, will lead again to a divergent result. This is the main difference with respect to the circle, maybe the sign of a conformal anomaly is acting at the quantum level.

4.2.1 The cut-offed circle

The first thing to do when trying to get rid of IR divergences is re-doing the same computations on the circle instead of the line. Since there's a conformal mapping between the two and the circle is compact[1], the IR problem should disappear. Indeed, performing the same computation on the circle, it's easy to see that the only divergences are the UV ones, symmetric in N_1 and N_2 and there's perfect cancellation between bosonic and fermionic contributions. Wonderful but the objection

[1]The one-loop correction of $\langle W\rangle$ vanishes on the circle.

is that the circle defined on $[-\pi, \pi]$ is equivalent to the infinite line. Than the equivalent of the cut-offed line is the cut-offed circle $[-\pi+\eta, \pi-\eta]$:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle_W = \langle W_{\pi,\pi-\eta} \underbrace{[W_{\pi-\eta,\sigma} \mathcal{O}_1 W_{\sigma,0} \mathcal{O}_2 W_{0,-\pi+\eta}]}_{line\ [-L,L]} W_{-\pi+\eta,-\pi} \rangle \tag{4.26}$$

It's convenient to compute corrections on the circle since, for the fermionic part, the outside contributions can be computed exactly (while on the line they will lead to divergences). The first order expansion is not sensible to this cutoff, in fact even on the line the integrals can be exactly solved. The main point of this procedure is highlighting the phenomenon that occurs in the second-order expansion of the Wilson loop: in the case of the line, the pieces leading to the infinite line are divergent, on the circle they are exact. The result on the circle can then be mapped on the line, giving a procedure to compute integrals that appear to be divergent at first sight.

Let's now evaluate the second order expansion terms: let's start from the pieces inside the square brackets

$$\int_\sigma^{\pi-\eta} d\tau_1 \int_\sigma^{\tau_1} d\tau_2 \partial_{\tau_2} \partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) = -\sin^{2\varepsilon}\left(\frac{\pi-\eta-\sigma}{2}\right) \tag{4.27}$$

$$\int_0^{\sigma} d\tau_1 \int_0^{\tau_1} d\tau_2 \partial_{\tau_2} \partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) = -\sin^{2\varepsilon}\left(\frac{\sigma}{2}\right) \tag{4.28}$$

$$\int_{-\pi+\eta}^{0} d\tau_1 \int_{-\pi+\eta}^{\tau_1} d\tau_2 \partial_{\tau_2} \partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) = -\sin^{2\varepsilon}\left(\frac{\pi-\eta}{2}\right) \tag{4.29}$$

$$\int_\sigma^{\pi-\eta} d\tau_1 \int_{-\pi+\eta}^{0} d\tau_2 \partial_{\tau_2} \partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) = \sin^{2\varepsilon}\left(\frac{\pi-\eta}{2}\right) - \sin^{2\varepsilon}\left(\frac{\sigma}{2}\right) \tag{4.30}$$

$$- \sin^{2\varepsilon}(\pi-\eta) + \sin^{2\varepsilon}\left(\frac{\pi-\eta+\sigma}{2}\right) \tag{4.31}$$

There are more integrals coming from the external parts

$$\int_{\pi-\eta}^{\pi} d\tau_1 \int_{\pi-\eta}^{\tau_1} d\tau_2 \partial_{\tau_2} \partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) = -\sin^{2\varepsilon}\left(\frac{\eta}{2}\right) \tag{4.32}$$

$$\int_{-\pi}^{-} \pi + \eta d\tau_1 \int_{-\pi}^{\tau_1} d\tau_2 \partial_{\tau_2} \partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) = -\sin^{2\varepsilon}\left(\frac{\eta}{2}\right) \tag{4.33}$$

and other five exchanges

$$\begin{aligned}\int_{\pi-\eta}^{\pi} d\tau_1 \int_{\sigma}^{\pi-\eta} d\tau_2 \partial_{\tau_2} \partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) &= \sin^{2\varepsilon}\left(\frac{\eta}{2}\right) - \sin^{2\varepsilon}\left(\frac{\pi-\sigma}{2}\right) \\ &+ \sin^{2\varepsilon}\left(\frac{\pi-\eta-\sigma}{2}\right)\end{aligned} \tag{4.34}$$

$$\begin{aligned}\int_{\pi-\eta}^{\pi} d\tau_1 \int_{-\pi+\eta}^{0} d\tau_2 \partial_{\tau_2}\partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) =& 1-\sin^{2\varepsilon}\left(\frac{\pi-\eta}{2}\right) \\ &-\sin^{2\varepsilon}\left(\pi-\frac{\eta}{2}\right)+\sin^{2\varepsilon}(\pi-\eta)\end{aligned} \tag{4.35}$$

$$\int_{\pi-\eta}^{\pi} d\tau_1 \int_{-\pi}^{-\pi+\eta} d\tau_2 \partial_{\tau_2}\partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) = 2\sin^{2\varepsilon}\left(\pi-\frac{\eta}{2}\right)-\sin^{2\varepsilon}(\pi-\eta) \tag{4.36}$$

$$\begin{aligned}\int_{\sigma}^{\pi-\eta} d\tau_1 \int_{-\pi}^{-\pi+\eta} d\tau_2 \partial_{\tau_2}\partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) =& \sin^{2\varepsilon}(\pi-\eta)-\sin^{2\varepsilon}\left(\frac{\pi-\eta+\sigma}{2}\right) \\ &-\sin^{2\varepsilon}\left(\pi-\frac{\eta}{2}\right)+\sin^{2\varepsilon}\left(\frac{\pi+\sigma}{2}\right)\end{aligned} \tag{4.37}$$

$$\int_{-\pi+\eta}^{0} d\tau_1 \int_{-\pi}^{-\pi+\eta} d\tau_2 \partial_{\tau_2}\partial_{\tau_1} \sin^{2\varepsilon}\left(\frac{\tau_1-\tau_2}{2}\right) = \sin^{2\varepsilon}\left(\frac{\pi-\eta}{2}\right)-1+\sin^{2\varepsilon}\left(\frac{\eta}{2}\right) \tag{4.38}$$

and from these is very easy to spot the map

$$(2L)^{2\varepsilon} \to \sin^{2\varepsilon}(\pi-\eta) \qquad (L-s)^{2\varepsilon} \to \sin^{2\varepsilon}\left(\frac{\pi-\eta-\sigma}{2}\right) \tag{4.39}$$

$$(L+s)^{2\varepsilon} \to \sin^{2\varepsilon}\left(\frac{\pi-\eta+\sigma}{2}\right) \qquad s^{2\varepsilon} \to \sin^{2\varepsilon}\left(\frac{\sigma}{2}\right) \tag{4.40}$$

Summing the extra-contribution we get

$$\sin^{2\varepsilon}(\pi-\eta)+\sin^{2\varepsilon}\left(\frac{\pi-\eta-\sigma}{2}\right)-\sin^{2\varepsilon}\left(\frac{\pi-\eta+\sigma}{2}\right) \tag{4.41}$$

In the line language is translated in

$$(2L)^{2\varepsilon}+(L-s)^{2\varepsilon}-(L+s)^{2\varepsilon} \tag{4.42}$$

The infinities we encounter in the line VEV seem to be an effect of the cut-off breaking gauge invariance. In particular, the finite size line is not a closed path, as the straight line closes at infinity.

An unanswered question remains whether the conformal anomaly is in action at the quantum level. Proving this point is quite involved and needs further investigations.

Two parameters family of Wilson Loops

So far we have dealt with supersymmetric Wilson Loops. In principle, we can consider non supersymmetric Wilson Loops. These operators are not protected and can trigger a RG flow in the 1d defect CFT. In YM, this flow can be described by the running of a parameter interpolating between the Wilson and Wilson-Maldacena Loop [99, 41, 42]. The one-parameter family can be viewed as a marginal deformation of the dCFT$_1$.

As we saw, in ABJ(M) we have to kinds of BPS loops: thus, we need two parameters to consider a complete interpolating object. In this sense, a two-parameter WL family can be obtained considering two parameters $\chi,\ \xi \in [0,1]$ such that

$$\mathcal{L}_{\chi,\xi} = \begin{pmatrix} A_\mu \dot{x}^\mu - \frac{2\pi i \ell}{k} |\dot{x}|\, \xi M_{\chi\ J}^{I} C_I \bar{C}^J & i\sqrt{\frac{2\pi\ell}{k}}\, |\dot{x}|\, \chi\xi\eta\bar{\Psi} \\ -i\sqrt{\frac{2\pi\ell}{k}}\, |\dot{x}|\, \chi\xi\Psi\bar{\eta} & \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i\ell}{k} |\dot{x}|\, \xi M_I^{\chi\ J} \bar{C}^J C_I \end{pmatrix} \tag{4.43}$$

where the matrix M is modified as

$$(M_\chi)^I_{\ J} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2\chi - 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.44}$$

For $\chi = 0$ we recover the coupling (2.24) for the bosonic loop and the fermions decouple. For $\chi = 1$ the scalar coupling matches (2.29) for the fermionic loop. For $\chi \neq 0$, we still have freedom from the ξ parameter: the limit $\xi = 0$ correspond to the non-SUSY case. In the special case $\chi = \xi = 1$ we obtain the 1/2-BPS Wilson Loop. If we set $\xi = 1$, we obtain a one-parameter family interpolating between the 1/6-BPS and the 1/2-BPS loop. The dual description of this object has been investigated in [100].

4.3 Deformations of the loop

We saw that the presence of a defect breaks translation invariance in its perpendicular directions, yielding the non-conservation of the stress-energy tensor, encoded in the *displacement* operator $\mathbb{D}$. When the defect is a Wilson Loop, the displacement operator measures the response of the loop under contour deformations. Therefore, the displacement is the operator that falls from the $\mathcal{P}$exp when considering small deformations of the loop. The computation is inspired by the analog one in [101], carried out in four dimensions. We present the computation setting again $\ell = 1$ and using the equations of motion from the rescaled action, but these choices will not affect the final results.

General contour

Rewrite the superconnection as

$$\mathcal{L} = \begin{pmatrix} (1) = A_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}|\, M^I_{\ J} C_I \bar{C}^J & (3) = i\sqrt{\frac{2\pi}{k}}\, |\dot{x}|\, \eta_I \bar{\psi}^I \\ (2) = -i\sqrt{\frac{2\pi}{k}}\, |\dot{x}|\, \psi_I \bar{\eta}^I & (4) = \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}|\, M_J^{\ I} \bar{C}^J C_I \end{pmatrix} \tag{4.45}$$

and expand the Path-exponential

$$\mathcal{P}\exp\left(-i\int ds\ \mathcal{L}\right) = \mathbb{1} - i\int ds\ \mathcal{L} - \int ds_1 \int_0^{s_1} ds_2 \mathcal{L}_1\mathcal{L}_2 + \dots \tag{4.46}$$

where we used the shorthand notation $\mathcal{L}_1 \equiv \mathcal{L}(s_1)$. Under the deformation $x \to x + \delta x$ we have, up to order δx^2,

$$1 \to A_0 + A_1 - \frac{2\pi i}{k}S \tag{4.47}$$

$$2 \to \mathfrak{C}_0 - i\sqrt{\frac{2\pi}{k}}\bar{\mathfrak{C}} \tag{4.48}$$

$$3 \to \mathfrak{B}_0 + i\sqrt{\frac{2\pi}{k}}\bar{\mathfrak{B}} \tag{4.49}$$

$$4 \to \hat{A}_0 + \hat{A}_1 - \frac{2\pi i}{k}\bar{S} \tag{4.50}$$

where

$$A_0 = \frac{A_\mu}{\sqrt{k}}\dot{x}^\mu - \frac{2\pi i}{k}\left|\dot{x}\right| M^I{}_J C_I \bar{C}^J, \qquad A_1 = \frac{A_\mu}{\sqrt{k}}\dot{\delta x}^\mu + \delta x^\sigma \partial_\sigma \frac{A_\mu}{\sqrt{k}}\dot{x}^\mu \tag{4.51}$$

$$S = M_J{}^I\left(\left|\dot{x}\right| C_I \delta x^\sigma \partial_\sigma \bar{C}^J + \left|\dot{x}\right| \delta x^\sigma \partial_\sigma C_I \bar{C}^J + \frac{\dot{\delta x}\cdot\dot{x}}{\left|\dot{x}\right|} C_I\bar{C}^J\right) \tag{4.52}$$

$$\mathfrak{B}_0 = i\sqrt{\frac{2\pi}{k}}\left|\dot{x}\right| \eta_I \bar{\psi}^I, \qquad \bar{\mathfrak{B}} = \left|\dot{x}\right| \eta_I \delta x^\sigma \partial_\sigma \bar{\psi}^I + \frac{\dot{\delta x}\cdot\dot{x}}{\left|\dot{x}\right|}\eta_I\bar{\psi}^I \tag{4.53}$$

$$\mathfrak{C}_0 = -i\sqrt{\frac{2\pi}{k}}\left|\dot{x}\right| \psi_I \bar{\eta}^I, \qquad \bar{\mathfrak{C}} = \left|\dot{x}\right| \delta x^\sigma \partial_\sigma \psi_I \bar{\eta}^I + \frac{\dot{\delta x}\cdot\dot{x}}{\left|\dot{x}\right|}\psi_I\bar{\eta}^I \tag{4.54}$$

$$\hat{A}_0 = \frac{\hat{A}_\mu}{\sqrt{k}}\dot{x}^\mu - \frac{2\pi i}{k}\left|\dot{x}\right| M_J{}^I \bar{C}^J C_I, \qquad \hat{A}_1 = \frac{\hat{A}_\mu}{\sqrt{k}}\dot{\delta x}^\mu + \delta x^\sigma \partial_\sigma \frac{\hat{A}_\mu}{\sqrt{k}}\dot{x}^\mu \tag{4.55}$$

$$\hat{S} = M_J{}^I\left(\left|\dot{x}\right| \bar{C}^J \delta x^\sigma \partial_\sigma C_I + \left|\dot{x}\right| \delta x^\sigma \partial_\sigma \bar{C}^J C_I + \frac{\dot{\delta x}\cdot\dot{x}}{\left|\dot{x}\right|}\bar{C}^J C_I\right) \tag{4.56}$$

$A_0, \hat{A}_0, \mathfrak{B}_0, \mathfrak{C}_0 \sim o(\delta x^0)$, while $A_1, \hat{A}_1, \bar{\mathfrak{B}}, \bar{\mathfrak{C}}, S, \hat{S} \sim o(\delta x)$.

Integrate by parts the first term in A_1:

$$\frac{1}{\sqrt{k}}\int d\tau A_\mu \frac{d}{d\tau}\delta x^\mu = \underbrace{A_\mu \delta x^\mu\big|_{extr}}_{=0} - \frac{1}{\sqrt{k}}\int d\tau \left(\frac{d}{d\tau}A_\mu\right)\delta x^\mu = -\int d\tau \dot{x}^\sigma \partial_\sigma \frac{A_\mu}{\sqrt{k}}\delta x^\mu \tag{4.57}$$

so

$$A_1 = -\dot{x}^\sigma\partial_\sigma \frac{A_\mu}{\sqrt{k}}\delta x^\mu + \delta x^\sigma \partial_\sigma \frac{A_\mu}{\sqrt{k}}\dot{x}^\mu = \frac{1}{\sqrt{k}}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\delta x^\mu \dot{x}^\nu \tag{4.58}$$

The same happens for $\hat{A}_1$. The second order term is

$$\mathcal{L}_D^{s_1}\cdot\mathcal{L}_D^{s_2} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \tag{4.59}$$

where

$$\textcircled{1} = A_0^{s_1} A_1^{s_2} + A_1^{s_1} A_0^{s_2} + \dots \tag{4.60}$$

$$\textcircled{2} = \mathfrak{C}_0^{s_1} A_1^{s_2} + \hat{A}_1^{s_1} \mathfrak{C}_0^{s_2} + \dots \tag{4.61}$$

$$\textcircled{3} = A_1^{s_1} \mathfrak{B}_0^{s_2} + \mathfrak{B}_0^{s_1} \hat{A}_1^{s_2} + \dots \tag{4.62}$$

$$\textcircled{4} = \hat{A}_0^{s_1} \hat{A}_1^{s_2} + \hat{A}_1^{s_1} \hat{A}_0^{s_2} + \dots \tag{4.63}$$

Where we wrote only the terms of order $o(\delta x^1)$ that can contribute to the first order expansion term. The "..." refers to terms involving more than one integral. Let's focus now on (4.60), the same will apply for the other 3 equations. The only two terms that are relevant for our purpose are $A_0^{s_1} A_1^{s_2}$ and $A_1^{s_1} A_0^{s_2}$

$$\int ds_1 \int^{s_1} ds_2 \, |\dot{x}_1| \, M^I{}_J C_{I,1} \bar{C}_1^J A_{\nu,2} \dot{\delta x}_2^{\nu} = \int ds \, |\dot{x}| \, M^I{}_J C_I \bar{C}^J A_\nu \delta x^\nu + \dots \tag{4.64}$$

$$\begin{aligned} &\int ds_1 A_{1,\mu} \delta x_1^\mu \int^{s_1} ds_2 \, |\dot{x}_2| \, M^I{}_J C_{I,2} \bar{C}_2^J \\ &= - \int ds_1 \delta x_1^\mu \frac{d}{ds_1} \left(A_{1,\mu} \int^{s_1} |\dot{x}_2| \, M^I{}_J C_{I,2} \bar{C}_2^J \right) + \dots \\ &= - \int ds \delta x^\mu A_\mu \, |\dot{x}| \, M^I{}_J C_I \bar{C}^J + \dots \end{aligned} \tag{4.65}$$

together with the terms restoring the non-abelian part of the field strength

$$\int ds_1 \int^{s_1} ds_2 A_{1,\mu} \dot{x}_1^\mu A_{2,\nu} \dot{\delta x}_2^{\nu} = \int ds A_\nu A_\mu \delta x^\mu \dot{x}^\nu + \dots \tag{4.66}$$

$$\int ds_1 \int^{s_1} ds_2 A_{1,\mu} \dot{\delta x}_1^{\mu} A_{2,\nu} \dot{x}_2^\nu = - \int ds A_\mu A_\nu \delta x^\mu \dot{x}^\nu + \dots \tag{4.67}$$

and so (restoring the constants in front of the terms)

$$A_0^{s_1} A_1^{s_2} + A_1^{s_1} A_0^{s_2} = -\frac{1}{k} \left[A_\mu, A_\nu \right] \delta x^\mu \dot{x}^n u + \frac{2\pi i}{k} \delta x^\mu \, |\dot{x}| \left[\frac{A_\mu}{\sqrt{k}}, M^I{}_J C_I \bar{C}^J \right] \tag{4.68}$$

$$\hat{A}_0^{s_1} \hat{A}_1^{s_2} + \hat{A}_1^{s_1} \hat{A}_0^{s_2} = -\frac{1}{k} \left[\hat{A}_\mu, \hat{A}_\nu \right] \delta x^\mu \dot{x}^n u + \frac{2\pi i}{k} \delta x^\mu \, |\dot{x}| \left[\frac{\hat{A}_\mu}{\sqrt{k}}, M^I{}_J C_I \bar{C}^J \right] \tag{4.69}$$

$$\mathfrak{C}_0^{s_1} A_1^{s_2} + \hat{A}_1^{s_1} \mathfrak{C}_0^{s_2} = -i \sqrt{\frac{2\pi}{k}} \, |\dot{x}| \, \delta x^\mu \left(\Psi \bar{\eta} \frac{A_\mu}{\sqrt{k}} - \frac{\hat{A}_\mu}{\sqrt{k}} \Psi \bar{\eta} \right) \tag{4.70}$$

$$A_1^{s_1} \mathfrak{B}_0^{s_2} + \mathfrak{B}_0^{s_1} \hat{A}_1^{s_2} = i \sqrt{\frac{2\pi}{k}} \, |\dot{x}| \, \delta x^\mu \left(\eta \bar{\Psi} \frac{\hat{A}_\mu}{\sqrt{k}} - \frac{A_\mu}{\sqrt{k}} \eta \bar{\Psi} \right) \tag{4.71}$$

Inserting these expression in the Path-exponential expansion, we can rewrite it as

$$\delta W = TrP \int ds \, (-i\mathbb{D}) \, exp \left(-i \int \mathcal{L} \right) \tag{4.72}$$

where

$$\mathbb{D} = -\delta x^\mu \left(\dot{x}^\nu \mathbb{F}_{\mu\nu} + i\,|\dot{x}|\,\mathcal{D}_\mu \mathbb{O}\right) - i\frac{\dot{x}\cdot\dot{\delta x}}{|\dot{x}|}\mathbb{O} \tag{4.73}$$

and the supermatrix form of the field strength is

$$\mathbb{F}_{\mu\nu} = \begin{pmatrix} F_{\mu\nu} & 0 \\ 0 & \hat{F}_{\mu\nu} \end{pmatrix} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i[\mathcal{A}_\mu, \mathcal{A}_\nu], \qquad \mathcal{A}_\mu = k^{-\frac{1}{2}} \begin{pmatrix} A_\mu & 0 \\ 0 & \hat{A}_\mu \end{pmatrix} \tag{4.74}$$

We can write the super-covariant derivative as

$$\mathcal{D}_\mu \mathbb{O} = \partial_\mu \mathbb{O} + i\,[\mathcal{A}_\mu, \mathbb{O}] \tag{4.75}$$

since

$$k^{-\frac{1}{2}} \left[\begin{pmatrix} A_\mu & 0 \\ 0 & \hat{A}_\mu \end{pmatrix}, \begin{pmatrix} \mathcal{O}^1 & \mathcal{O}^2 \\ \mathcal{O}^3 & \mathcal{O}^4 \end{pmatrix} \right] = k^{-\frac{1}{2}} \begin{pmatrix} [A_\mu, \mathcal{O}^1] & A_\mu \mathcal{O}^2 - \mathcal{O}^2 \hat{A}_\mu \\ \hat{A}_\mu \mathcal{O}^3 - \mathcal{O}^3 A_\mu & \left[\hat{A}_\mu, \mathcal{O}^4\right] \end{pmatrix}. \tag{4.76}$$

then our operator $\mathbb{O}$ in (4.73) is

$$\mathbb{O} = \begin{pmatrix} -\frac{2\pi}{k} M^I{}_J C_I \bar{C}^J & \sqrt{\frac{2\pi}{k}} \eta_I \bar{\Psi}^I \\ -\sqrt{\frac{2\pi}{k}} \Psi_I \bar{\eta}^I & -\frac{2\pi}{k} M_J{}^I \bar{C}^J C_I \end{pmatrix} \tag{4.77}$$

Wilson Line

All the computations so far are done for general contour. Now we want to specialize on the line case, along the third direction as $x^\mu(s) = (0,0,s)$. We have to consider the following deformation $\delta x^\mu = (\varepsilon^1(s), \varepsilon^2(s), 0)$. The operator (4.77) becomes

$$\mathbb{O}_{line} = \begin{pmatrix} \frac{2\pi}{k}\left(Z\bar{Z} - Y_a \bar{Y}^a\right) & 2\sqrt{\frac{\pi}{k}}\bar{\psi}_1 \\ -2i\sqrt{\frac{\pi}{k}}\psi^1 & \frac{2\pi}{k}\left(\bar{Z}Z - \bar{Y}^a Y_a\right) \end{pmatrix} \tag{4.78}$$

and the displacement operator takes the following expression

$$\mathbb{D}_{line} = -\varepsilon^k \left(\mathbb{F}_{k3} + i\mathcal{D}_k \mathbb{O}_{line}\right) \qquad k = 1, 2. \tag{4.79}$$

Expanding in the supermatrix form

$$\begin{aligned}\mathbb{F}_{13} &= \begin{pmatrix} F_{13} & 0 \\ 0 & \hat{F}_{13} \end{pmatrix} = k^{-\frac{1}{2}} \begin{pmatrix} \partial_1 A_3 - \partial_3 A_1 + \frac{i}{\sqrt{k}}[A_1, A_3] & 0 \\ 0 & \partial_1 \hat{A}_3 - \partial_3 \hat{A}_1 + \frac{i}{\sqrt{k}}[\hat{A}_1, \hat{A}_3] \end{pmatrix} \\ &= \begin{pmatrix} \frac{2\pi}{k}\left(C_I D^2 \bar{C}^I - D^2 C_I \bar{C}^I\right) + \frac{2\pi i}{k}\bar{\Psi}^I \gamma^2 \Psi_I & 0 \\ 0 & \frac{2\pi}{k}\left(D^2 \bar{C}^I C_I - \bar{C}^I D^2 C_I\right) - \frac{2\pi i}{k}\Psi_I \gamma^2 \bar{\Psi}^I \end{pmatrix}\end{aligned} \tag{4.80}$$

where in the last equality we used the EOM (A.9). Same thing for the other component

$$\mathbb{F}_{23}=\begin{pmatrix}F_{23} & 0\\ 0 & \hat{F}_{23}\end{pmatrix}=k^{-\frac{1}{2}}\begin{pmatrix}\partial_2 A_3-\partial_3 A_2+\frac{i}{\sqrt{k}}[A_2,A_3] & 0\\ 0 & \partial_2\hat{A}_3-\partial_3\hat{A}_2+\frac{i}{\sqrt{k}}[\hat{A}_2,\hat{A}_3]\end{pmatrix}$$
$$=\begin{pmatrix}\frac{2\pi}{k}\left(D^1C_I\bar{C}^I-C_ID^1\bar{C}^I\right)-\frac{2\pi i}{k}\bar{\Psi}^I\gamma^1\Psi_I & 0\\ 0 & \frac{2\pi}{k}\left(\bar{C}^ID^1C_I-D^1\bar{C}^IC_I\right)+\frac{2\pi i}{k}\Psi_I\gamma^1\bar{\Psi}^I\end{pmatrix} \tag{4.81}$$

It's convenient to write the displacement in the complex components. Since

$$\mathcal{D}=D^1-iD^2 \tag{4.82}$$

we choose the deformation parameters to be $\varepsilon^k=(i,1)$, such that

$$\mathbb{D}=-\left(\mathbb{F}_{23}+i\mathbb{F}_{13}\right)+\mathcal{D}\mathbb{O}_{line} \tag{4.83}$$

with

$$\mathcal{D}\mathbb{O}_{line}=\begin{pmatrix}\frac{2\pi}{k}\left(\mathcal{D}Z\bar{Z}+Z\mathcal{D}\bar{Z}-\mathcal{D}Y_a\bar{Y}^a-Y_a\mathcal{D}\bar{Y}^a\right) & 2\sqrt{\frac{\pi}{k}}\mathcal{D}\bar{\psi}_1\\ -2i\sqrt{\frac{\pi}{k}}\mathcal{D}\psi^1 & \frac{2\pi}{k}\left(\mathcal{D}\bar{Z}Z+\bar{Z}\mathcal{D}Z-\mathcal{D}\bar{Y}^aY_a-\bar{Y}^a\mathcal{D}Y_a\right)\end{pmatrix} \tag{4.84}$$

and

$$\mathbb{F}_{23}+i\mathbb{F}_{13}=\frac{2\pi}{k}\begin{pmatrix}\mathcal{D}C_I\bar{C}^I-C_I\mathcal{D}\bar{C}^I-\bar{\Psi}^I\left(i\gamma^1+\gamma^2\right)\Psi_I & 0\\ 0 & \bar{C}^I\mathcal{D}C_I-\mathcal{D}\bar{C}^IC_I+\Psi_I\left(i\gamma^1+\gamma^2\right)\bar{\Psi}^I\end{pmatrix} \tag{4.85}$$

and summing the two pieces

$$\mathbb{D}=\begin{pmatrix}\frac{4\pi}{k}\left(Z\mathcal{D}\bar{Z}-\mathcal{D}Y_a\bar{Y}^a+i\bar{\psi}_1\psi^2+i\bar{\chi}_1^a\chi_a^2\right) & 2\sqrt{\frac{\pi}{k}}\mathcal{D}\bar{\psi}_1\\ -2i\sqrt{\frac{\pi}{k}}\mathcal{D}\psi^1 & \frac{4\pi}{k}\left(\mathcal{D}\bar{Z}Z-\bar{Y}^a\mathcal{D}Y_a-i\psi^2\bar{\psi}_1-i\chi_a^2\bar{\chi}_1^a\right)\end{pmatrix} \tag{4.86}$$

From the previous expression we can separate two pieces

$$\mathbb{D}=\mathbb{D}_E+\tilde{\mathbb{D}} \tag{4.87}$$

where, after using (A.11) for the left-bottom block,

$$\mathbb{D}_E=\begin{pmatrix}\frac{4\pi}{k}\left(Z\mathcal{D}\bar{Z}-\mathcal{D}Y_a\bar{Y}^a+i\bar{\chi}_1^a\chi_a^2\right) & 2\sqrt{\frac{\pi}{k}}\mathcal{D}\bar{\psi}_1\\ 8i\left(\frac{\pi}{k}\right)^{\frac{3}{2}}\left(\bar{Y}^aZ\chi_a^2-\chi_a^2Z\bar{Y}^a+\varepsilon_{abc}\bar{Y}^a\bar{\chi}_1^b\bar{Y}^c\right) & \frac{4\pi}{k}\left(\mathcal{D}\bar{Z}Z-\bar{Y}^a\mathcal{D}Y_a-i\chi_a^2\bar{\chi}_1^a\right)\end{pmatrix} \tag{4.88}$$

is the exact part of the Displacement operator, and we get an extra piece

$$\begin{aligned}\tilde{\mathbb{D}}&=\begin{pmatrix}\frac{4\pi i}{k}\bar{\psi}_1\psi^2 & 0\\ -2i\sqrt{\frac{\pi}{k}}D_3\psi^2+4i\left(\frac{\pi}{k}\right)^{\frac{3}{2}}\left[\hat{\ell}_B\psi^2-\psi^2\ell_B\right] & -\frac{4\pi i}{k}\psi^2\bar{\psi}_1\end{pmatrix}\\ &=2\sqrt{\frac{\pi}{k}}\left(-i\mathcal{D}_3\begin{pmatrix}0 & 0\\ \psi^2 & 0\end{pmatrix}+\left[\mathcal{L}_B+\mathcal{L}_F,\begin{pmatrix}0 & 0\\ \psi^2 & 0\end{pmatrix}\right]\right)\end{aligned} \tag{4.89}$$

and due to (A.15) we see that it combines in the complete covariant derivative on the Wilson Loop (2.27)

$$\tilde{\mathbb{D}} = -2i\sqrt{\frac{\pi}{k}}\mathfrak{D}_3 \begin{pmatrix} 0 & 0 \\ \psi^2 & 0 \end{pmatrix} \tag{4.90}$$

This shows $\tilde{D}$ can be neglected.

4.3.1 Covariant supercharges in supermatrix formalism

The supermatrix form of the Wilson Loop forces us to write every object interacting with the Wilson Loop in terms of supermatrices. Considering the supercharges, we know how they act on the ABJ(M) fields of the theory. We want to generalize their action on supermatrices.

The supermatrix representation of supercharges follows from (D.11)

$$\mathbb{Q}^a = \begin{pmatrix} Q^a & 0 \\ 0 & -Q^a \end{pmatrix} \qquad \bar{\mathbb{Q}}_a = \begin{pmatrix} \bar{Q}_a & 0 \\ 0 & -\bar{Q}_a \end{pmatrix} \tag{4.91}$$

since they are parity odd. It's easy to see that they respect the commutation rules; for instance, let $\mathbb{X}$ be an even supermatrix, then $\mathbb{Q}$ will act on $\mathbb{X}$ with the commutator

$$[\mathbb{Q}, \mathbb{X}] = \begin{pmatrix} [Q, X_1] & \{Q, X_2\} \\ -\{Q, X_3\} & -[Q, X_4] \end{pmatrix} \tag{4.92}$$

and for an odd supermatrix $\mathbb{Y}$ as

$$\{\mathbb{Q}, \mathbb{Y}\} = \begin{pmatrix} \{Q, Y_1\} & [Q, Y_2] \\ -[Q, Y_3] & -\{Q, Y_4\} \end{pmatrix} \tag{4.93}$$

The action on each entries is the correct one with respect to the nature of the field on which the supercharge is acting.

We want to compute the action of the supercharges on the Wilson Line superconnection $\mathcal{L}$. It's convenient to split it into three parts $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_B + \mathcal{L}_F$and when the contour is the line takes the following expression

$$\mathcal{L} = \begin{pmatrix} \ell A_3 & 0 \\ 0 & \ell \hat{A}_3 \end{pmatrix} + \frac{2\pi i\ell}{k}\begin{pmatrix} Z\bar{Z} - Y_a\bar{Y}^a & 0 \\ 0 & \bar{Z}Z - \bar{Y}^a Y_a \end{pmatrix} + 2\sqrt{\frac{\pi\ell}{k}}\begin{pmatrix} 0 & i\bar{\psi}_{(1)} \\ \psi^{(1)} & 0 \end{pmatrix} \tag{4.94}$$

All the components are even supermatrices. Applying rule (4.92) on each piece we obtain

$$[\mathbb{Q}^a, \mathcal{L}_A] = \begin{pmatrix} -\frac{2\pi i\ell}{k}\left(\bar{\psi}_1\bar{Y}^a - \bar{\chi}_1^a\bar{Z} + \varepsilon^{abc}Y_b\chi_c^2\right) & 0 \\ 0 & \frac{2\pi i\ell}{k}\left(\bar{Y}^a\bar{\psi}_1 - \bar{Z}\bar{\chi}_1^a + \varepsilon^{abc}\chi_c^2 Y_b\right) \end{pmatrix} \tag{4.95}$$

$$[\mathbb{Q}^a, \mathcal{L}_B] = \begin{pmatrix} \frac{2\pi i \ell}{k}\left(-\bar{\psi}_1 \bar{Y}^a - \bar{\chi}_1^a \bar{Z} + \varepsilon^{abc} Y_b \chi_c^2\right) & 0 \\ 0 & -\frac{2\pi i}{k}\left(-\bar{Y}^a \bar{\psi}_1 - \bar{Z}\bar{\chi}_1^a + \varepsilon^{abc}\chi_c^2 Y_b\right) \end{pmatrix} \tag{4.96}$$

$$[\mathbb{Q}^a, \mathcal{L}_F] = \begin{pmatrix} 0 & 0 \\ 2\sqrt{\frac{\pi\ell}{k}}\left(iD_3\bar{Y}^a + \frac{2\pi i}{k}\left(\bar{Y}^a \ell_B - \hat{\ell}_B \bar{Y}^a\right)\right) & 0 \end{pmatrix} \tag{4.97}$$

that gives

$$[\mathbb{Q}^a, \mathcal{L}] = \begin{pmatrix} -\frac{4\pi i\ell}{k}\bar{\psi}_1 \bar{Y}^a & 0 \\ 2\sqrt{\frac{\pi\ell}{k}}\left(iD_3\bar{Y}^a + \frac{2\pi i}{k}\left(\bar{Y}^a \ell_B - \hat{\ell}_B \bar{Y}^a\right)\right) & \frac{4\pi i\ell}{k}\bar{Y}^a\bar{\psi}_1 \end{pmatrix} \tag{4.98}$$

This result is compatible with a supergauge transformation, as

$$[\mathbb{Q}, \mathcal{L}] = i\partial_3 G^a + [G^a, \mathcal{L}] \tag{4.99}$$

where G^a is a supergauge odd supermatrix of the form

$$G^a = 2\sqrt{\frac{\pi\ell}{k}}\begin{pmatrix} 0 & 0 \\ \bar{Y}^a & 0 \end{pmatrix} \tag{4.100}$$

The same holds for the variation w.r.t. $\bar{Q}_a$, and after defining the other supergauge matrix

$$\bar{G}_a = -2\sqrt{\frac{\pi\ell}{k}}\begin{pmatrix} 0 & Y_a \\ 0 & 0 \end{pmatrix} \tag{4.101}$$

we can write

$$\left[\bar{\mathbb{Q}}_a, \mathcal{L}\right] = i\partial_3 \bar{G}_a + \left[\bar{G}_a, \mathcal{L}\right] \tag{4.102}$$

The two new matrices are related by

$$\bar{G}_a = -\mathfrak{P}\left(G^a\right)^{PL}\mathfrak{P}^{-1} \tag{4.103}$$

and vice versa. By looking at the transformation rules (4.99) and (4.102), it's easy to define the covariant supercharges

$$\mathbb{Q}^a_{cov}(\cdot) \equiv [\mathbb{Q}^a_{cov}, (\cdot)\} = [\mathbb{Q}^a - G^a, (\cdot)\} \tag{4.104}$$

$$\bar{\mathbb{Q}}_{cov,a}(\cdot) \equiv \left[\bar{\mathbb{Q}}_{cov,a}, (\cdot)\right\} = \left[\bar{\mathbb{Q}}_a - \bar{G}_a, (\cdot)\right\} \tag{4.105}$$

where $(\cdot)$ means applied to an operator $\cdot$ with parity $|\cdot|$. Multiplying by the susy transformation parameters we recover the standard covariant susy transformation

$$\Delta_{cov}(\cdot) = \left[\theta_a \mathbb{Q}^a + \begin{pmatrix} 0 & 0 \\ \theta_a \bar{Y}^a & 0 \end{pmatrix}, (\cdot)\right] \tag{4.106}$$

$$\bar{\Delta}_{cov}(\cdot) = \left[\bar{\theta}^a \bar{\mathbb{Q}}_a + \begin{pmatrix} 0 & \bar{\theta}^a Y_a \\ 0 & 0 \end{pmatrix}, (\cdot)\right] \tag{4.107}$$

The covariant supercharges defined in (4.104) and (4.105) must transform under parity-like respectively as G^a and $\bar{G}_a$, i.e.

$$\bar{\mathbb{Q}}_a = -\mathfrak{P}\,(\mathbb{Q}^a)^{PL}\,\mathfrak{P}^{-1} \tag{4.108}$$

and vice versa. From (4.108) we can read the transformation rules for the original supercharges

$$(Q^a)^{PL} = \bar{Q}_a \qquad \left(\bar{Q}_a\right)^{PL} = Q^a \tag{4.109}$$

4.4 Displacement supermultiplet

The displacement operator has quantum numbers $[2,3,0,0]$. We can reach these numbers by a chain of three supercharges action on a superprimary operator with quantum numbers $\left[\frac{1}{2},\frac{3}{2},0,0\right]$. The operator we can build with these quantum numbers is

$$\mathbb{Z} = \phi \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \tag{4.110}$$

where ϕ is a constant we will fix. The displacement is obtained as

$$\mathbb{D} = \frac{1}{3!}\varepsilon_{abc}\mathbb{Q}^a_{cov}\mathbb{Q}^b_{cov}\mathbb{Q}^c_{cov}\mathbb{Z} \tag{4.111}$$

In order to fix the constant ϕ, we have to compare the action of $\mathbb{Q}^a_{cov}\mathbb{Z}$ with the operator coming from the variation of the WL w.r.t. the generator J^a_1[2]. Let's start from the last one:

$$\mathbb{O}^a := i\delta_{J^a_1}\mathcal{L} = -2\sqrt{\frac{\pi\ell}{k}}\begin{pmatrix} 2\sqrt{\frac{\pi\ell}{k}}Z\bar{Y}^a & \bar{\chi}^a_1 \\ 0 & 2\sqrt{\frac{\pi\ell}{k}}\bar{Y}^a Z \end{pmatrix} \tag{4.112}$$

The former is

$$\mathbb{Q}^a_{cov}\mathbb{Z} = -\phi\begin{pmatrix} 2\sqrt{\frac{\pi\ell}{k}}Z\bar{Y}^a & \bar{\chi}^a_1 \\ 0 & 2\sqrt{\frac{\pi\ell}{k}}\bar{Y}^a Z \end{pmatrix} \tag{4.113}$$

Choose $\phi = 2\sqrt{\frac{\pi\ell}{k}}$ then

$$\mathbb{Z} = 2\sqrt{\frac{\pi\ell}{k}}\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix} \qquad \textit{and} \qquad \mathbb{Q}^a_{cov}\mathbb{Z} = \mathbb{O}^a \tag{4.114}$$

Acting again with $\mathbb{Q}^a_{cov}$ on this new operator

$$\mathbb{Q}^a_{cov}\mathbb{O}^b = \varepsilon^{abc}\mathbb{L}_c \tag{4.115}$$

[2]The generator J acts on $\mathcal{L}$ as a supermatrix, following (D.11) with even parity.

with

$$\mathbb{L}_c = -2\sqrt{\frac{\pi\ell}{k}} \begin{pmatrix} 2\sqrt{\frac{\pi\ell}{k}}\left(-\varepsilon_{crs}\bar{\chi}_1^r\bar{Y}^s - Z\chi_c^2\right) & -iDY_c \\ -\frac{4\pi\ell}{k}\left(\varepsilon_{crs}\bar{Y}^r Z\bar{Y}^s\right) & -2\sqrt{\frac{\pi\ell}{k}}\left(-\varepsilon_{crs}\bar{Y}^s\bar{\chi}_1^r - \chi_c^2 Z\right) \end{pmatrix} \tag{4.116}$$

Last step

$$\mathbb{D} = \frac{1}{3!}\varepsilon_{abc}\varepsilon^{bcd}\mathbb{Q}^a_{cov}\mathbb{L}_d = \frac{1}{3}\mathbb{Q}^a_{cov}\mathbb{L}_a \tag{4.117}$$

and we get

$$\mathbb{D} = i\begin{pmatrix} \frac{4\pi\ell}{k}\left(ZD\bar{Z} - DY_g\bar{Y}^g + i\bar{\chi}_1^g\chi_g^2\right) & 2\sqrt{\frac{\pi\ell}{k}}D\bar{\psi}_1 \\ 8i\left(\frac{\pi\ell}{k}\right)^{\frac{3}{2}}\left(\bar{Y}^g Z\chi_g^2 - \chi_g^2 Z\bar{Y}^g + \varepsilon_{abc}\bar{Y}^a\bar{\chi}_1^b\bar{Y}^c\right) & \frac{4\pi\ell}{k}\left(D\bar{Z}Z - \bar{Y}^g DY_g - i\chi_g^2\bar{\chi}_1^g\right) \end{pmatrix} \tag{4.118}$$

The results is compatible with $\mathbb{D}_E$ of (4.88) ($\ell = 1$). The term $\tilde{\mathbb{D}}$ does not appear, sign that the covariantized supercharges are blind to exact-terms.

To conclude the section, we just make an observation. When fixing the coefficient of the fermions transformation under parity-like symmetry, for consistency we required (4.12) to be satisfied, precisely with the strange minus sign. Indeed, the minus sign is perfectly consistent and we can see it by looking at (4.114). Under parity-like we have

$$(\mathbb{O}^a)^{PL} = (\{\mathbb{Q}^a_{cov}, \mathbb{Z}\})^{PL} = \left\{(\mathbb{Q}^a_{cov})^{PL}, (\mathbb{Z})^{PL}\right\} = -\left\{\bar{\mathbb{Q}}_a^{cov}, \bar{\mathbb{Z}}\right\} = -\bar{\mathbb{O}}_a \tag{4.119}$$

4.5 Bremsstrahlung function

We have seen that, for BPS Wilson Loops, we can compute the Bremsstrahlung function considering several different configurations. In this section we want to compute perturbatively the Bremsstrahlung function by insertions of superprimary operators in the Displacement supermultiplet.

The Bremsstrahlung function is the coefficient of the displacement correlator. Computing it with quantum correction in a perturbation theory framework is a quite hard task, so, taking advantage of the fact that correlators of operators in the same supermultiplet are proportional to the same coefficient, we switch in computing the correlator of the superprimaries in the displacement supermultiplet.

In [70], they argue that the two point function $\langle\mathbb{Z}\bar{\mathbb{Z}}\rangle$ should be proportional to the Bremsstrahlung function in the following way

$$\langle\mathbb{Z}_\sigma\bar{\mathbb{Z}}_0\rangle = \frac{C_\Phi}{|\sigma|}, \qquad C_\Phi = 2B_{1/2} \tag{4.120}$$

We also have a proposal for it (up to 5 loops) from matrix model computation on the latitude Wilson Loop [60], that is

$$\begin{aligned} B_{1/2} &= \frac{N_1N_2}{4k\,(N_1+N_2)} - \frac{\pi^2 N_1 N_2\,(N_1N_2-3)}{24k^3\,(N_1+N_2)} + o(k^{-5}) \\ &= \frac{N_1N_2}{4k\,(N_1+N_2)}\left(1-\frac{\pi^2}{6k^2}\,(N_1N_2-3)+o(k^{-5})\right) \end{aligned} \tag{4.121}$$

(we don't need the five loops correction now). In the ABJM case the Bremsstrahlung function has the following form [68, 102]

$$B_{1/2} = \frac{N}{8k} - \frac{\pi^2 N\,(N^2-3)}{48k^3} + o\left(k^{-5}\right) = \frac{N}{8k}\left(1-\frac{\pi^2}{6k^2}\left(N^2-3\right)\right) + o\left(k^{-5}\right) \tag{4.122}$$

4.5.1 Two-pt function $\langle \mathbb{Z}\bar{\mathbb{Z}}\rangle$

We will use the definition of the operators as in (4.114) and the rescaled action, but this time we not fix the value of ℓ. Instead, we define a new, parity-invariant coupling

$$\kappa = \frac{\ell}{k} \tag{4.123}$$

Tree Level

The tree-level contribution is straightforward

$$\langle\langle \mathbb{Z}_\sigma \bar{\mathbb{Z}}_0\rangle\rangle_W^{(0)} = \frac{\kappa}{2}\frac{N_1N_2}{N_1+N_2}\frac{1}{2\sin\frac{\sigma}{2}} \tag{4.124}$$

and it's easy to see that is in accordance with (4.120) and the first term of (4.121).

One Loop

The 1-loop bosonic interactions are obtained by expanding the Path-Exponential at first order; in total we get three contribution. Let's start from the first one

$$\begin{aligned} &-i\int_{-\pi}^{0} d\tau\,\mathrm{Tr}[\mathbb{Z}_\sigma\bar{\mathbb{Z}}_0\mathcal{L}_\tau] \\ &= -2\pi i\kappa\int_{-\pi}^{0} d\tau\,\mathrm{Tr}\left[\begin{pmatrix}0 & Z\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ \bar{Z} & 0\end{pmatrix}\begin{pmatrix}\mathcal{A} & i\sqrt{2\pi\kappa}|\dot{x}|\eta_I\bar{\psi}^I\\ -i\sqrt{2\pi\kappa}|\dot{x}|\psi_I\bar{\eta}^I & \hat{\mathcal{A}}\end{pmatrix}\right] \end{aligned} \tag{4.125}$$

where $\mathcal{A} = \frac{1}{\sqrt{k}}A_\mu\dot{x}^\mu - 2\pi i\kappa|\dot{x}|M_J^I C_I\bar{C}^J$ and $\hat{\mathcal{A}} = \frac{1}{\sqrt{k}}\hat{A}_\mu\dot{x}^\mu - 2\pi i\kappa|\dot{x}|M_J^I\bar{C}^J C_I$. The previous integral becomes

$$\begin{aligned} -2\pi i\kappa\int_{-\pi}^{0} d\tau\,\mathrm{Tr}[Z_\sigma\bar{Z}_0\mathcal{A}_\tau] &= -2\pi i\kappa\int_{-\pi}^{0} d\tau\,\mathrm{Tr}[Z_\sigma\bar{Z}_0\,(2\pi i\kappa)\left(Z\bar{Z}-Y_a\bar{Y}^a\right) \\ &= 4\pi^2\kappa^2\int_{-\pi}^{0} d\tau\,\mathrm{Tr}\left[Z_\sigma\bar{Z}_0 Z_\tau\bar{Z}_\tau\right] \end{aligned} \tag{4.126}$$

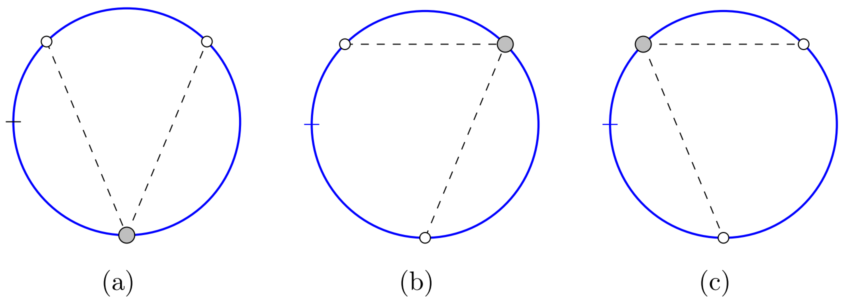


Figure 4.3: Three bosonic interaction first order.

Do the contractions

$$(2\pi\kappa)^2 N_1^2 N_2 \left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2 \int_{-\pi}^{0} \frac{d\tau}{\left(4\sin\frac{\sigma-\tau}{2}\sin\frac{-\tau}{2}\right)^{1-2\varepsilon}} \tag{4.127}$$

In the same way we get the other two contributions

$$4\pi^2\kappa^2 \int_0^\sigma d\tau Tr\left[Z_\sigma \bar{Z}_\tau Z_\tau \bar{Z}_0\right] = (2\pi\kappa)^2 N_1 N_2^2 \left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2 \int_0^\sigma \frac{d\tau}{\left(4\sin\frac{\sigma-\tau}{2}\sin\frac{\tau}{2}\right)^{1-2\varepsilon}} \tag{4.128}$$

$$4\pi^2\kappa^2 \int_\sigma^\pi d\tau \,\mathrm{Tr}\left[Z_\tau \bar{Z}_\tau Z_\sigma \bar{Z}_0\right] = (2\pi\kappa)^2 N_1^2 N_2 \left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2 \int_\sigma^\pi \frac{d\tau}{\left(4\sin\frac{\tau-\sigma}{2}\sin\frac{\tau}{2}\right)^{1-2\varepsilon}} \tag{4.129}$$

Using the results of the appendix C.1 we get the final bosonic contribution

$$(2\pi\kappa)^2 N_1 N_2 \left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2 \frac{2^{4\varepsilon-1}\sqrt{\pi}\left(\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}\tan\frac{\sigma}{4}}{\left(1+\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}} \frac{\Gamma(2\varepsilon)}{\Gamma\left(\frac{1}{2}+2\varepsilon\right)}$$
$$\times\left[N_1\, {}_2F_1\left(\frac{1}{2},2\varepsilon;\frac{1}{2}+2\varepsilon;-\cot^2\left(\frac{\sigma}{4}\right)\right)\tan^{-4\varepsilon}\left(\frac{\sigma}{4}\right)+N_2\, {}_2F_1\left(\frac{1}{2},2\varepsilon;\frac{1}{2}+2\varepsilon;-\tan^2\left(\frac{\sigma}{4}\right)\right)\right] \tag{4.130}$$

The series expansion in ε is

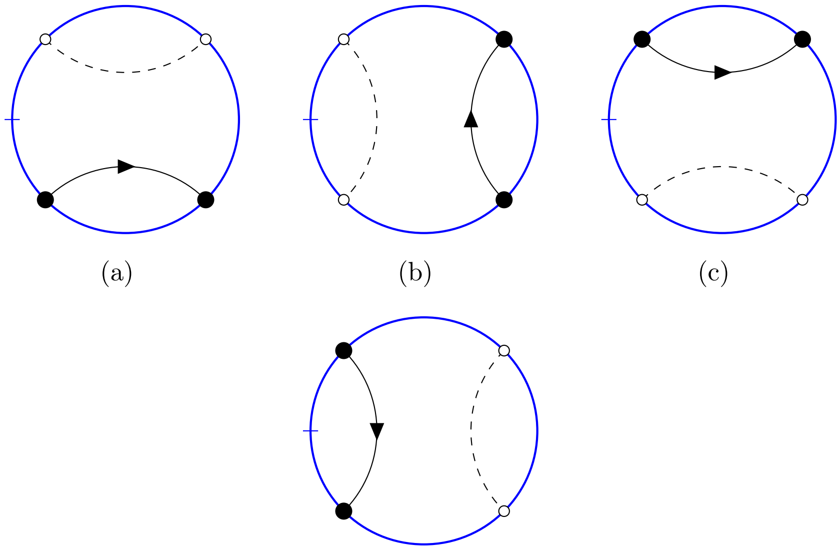
$$\kappa^2\frac{N_1N_2}{2\sin\left(\frac{\sigma}{2}\right)}\left[\frac{1}{2\varepsilon}+\left(2\log\left(\sin\left(\frac{\sigma}{2}\right)\right)-\gamma+\log(\pi)-2\psi^{(0)}\left(\frac{1}{2}\right)\right)+O\left(\varepsilon^1\right)\right] \tag{4.131}$$

γ being the Euler-Mascheroni constant.

To get the fermionic interaction, expand at second order the Wilson Lines, obtaining four contributions

$$-2\pi^2\kappa^2\int_{-\pi}^{0} d\tau_1 \int_{-\pi}^{\tau_1} d\tau_2 Tr\left[Z_\sigma \bar{Z}_0 \left(\eta\bar{\psi}\right)_1 \left(\psi\bar{\eta}\right)_2\right] = -\mathfrak{F}(k,\varepsilon) N_1 N_2^2 \tag{4.132}$$

$$-2\pi^2\kappa^2\int_0^\sigma d\tau_1 \int_0^{\tau_1} d\tau_2 Tr\left[Z_\sigma \left(\psi\bar{\eta}\right)_1 \left(\eta\bar{\psi}\right)_2 \bar{Z}_0\right] = -\mathfrak{F}(k,\varepsilon) N_1^2 N_2 \sin^{2\varepsilon}\left(\frac{\sigma}{2}\right) \tag{4.133}$$



(a) (b) (c)

(d)

Figure 4.4: Three bosonic interaction first order.

$$-2\pi^2\kappa^2\int_\sigma^\pi d\tau_1\int_\sigma^{\tau_1} d\tau_2 Tr\left[\left(\eta\bar{\psi}\right)_1\left(\psi\bar{\eta}\right)_2 Z_\sigma\bar{Z}_0\right] = -\mathfrak{F}(k,\varepsilon)N_1N_2^2\cos^{2\varepsilon}\left(\frac{\sigma}{2}\right) \tag{4.134}$$

$$\begin{aligned} -2\pi^2\kappa^2\int_{-\pi}^0 d\tau_1\int_\sigma^\pi d\tau_2\,\mathrm{Tr}\left[\left(\psi\bar{\eta}\right)_2 Z_\sigma\bar{Z}_0\left(\eta\bar{\psi}\right)_1\right] =& \\ =\mathfrak{F}(k,\varepsilon)N_1N_2^2\left[1-\sin^{2\varepsilon}\left(\frac{\sigma}{2}\right)+\cos^{2\varepsilon}\left(\frac{\sigma}{2}\right)\right]& \end{aligned} \tag{4.135}$$

where we have factorize the constant in front

$$\mathfrak{F}(k,\varepsilon) = (2\pi\kappa)^2\frac{1}{\left(2\sin\frac{\sigma}{2}\right)^{1-2\varepsilon}}\left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2\frac{2^{2\varepsilon}}{\varepsilon} \tag{4.136}$$

Summing up, the fermions contribution is

$$-(2\pi\kappa)^2 N_1N_2\left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2\frac{\left(2\sin\left(\frac{\sigma}{2}\right)\right)^{4\varepsilon-1}}{\varepsilon} \tag{4.137}$$

Expansion around ε reads

$$-\kappa^2\frac{N_1N_2}{2\sin\left(\frac{\sigma}{2}\right)}\left[\frac{1}{2\varepsilon}+\left(2\log\left(\sin\left(\frac{\sigma}{2}\right)\right)+\gamma+\log(16\pi)\right)+O\left(\varepsilon^1\right)\right] \tag{4.138}$$

and summing (4.131) and (4.138) the one loop correction vanishes, in agreement with (4.121). Notice that even the contributions of the finite part perfectly cancel. In particular, the constant terms vanish due to the special value of the digamma function

$$\psi^{(0)}\left(\frac{1}{2}\right) = -\gamma-2\log(2) \tag{4.139}$$

Two Loops

As happened for the topological operators, the first non-vanishing correction should appear at order k^{-2}. The computation of the diagrams is a work in progress and will appear soon [6]. Here we present the diagrams and comment on the various topology we find.

Correction to the Tree Level The first contribution at order k^{-2} comes from the two-loop correction of the tree-level and it is already computed in (C.17).

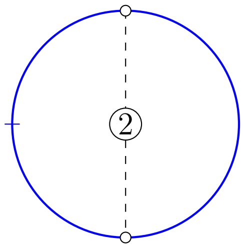


Figure 4.5: Two-loops correction to the tree-level.

First order In the first-order expansion, taking some vertices from the action, we have three non-vanishing diagrams, see figure 4.6. It is worth noticing that similar diagrams appear in the two-loop expansion of the topological operators. Moreover, they are the analogue of diagrams 3.2(j), 3.2(k) and 3.2(b). In particular, all those diagrams are finite and give similar coefficients proportional to $(\pi^2 - 12)$.

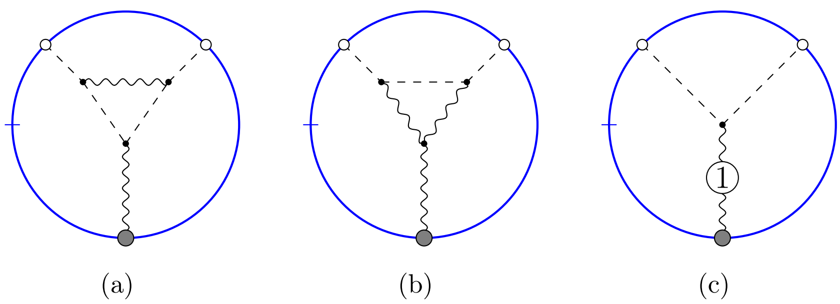


Figure 4.6: Diagrams at order k^{-2} from the first order expansion.

Second order In the second order expansion in principle we have seven type of diagrams (see figure 4.7), all with a new topology with respect to the topological case, except for the diagram 4.7(b), similar to 3.2(g), that even in this case gives a finite contribution. For the others, we expect they diverge.

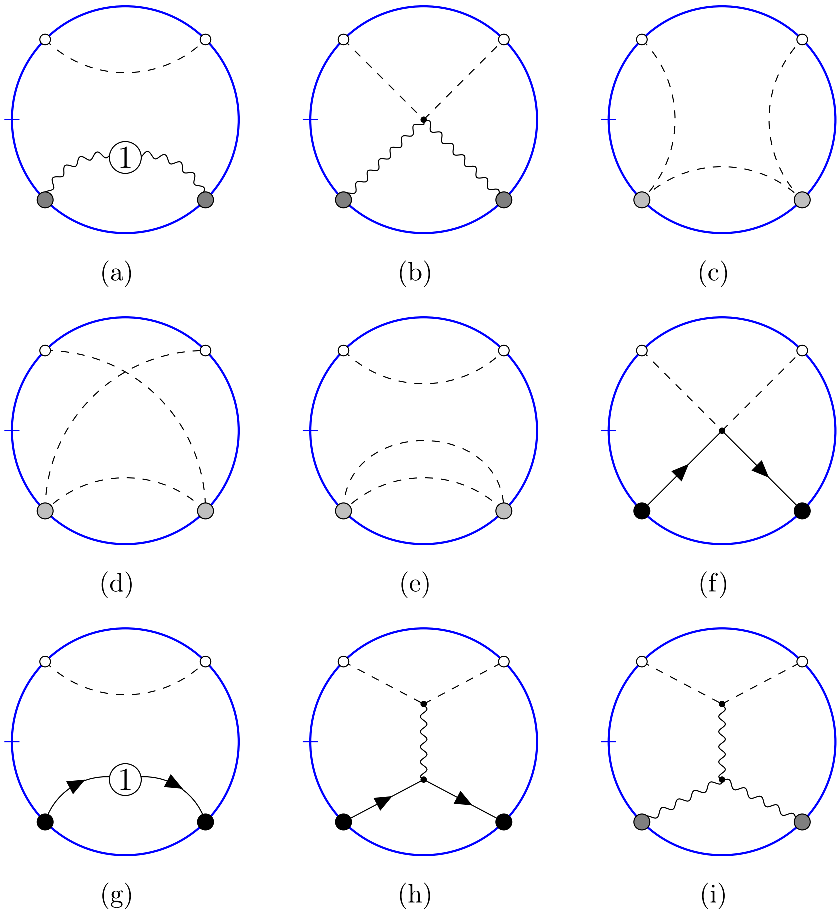


Figure 4.7: Diagrams appearing in the second order expansion at $o\left(k^{-2}\right)$.

Third and fourth order In these cases we have a limited number of possibilities since the level k^{-2} is almost or completely saturated by the various connection coming from the path-order exponential expansion. We expect all these diagrams to give divergent contributions.

4.6 Operators associated to the $U(1)_B$ symmetry

We saw in the previous chapter that the the algebra preserved by the line inside $\mathfrak{osp}(6|4)$ is $\mathfrak{su}(1,1|3) \oplus \mathfrak{u}(1)_B$. The $\mathfrak{su}(1,1|3)$ maximal bosonic subalgebra is $\mathfrak{su}(1,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)$. If we turn on the Wilson Line, the $\mathfrak{su}(1,1|3)$ algebra is still preserved, while the $\mathfrak{u}(1)_B$ apparently do not[3]. Nevertheless, it is worth to study the operator related to this apparent breaking. To obtain this operator, we consider the superconnection variation under the generator of the $\mathfrak{u}(1)_B$ symmetry

$$B = M_{12} + 2iJ_1{}^1 \tag{4.140}$$

[3]The variation of $\mathcal{L}$ under $\mathfrak{u}(1)_b$ can be reabsorbed by a global gauge transformation on the fermions [103]

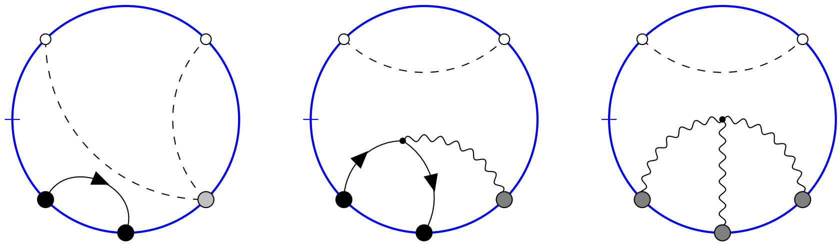


Figure 4.8: Diagrams appearing at k^{-2} from the third order expansion.

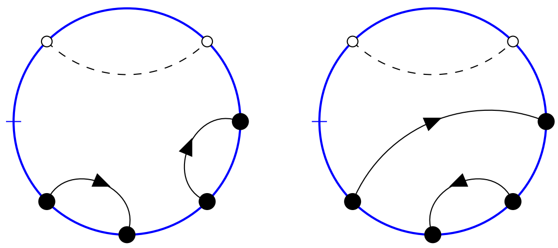


Figure 4.9: Diagrams appearing at k^{-2} in the fourth order expansion of the Wilson Loop.

It acts on the scalar fields as

$$\begin{aligned} \delta_{J_1{}^1} Z &= -\frac{3}{4} Z & \delta_{J_1{}^1} Y_a &= \frac{1}{4} Y_a \\ \delta_{J_1{}^1} \bar{Z} &= \frac{3}{4} \bar{Z} & \delta_{J_1{}^1} \bar{Y}^a &= -\frac{1}{4} \bar{Y}^a \end{aligned} \Rightarrow \delta_{J_1{}^1} C\bar{C} = 0 \tag{4.141}$$

so $\delta_B C\bar{C} = 0$. We have the same for the third component of the gauge field, i.e. $\delta_B A_3 = 0$. For the fermions we have a contribution from the rotations and another one from the R-symmetry:

$$\begin{aligned} \delta_{M_{12}} \psi &= \frac{i}{2} \psi & \delta_{J_1{}^1} \psi &= -\frac{3}{4} \psi \\ \delta_{M_{12}} \bar{\psi} &= -\frac{i}{2} \bar{\psi} & \delta_{J_1{}^1} \bar{\psi} &= \frac{3}{4} \bar{\psi} \end{aligned} \tag{4.142}$$

such that

$$\delta_b \bar{\psi}^1_+ = i \bar{\psi}^1_+ \qquad \delta_b \psi_1^+ = -i \psi_1^+ \tag{4.143}$$

so

$$\mathbb{P} := i\delta_B \mathcal{L} = i\,[\mathbb{B}, \mathcal{L}] = 2\sqrt{\pi\kappa} \begin{pmatrix} 0 & -i\bar{\psi}^1_+ \\ \psi_1^+ & 0 \end{pmatrix} \tag{4.144}$$

where $\mathbb{B}$ is the supermatrix representation of B. If we define the following operators

$$\mathbb{Y}_a = 2\sqrt{\pi\kappa} \begin{pmatrix} 0 & Y_a \\ 0 & 0 \end{pmatrix} \qquad \bar{\mathbb{Y}}^a = 2\sqrt{\pi\kappa} \begin{pmatrix} 0 & 0 \\ \bar{Y}^a & 0 \end{pmatrix} \tag{4.145}$$

and act with the supercharges

$$\mathbb{Q}^a_{cov}\mathbb{Y}_b = 2\sqrt{\pi\kappa}\delta^a{}_b \begin{pmatrix} 0 & \bar{\psi}_1 \\ 0 & 0 \end{pmatrix} - 4\pi\kappa \begin{pmatrix} Y_b\bar{Y}^a & 0 \\ 0 & \bar{Y}^a Y_b \end{pmatrix} \tag{4.146}$$

$$\bar{\mathbb{Q}}_{cov,a}\mathbb{Y}_b = -2i\sqrt{\pi\kappa}\varepsilon_{abc} \begin{pmatrix} 0 & \bar{\chi}^c_2 \\ 0 & 0 \end{pmatrix} \tag{4.147}$$

$$\mathbb{Q}^a_{cov}\bar{\mathbb{Y}}^b = 2\sqrt{\pi\kappa}\varepsilon^{abc} \begin{pmatrix} 0 & 0 \\ \chi^2_c & 0 \end{pmatrix} \tag{4.148}$$

$$\bar{\mathbb{Q}}_{cov,a}\bar{\mathbb{Y}}^b = 2\sqrt{\pi\kappa}\delta_a{}^b \begin{pmatrix} 0 & 0 \\ i\psi^1 & 0 \end{pmatrix} + 4\pi\kappa \begin{pmatrix} Y_a\bar{Y}^b & 0 \\ 0 & \bar{Y}^b Y_a \end{pmatrix} \tag{4.149}$$

so the following holds

$$\mathbb{P} = -\frac{i}{3}\left(\mathbb{Q}^a_{cov}\mathbb{Y}_a + \bar{\mathbb{Q}}_{cov,a}\bar{\mathbb{Y}}^a\right) \tag{4.150}$$

In principle, for the same argument on the Displacement, we expect $\mathbb{P}$ to be a protected operator since it comes from a broken symmetry. But (4.150) doesn't imply that the superprimaries $\mathbb{Y}$, $\bar{\mathbb{Y}}$ are protected. As a check, we perform a one-loop computation of the two-point function of these operators.

4.6.1 Two-pt function $\langle\mathbb{Y}\bar{\mathbb{Y}}\rangle$ and anomalous dimension

As we did for the Bremsstrahlung, we compute the correlation function on the circle. In this case, we have $SU(3)$ indices, but the integrals will be the same. For simplicity, again we insert $\bar{\mathbb{Y}}$ in $\tau = 0$ while $\mathbb{Y}$ in $\tau = \sigma > 0$.

Tree Level

The tree level computation is again straightforward

$$\langle\langle\mathbb{Y}_a(\sigma)\bar{\mathbb{Y}}^b(0)\rangle\rangle_{\mathcal{W}} = \frac{\langle\mathrm{Tr}\left[Y_a(\sigma)\bar{Y}^b(0)\right]\rangle}{\langle\mathcal{W}\rangle} = \kappa\frac{N_1N_2}{N_1+N_2}\frac{\delta_a{}^b}{2\sin\frac{\sigma}{2}} \tag{4.151}$$

1-loop bosonic

Here lies the main difference w.r.t. the Bremsstrahlung computation: the $\mathbb{Y}$'s couple with the $SU(3)$ part of the term $(Z\bar{Z} - Y_a\bar{Y}^a)$ in the superconnection, changing the sign of the integrals. Indeed, e.g.

$$-i\int_{-\pi}^{0} d\tau\,\mathrm{Tr}\left[\mathbb{Y}_a\bar{\mathbb{Y}}^b\mathcal{L}_\tau\right] \tag{4.152}$$

reduce to

$$-8\pi^2\kappa^2\int_{-\pi}^{0} d\tau\,\mathrm{Tr}\left[Y_a\bar{Y}^bY_c\bar{Y}^c\right] \tag{4.153}$$

and after the contractions reads

$$-2\delta_a{}^b (2\pi\kappa)^2 N_1^2 N_2 \left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2 \int_{-\pi}^{0} \frac{d\tau}{\left(4\sin\frac{\sigma-\tau}{2}\sin\frac{-\tau}{2}\right)^{1-2\varepsilon}} \tag{4.154}$$

The same happens for the other two contributions, and the result for the bosonic part is

$$-2\delta_a{}^b \left(\frac{2\pi}{k}\right)^2 N_1 N_2 \left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2 \frac{2^{4\varepsilon-1}\sqrt{\pi}\left(\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}\tan\frac{\sigma}{4}}{\left(1+\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}} \frac{\Gamma(2\varepsilon)}{\Gamma\left(\frac{1}{2}+2\varepsilon\right)} \times$$
$$\left[N_1\, {}_2F_1\left(\frac{1}{2}, 2\varepsilon; \frac{1}{2}+2\varepsilon; -\cot^2\left(\frac{\sigma}{4}\right)\right)\tan^{-4\varepsilon}\left(\frac{\sigma}{4}\right) + N_2\, {}_2F_1\left(\frac{1}{2}, 2\varepsilon; \frac{1}{2}+2\varepsilon; -\tan^2\left(\frac{\sigma}{4}\right)\right)\right] \tag{4.155}$$

1-loop fermionic

Since the fermions coming from the second order expansion do not interact with the scalars, the fermionic contribution is the same as for the Bremsstrahlung

$$-2\delta_a{}^b (2\pi\kappa)^2 N_1 N_2 \left(\frac{\Gamma\left(\frac{1}{2}-\varepsilon\right)}{4\pi^{\frac{3}{2}-\varepsilon}}\right)^2 \frac{\left(2\sin\left(\frac{\sigma}{2}\right)\right)^{4\varepsilon-1}}{\varepsilon} \tag{4.156}$$

Anomalous dimension

The divergences in this case don't mutually cancel out. Indeed, if we sum the two contributions we get

$$-\kappa^2\frac{N_1N_2}{2\varepsilon\sin\left(\frac{\sigma}{2}\right)} + \kappa^2\frac{N_1N_2\left(-4\log\left(\sin\left(\frac{\sigma}{2}\right)\right) - \log\left(16\pi^2\right) + 2\psi^{(0)}\left(\frac{1}{2}\right)\right)}{2\sin\left(\frac{\sigma}{2}\right)} + O\left(\varepsilon\right) \tag{4.157}$$

If we take a closer look to the finite part of (4.157), and in particular at the term with the logarithm of the sine, it's easy to convince ourselves that it is compatible with an expansion like

$$\frac{1}{\left(\sin\frac{\sigma}{2}\right)^{\Delta^{(0)}+\gamma^{(1)}}} \sim \frac{1}{\left(\sin\frac{\sigma}{2}\right)^{\Delta^{(0)}}}\left(1-\gamma^{(1)}\log\sin\frac{\sigma}{2} + o\left(\gamma^2\right)\right) \tag{4.158}$$

Together with the tree-level, we have

$$\langle\langle \mathbb{Y}_a(\sigma)\bar{\mathbb{Y}}^b(0)\rangle\rangle_{\mathcal{W}}^{(0+(1)} = \kappa\frac{N_1N_2}{N_1+N_2}\frac{\delta_a{}^b}{2\sin\frac{\sigma}{2}}\times$$
$$\left[1-\kappa\frac{N_1+N_2}{\varepsilon} + \kappa\left(N_1+N_2\right)\left(-4\log\left(\sin\left(\frac{\sigma}{2}\right)\right) - \log\left(256\pi^2\right) - 2\gamma\right) + O\left(\varepsilon\right)\right] \tag{4.159}$$

The anomalous dimension is easily recognized as

$$\gamma^{(1)} = 4\kappa\left(N_1+N_2\right) \tag{4.160}$$

The divergence appearing at one-loop is a clear sign, unexpected though, of the recombination. We thought the operators $\mathbb{Y}$, $\bar{\mathbb{Y}}$ to be protected, but the quantum correction show it is only at the classical level. Moreover, the supersymmetry is broken by the particular shape of the coupling matrix $M_I{}^J$, keeping the $SU(3)$ singlet $\mathbb{Z}$ protected while removing protection from the triplet. In the following, we try to explore the long supermultiplet generated from $\mathbb{Y}$, $\bar{\mathbb{Y}}$.

At this stage, one could also compute the anomalous dimension by taking the derivative of the renormalizing constant [104]. A possible mismatch in the sign can change the interpretation of what we think are bare or renormalized fields.

4.6.2 Multiplet recombination

The appearance of an anomalous dimension is a sign of the multiplet recombination phenomenon: due to the interactions, short multiplets recombine giving long multiplets. In a general QFT, this phenomenon occurs when considering relevant deformation of the theory using a conformal primary operator, triggering an RG-flow. At the fixed point, the conformal operator deforming the theory becomes a descendant of a field of the original theory [105].

We want to investigate the difference in the supermultiplets as long as we turn on or off the coupling with the Wilson Line (the free theory case corresponds to the limit $k \to \infty$, or $\kappa \to 0$), using representation theory arguments.

For the free theory, we can express the supercharges as (4.91) even if there's no need to use the supermatrix description. Instead, when the Wilson Loop is turned on, the charges get modified as (4.104) and (4.105).

Consider now the operator $\mathbb{Y}_a$, as defined in (4.145). From [103], we know this is the highest weight state of a short multiplet. Indeed

$$\mathbb{Q}^a \mathbb{Q}^b \mathbb{Y}^c = 0 \tag{4.161}$$

Let's focus on the first step

$$\{\mathbb{Q}^a, \mathbb{Y}_b\} = \begin{pmatrix} 0 & [Q^a, Y_b] \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta^a_b \bar{\psi}_1 \\ 0 & 0 \end{pmatrix} \tag{4.162}$$

From the point of view of group theory, this anticommutator is $\bar{\mathbf{3}} \times \mathbf{3} = \mathbf{8} + \mathbf{1}$, but in (4.162) the adjoint part does not appear, only the singlet shows up. This means that the $\mathbf{8}$ is a null state in the free theory, but when we turn on the WL we get

$$\{\mathbb{Q}^a_{cov}, \mathbb{Y}_b\} = \begin{pmatrix} -2\sqrt{\frac{\pi}{k}} Y_b \bar{Y}^a & [Q^a, Y_b] \\ 0 & -2\sqrt{\frac{\pi}{k}} \bar{Y}^a Y_b \end{pmatrix} =$$

$$= -2\sqrt{\frac{\pi}{k}} \begin{pmatrix} Y_b \bar{Y}^a - \frac{1}{3}\delta^a_b Y_c \bar{Y}^c & 0 \\ 0 & \bar{Y}^a Y_b - \frac{1}{3}\delta^a_b \bar{Y}^c Y_c \end{pmatrix} + \delta^a_b \begin{pmatrix} -\frac{2}{3}\sqrt{\frac{\pi}{k}} Y_c \bar{Y}^c & \bar{\psi}_1 \\ 0 & -\frac{2}{3}\sqrt{\frac{\pi}{k}} \bar{Y}^c Y_c \end{pmatrix} \tag{4.163}$$

where we have separeted the traceless symmetric part from the singlet part. Notice that the presence of the Wilson Line turn on the adjoint traceless part (that was null in the free case) and change the shape of the singlet.

On the other hand, for the action of $\bar{\mathbb{Q}}$ on $\mathbb{Y}$, there is no difference between free and interacting cases

$$\{\bar{\mathbb{Q}}^{cov}_a, \mathbb{Y}_b\} = \begin{pmatrix} 0 & [\bar{Q}_a, Y_b] \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\varepsilon_{abc}\bar{\chi}^c_2 \\ 0 & 0 \end{pmatrix} = \left\{\bar{\mathbb{Q}}_a, \mathbb{Y}_b\right\} \tag{4.164}$$

Again, using the group representation argument, this should be a $\mathbf{3}\times\mathbf{3} = \mathbf{6}+\bar{\mathbf{3}}$, but in this case the covariant part doesn't produce the symmetric state. In particular, this state should have quantum numbers [1,1,2,0] and can't be built out of the ABJ fields. The only consistent way to create an operator with those quantum numbers works only in the $U(2) \times U(2)$ case and is

$$(Y_A)_a^{\ \hat{a}}(Y_B)_b^{\ \hat{b}}\varepsilon^{ab}\,\varepsilon_{\hat{a}\hat{b}}$$

and looks like a monopole operator.

It is easy to see that, in the interacting case, the non-linear piece in the covariant supercharges prevents the supermultiplet to reach a null state: we obtained a long supermultiplet. Moreover, operators sitting in the same long supermultiplet share the same anomalous dimension of the superprimary since

$$Q\mathcal{O}_{\Delta+\gamma} \sim \mathcal{O}'_{\Delta+\frac{1}{2}+\gamma} \tag{4.165}$$

Thus, it will be interesting to check (4.165) by direct perturbative computations.

Conclusions

The imagination of nature is far,
far greater than the imagination of man.

R. P. Feynman

In this thesis, we made progress in the analysis of one-dimensional sectors in ABJ(M), both in the free theory and in the dCFT living on a 1/2-BPS Wilson Line. In the free theory, we have identified the topological sector; since the presence of the Chern–Simons term prevents us to write a one-dimensional path integral, i.e. a quantum mechanics, we have applied the topological twist. The maximum dimension of the twist allowed by the symmetries is one. We have found the explicit expression for the topological operator in terms of ABJ(M) fields and we computed its 2-, 3- and 4-point functions at tree level, finding in particular that the three-point function is always vanishing. We have seen that this feature agrees with a general rule for higher-dimensional operators.

At one loop, we found that all quantum corrections vanish due to the geometrical shape of the integrals, in agreement with parity-like transformations (odd powers in the coupling constant will produce a sign change). For the 2-point function, we pushed the computation to two loops, computing the diagrams recurring to known master integrals. The sum of all contributions is finite and parity-like invariant. The prefactor $\pi^2/6$ looks like the Riemann Zeta function $\zeta(2)$: it often appears in QFT loops computation. If so, the four loops correction should have a $\zeta(3)$ as a prefactor.

From the localization point of view, it has been argued that correlators of the superprimary operators are captured by derivatives of a mass deformed Matrix model with respect to the mass parameters [93] and after setting the mass parameters to zero. A relation between this matrix model and the central charge of the theory has been conjectured. By direct computation, expanding the matrix model at weak coupling, we matched our perturbative computation and proved the relation for the central charge. As a by-product, we have computed the two-loops correction to the ABJ(M) central charge. The central charge is a function that interpolates between the weak and strong coupling.

We have explored the possibility to have a non-trivial defect realizing the line: the case we considered is a one-dimensional defect realized by Wilson Loop preserving half of the supercharges. The gauge connection has to be specified in terms of an augmented gauge supergroup $U(N_1|N_2)$ and it is built with all the ABJ(M) fields, including fermions (thus the name "fermionic loop"). Here is the main difference with the four dimensions: in the latter, we can couple only a scalar to the gauge fields, while in ABJ(M) all the scalars enter with a coupling matrix ${M_I}^J$. This matrix turns out to be crucial when computing defect correlators.

The supermatrix structure has been obscure since the discovery of the fermionic loop. In particular, all the quantities living on the loop have to be written in a supermatrix form. The supercharges have a peculiar shape in this representation, which allows recovering the correct commutation relations on the single fields. Moreover, the supercharges preserved by the Wilson Line are slightly modified, in order to respect the SUSY condition of the loop. We call these new set covariant supercharges $\mathbb{Q}_{cov}$. In particular, we noticed that the addition of the new term light-up some states that were hidden without the Wilson Line. This phenomenon is the recombination of some supermultiplets, i.e. short supermultiplets become long when the line is turned on. On the other hand, group theory arguments predict the existence of other states, invisible to our multiplet analysis, since they don't have a realization in terms of fields. We think they are related to monopole operators.

We have found that new superprimary operators are present, whose scaling dimension is $\Delta = \frac{1}{2}$ and they represent fermions. One of these, denoted with $\mathbb{Z}$, is the superprimary operator of the displacement supermultiplet. We have shown that we can generate the whole multiplet by multiple actions of the covariant supercharges on the superprimary $\mathbb{Z}$. The Displacement operator obtained in this way agrees with the expression resulting from the contour deformations (aka wavy line).

Another issue of the fermionic loop is the difference in the one-loop correction to the VEV between the line and the circle. The two contours are conformally equivalent, so their VEV has to be the same. For the line, the fermionic diagram needs an IR cut-off to not diverge; after computing the integral, the IR divergence remains. This is in contrast with the zero one-loop correction of the circle, due to the compactness of the contour. There are two possible reasons: either there is a conformal anomaly acting at the quantum level, or the infinities of the line are due to the IR cut-off that breaks gauge invariance. We have tried to explore the second option, defining a cut-offed circle Wilson Loop, finding a parallel with the quantities resulting in the line computations. The compactness of the circle allows computing the possible shape of the pieces we are cutting away on the line, showing that possible divergencies mutually cancel out. Still, the option of the conformal anomaly is not excluded a priori, and further investigations are needed.

Since the operator $\mathbb{Z}$ is the superprimary of the Displacement multiplet, the coefficient of its two-point correlator is proportional to the Bremsstrahlung function $B_{1/2}$. We already know its value from the fermionic latitude in ABJM, and we have only a conjecture on the general ABJ expression. In particular, the one-loop is vanishing, as expected since in principle it is a protected quantity. We computed perturbatively the 2-point correlation function $\langle \mathbb{Z}\bar{\mathbb{Z}} \rangle_{\mathcal{W}}$: the tree level and the one-loop correction match the previous results. Computing the two-loop contribution is a work in progress. Some of the diagrams involved possess a very similar topology of the topological two-loops diagrams and new ones arise.

The topological sector for the Wilson Line is still elusive. Indeed, the topological twist can be performed as the topological line case, as we have the same superalgebra and in principle it will give us the same selection rules. However, the presence of an interacting defect forces the multiplets to recombine. This effect is manifest in various situations:

- no operators in the displacement multiplet satisfy the twist selection rules, in contrast to what happen in four dimensions;
- superprimaries $\mathbb{Y}$, $\bar{\mathbb{Y}}$ are protected in the free theory, but the correlator $\langle \mathbb{Y}\bar{\mathbb{Y}} \rangle_{\mathcal{W}}$ is divergent at one loop. The cancellation that we have for the Bremmsstrahlung function is now missing due to the change of sign in the coupling matrix $M_I{}^J$. The superprimaries $\mathbb{Y}$ now belong to a long multiplet that is no more protected by supersymmetry. They acquire an anomalous dimension, easily computed looking at the coefficient of the log in the finite part. The anomalous dimension has to be the same through all the long multiplet;
- The supermatrix analogue of the topological operator is no more a superprimary operator, but can be obtained by the action of a supercharge on, for instance, $\mathbb{Y}$.

The presence of the one-loop divergence is surprising because we believed those operators to be protected by SUSY. Indeed, they have to break all the charges at the quantum level. We are looking for a SUSY breaking mechanism in this case.

Thus, in the search of the topological sector, we need to further investigate the multiplets allowed on the Wilson Line and look if some the new multiplets match the twist selection rules.

Future directions

The essence of science is whenever you solve a problem, tons of new questions arise. Therefore, even if we have clarified some aspects regarding one-dimensional sectors in ABJ(M) and in particular for the fermionic Wilson Loop, there are still a lot of obscure points to investigate. Indeed, it is not clear yet the operators entering the new long supermultiplet: we stress that this is an effect due to only the interaction of the loop, not visible from the one-dimensional superconformal algebra. In particular, operators in the same long multiplet share the same anomalous dimension with the superprimary and we can use their $\gamma^{(1)}$ to classify operators. For higher scaling dimension operators the computation of $\gamma^{(1)}$ by perturbation theory can become cumbersome and it will require a non-perturbative approach. This can be accomplished by studying three- and four-point correlation functions using bootstrap techniques combined with localization and Mellin space analysis. For the three-point function in ABJ(M), some progress has been made recently in [106].

On one hand, the difference in the VEV between line and circle remains an open question and appears in more general situations, see e.g. [39]. On the other hand, logarithmic divergences appearing in the VEV for straight contours can be reabsorbed in a parameter defining a family of Wilson Loops interpolating between non-SUSY and SUSY configuration [41]. A similar family can be defined in ABJ(M) as well with two additional parameters: it would be interesting to study its properties in relation to the renormalization group flow and compute its defect entropy function, as in [39].

From a more general point of view, in ABJ(M) we have explicitly constructed only two types of Wilson loops: the bosonic $\frac{1}{6}$-BPS and the fermionic one, related by a cohomology transformation. We know that there should exist other classes of loops [59], but the explicit description is still missing. One way to attack the problem is trying to define a quantum mechanics, i.e. a path-integral depending on one-dimensional degrees of freedom that gives the fermionic loop once the degrees of freedom are integrated out. This description allows the study of even more types of extended operators, like monopoles, and, if present, to directly find the topological sector.

In addition, we know little about the interactions with the bulk. For a well-defined interaction, the supermatrix structure imposed by the fermionic loop has to be extended outside the defect compatibly with gauge-invariance. We can take advantage of exact results in particular kinematic configurations [107, 108] in $\mathcal{N} = 4$ SYM and try to generalize them to the ABJ(M) case.

Appendix

A

Conventions and Feynman Rules

A.1 Notations and conventions

We work in euclidean space with coordinates $x^\mu = (x^1, x^2, x^3)$ and metric $\delta_{\mu\nu}$. Gamma matrices satisfying the usual Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$, are chosen to be the Pauli matrices

$$(\gamma^\mu)_\alpha^{\ \beta} \equiv (\sigma^\mu)_\alpha^{\ \beta} \qquad \mu = 1, 2, 3 \tag{A.1}$$

Standard relations which are useful for perturbative calculations are

$$\gamma^\mu\gamma^\nu = \delta^{\mu\nu} + i\varepsilon^{\mu\nu\rho}\gamma_\rho \tag{A.2}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = \delta^{\mu\nu}\gamma^\rho - \delta^{\mu\rho}\gamma^\nu + \delta^{\nu\rho}\gamma^\mu + i\varepsilon^{\mu\nu\rho} \tag{A.3}$$

Moreover, we define $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Spinor indices are raised and lowered according to the following rules

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$$

with $\varepsilon^{12} = -\varepsilon_{12} = 1$. Consequently, we define $(\gamma^\mu)_{\alpha\beta} \equiv \varepsilon_{\beta\gamma}(\gamma^\mu)_\alpha^{\ \gamma} = (-\sigma^3, i\mathrm{I}, \sigma^1)$ and $(\gamma^\mu)^{\alpha\beta} \equiv \varepsilon^{\alpha\gamma}(\gamma^\mu)_\gamma^{\ \beta} = (\sigma^3, i\mathrm{I}, -\sigma^1)$. They satisfy $(\gamma^\mu)_{\alpha\beta} = (\gamma^\mu)_{\beta\alpha}$ and $(\gamma^\mu)^{\alpha\beta} = (\gamma^\mu)^{\beta\alpha}$.

A.1.1 ABJ(M) rescaled action

For the perturbative computations we find useful to consider a convenient rescaling of the gauge fields and the corresponding ghosts as

$$A \to \frac{1}{\sqrt{k}}A,\ \hat{A} \to \frac{1}{\sqrt{k}}\hat{A}, \qquad c \to \frac{1}{\sqrt{k}}c,\ \hat{c} \to \frac{1}{\sqrt{k}}\hat{c}$$

The covariant derivatives (1.60) change as

$$D_\mu C_I = \partial_\mu C_I + \frac{i}{\sqrt{k}} A_\mu C_I - \frac{i}{\sqrt{k}} C_I \hat{A}_\mu, \qquad D_\mu \bar{C}^I = \partial_\mu \bar{C}^I + \frac{i}{\sqrt{k}} \hat{A}_\mu \bar{C}^I - \frac{i}{\sqrt{k}} \bar{C}^I A_\mu$$
$$D_\mu \bar{\psi}^I = \partial_\mu \bar{\psi}^I + \frac{i}{\sqrt{k}} A_\mu \bar{\psi}^I - \frac{i}{\sqrt{k}} \bar{\psi}^I \hat{A}_\mu, \qquad D_\mu \psi_I = \partial_\mu \psi_I + \frac{i}{\sqrt{k}} \hat{A}_\mu \psi_I - \frac{i}{\sqrt{k}} \psi_I A_\mu \tag{A.4}$$

the Euclidean gauge-fixed action is then given by

$$S_{\text{CS}} = -\tfrac{i}{4\pi} \int d^3x\ \varepsilon^{\mu\nu\rho} \left[\text{Tr} \left(A_\mu \partial_\nu A_\rho + \tfrac{2i}{3\sqrt{k}} A_\mu A_\nu A_\rho \right) - \text{Tr} \left(\hat{A}_\mu \partial_\nu \hat{A}_\rho + \tfrac{2i}{3\sqrt{k}} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho \right) \right] \tag{A.5}$$

$$S_{\text{gf}} = \frac{1}{4\pi} \int d^3x\ \text{Tr} \left[\frac{1}{\xi} (\partial_\mu A^\mu)^2 + \partial_\mu \bar{c} D^\mu c - \frac{1}{\xi} \left(\partial_\mu \hat{A}^\mu \right)^2 - \partial_\mu \bar{\hat{c}} D^\mu \hat{c} \right] \tag{A.6}$$

$$S_{\text{mat}} = \int d^3x\ \text{Tr} \left[D_\mu C_I D^\mu \bar{C}^I - i \bar{\psi}^I \gamma^\mu D_\mu \psi_I \right] = \tag{A.7}$$

$$\begin{aligned} = \int d^3x\ \text{Tr} \Big[& \partial_\mu C_I \partial^\mu \bar{C}^I - i \bar{\psi}^I \gamma^\mu \partial_\mu \psi_I + \frac{1}{\sqrt{k}} \left(\bar{\psi}^I \gamma^\mu \hat{A}_\mu \psi_I - \bar{\psi}^I \gamma^\mu \psi_I A_\mu \right) \\ & + \frac{i}{\sqrt{k}} \left(A_\mu C_I \partial^\mu \bar{C}^I - C_I \hat{A}_\mu \partial^\mu \bar{C}^I - \partial_\mu C_I \bar{C}^I A^\mu + \partial_\mu C_I \hat{A}^\mu \bar{C}^I \right) \\ & + \frac{1}{k} \left(A_\mu C_I \bar{C}^I A^\mu - A_\mu C_I \hat{A}^\mu \bar{C}^I - C_I \hat{A}_\mu \bar{C}^I A^\mu + C_I \hat{A}_\mu \hat{A}^\mu \bar{C}^I \right) \Big] \end{aligned} \tag{A.8}$$

while the interaction part remains unchanged.

A.1.2 Equations of motion

The EOMs for our choice of the Clifford Algebra are

$$\varepsilon^{\mu\nu\rho} \left(\partial_\nu A_\rho + \frac{i}{\sqrt{k}} A_\nu A_\rho \right) = \frac{2\pi}{\sqrt{k}} \left(D^\mu C_I \bar{C}^I - C_I D^\mu \bar{C}^I \right) - \frac{2\pi i}{\sqrt{k}} \bar{\Psi}^I \gamma^\mu \Psi_I \tag{A.9}$$

$$\varepsilon^{\mu\nu\rho} \left(\partial_\nu \hat{A}_\rho + \frac{i}{\sqrt{k}} \hat{A}_\nu \hat{A}_\rho \right) = \frac{2\pi}{\sqrt{k}} \left(\bar{C}^I D^\mu C_I - D^\mu \bar{C}^I C_I \right) + \frac{2\pi i}{\sqrt{k}} \Psi_I \gamma^\mu \bar{\Psi}^I \tag{A.10}$$

$$\begin{aligned} i \left(\gamma^\mu D_\mu \right)_\alpha^{\ \beta} \Psi_{A,\beta} = -\frac{2\pi i}{k} & \left[-\bar{C}^I C_I \varepsilon_{\sigma\alpha} \Psi_A^\sigma - \varepsilon_{\alpha\sigma} \Psi_A^\sigma C_I \bar{C}^I + 2\varepsilon_{\alpha\sigma} \Psi_L^\sigma C_A \bar{C}^J \right. \\ & \left. + 2\varepsilon_{\sigma\alpha} \bar{C}^I C_A \Psi_I^\sigma + 2\varepsilon_{AIJK} \bar{C}^K \bar{\Psi}_\alpha^J \bar{C}^K \right] \end{aligned} \tag{A.11}$$

For $\alpha = 1$ we have

$$\begin{aligned} D_3 \psi_1 + \mathcal{D} \psi_2 = \mathcal{D} \psi^1 - D_3 \psi^2 = -\frac{2\pi}{k} & \left[\psi^2 C_I \bar{C}^I - \bar{C}^I C_I \psi^2 + 2\bar{C}^I Z \Psi_I^2 \right. \\ & \left. -2\Psi_I^2 Z \bar{C}^I + 2\varepsilon_{abc} \bar{Y}^a \bar{\chi}_1^b \bar{Y}^c \right] \end{aligned} \tag{A.12}$$

such that

$$\mathcal{D} \psi^1 = D_3 \psi^2 - \frac{2\pi}{k} \left[\psi^2 C_I \bar{C}^I - \bar{C}^I C_I \psi^2 + 2\bar{C}^I Z \Psi_I^2 - 2\Psi_I^2 Z \bar{C}^I + 2\varepsilon_{abc} \bar{Y}^a \bar{\chi}_1^b \bar{Y}^c \right] \tag{A.13}$$

In the $SU(3)$ formalism

$$\begin{aligned} -2i\sqrt{\frac{\pi}{k}}\mathcal{D}\psi^1 =& -2i\sqrt{\frac{\pi}{k}}\left(D_3\psi^2+\frac{2\pi}{k}\left(\psi^2\ell_B-\hat{\ell}_B\psi^2\right)\right) \\ &+8i\left(\frac{\pi}{k}\right)^{\frac{3}{2}}\left(\bar{Y}^a Z\chi^2_A-\chi^2_a Z\bar{Y}^a+\varepsilon_{abc}\bar{Y}^a\bar{\chi}^b_1\bar{Y}^c\right) \end{aligned} \tag{A.14}$$

The derivative D_3 in the first line can be lifted to the supermatrix form, as defined in (4.75). In particular, in the line case we have $k^{-\frac{1}{2}}\mathcal{A}_\mu=\mathcal{L}_A$ such that

$$D_3\psi^2=\mathcal{D}_3\begin{pmatrix}0&0\\\psi^2&0\end{pmatrix}=\partial_3\begin{pmatrix}0&0\\\psi^2&0\end{pmatrix}+i\left[\mathcal{L}_A,\begin{pmatrix}0&0\\\psi^2&0\end{pmatrix}\right] \tag{A.15}$$

A.2 Feynman Rules

From the action we have the propagators for the fields

- scalars

$$\langle (C_I)_i{}^{\hat{j}}(x)\ (\bar{C}^J)_{\hat{k}}{}^l(y)\rangle^{(0)}=\delta^J_I\delta^l_i\delta^{\hat{j}}_{\hat{k}}\ \frac{\Gamma(\frac{1}{2}-\varepsilon)}{4\pi^{\frac{3}{2}-\varepsilon}}\frac{1}{|x-y|^{1-2\varepsilon}} \tag{A.16}$$

- fermions

$$\langle (\psi_{\alpha I})_{\hat{i}}{}^j(x)\ (\bar{\psi}^{J\beta})^{\hat{l}}{}_k(y)\rangle^{(0)}=\delta^J_I\delta^{\hat{l}}_{\hat{i}}\delta^j_k\ i\,\frac{\Gamma(\frac{3}{2}-\varepsilon)}{2\pi^{\frac{3}{2}-\varepsilon}}\ (\gamma^\mu)_\alpha{}^\beta\ \frac{(x-y)_\mu}{|x-y|^{3-2\varepsilon}} \tag{A.17}$$

  and one-loop correction

$$\langle (\psi_{\alpha I})_{\hat{i}}{}^j(x)\ (\bar{\psi}^{J\beta})^{\hat{l}}{}_k(y)\rangle^{(1)}=-\delta^J_I\delta^{\hat{l}}_{\hat{i}}\delta^j_k\ \delta_\alpha{}^\beta\ \left(\frac{N_1-N_2}{k}\right)\ i\,\frac{\Gamma^2(\frac{1}{2}-\varepsilon)}{8\pi^{2-2\varepsilon}}\ \frac{1}{|x-y|^{2-4\varepsilon}} \tag{A.18}$$

- gauge fields

$$\langle (A_\mu)_i{}^j(x)\ (A_\nu)_k{}^l(y)\rangle^{(0)}=\delta^l_i\delta^j_k\ i\,\frac{\Gamma(\frac{3}{2}-\varepsilon)}{\pi^{\frac{1}{2}-\varepsilon}}\ \varepsilon_{\mu\nu\rho}\ \frac{(x-y)^\rho}{|x-y|^{3-2\varepsilon}} \tag{A.19}$$

$$\langle (\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}(x)\ (\hat{A}_\nu)_{\hat{k}}{}^{\hat{l}}(y)\rangle^{(0)}=-\,\delta^{\hat{l}}_{\hat{i}}\delta^{\hat{j}}_{\hat{k}}\ i\,\frac{\Gamma(\frac{3}{2}-\varepsilon)}{\pi^{\frac{1}{2}-\varepsilon}}\ \varepsilon_{\mu\nu\rho}\ \frac{(x-y)^\rho}{|x-y|^{3-2\varepsilon}} \tag{A.20}$$

  and one-loop corrections

$$\langle (A_\mu)_i{}^j(x)\ (A_\nu)_k{}^l(y)\rangle^{(1)}=\delta^l_i\delta^j_k\ \frac{N_2}{k}\ \frac{\Gamma^2(\frac{1}{2}-\varepsilon)}{\pi^{1-2\varepsilon}}\left(\frac{\delta_{\mu\nu}}{|x-y|^{2-4\varepsilon}}-\partial_\mu\partial_\nu\frac{|x-y|^{4\varepsilon}}{4\varepsilon(1+2\varepsilon)}\right) \tag{A.21}$$

$$\langle (\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}(x)\ (\hat{A}_\nu)_{\hat{k}}{}^{\hat{l}}(y)\rangle^{(1)}=\delta^l_i\delta^j_k\ \frac{N_1}{k}\ \frac{\Gamma^2(\frac{1}{2}-\varepsilon)}{\pi^{1-2\varepsilon}}\left(\frac{\delta_{\mu\nu}}{|x-y|^{2-4\varepsilon}}-\partial_\mu\partial_\nu\frac{|x-y|^{4\varepsilon}}{4\varepsilon(1+2\varepsilon)}\right) \tag{A.22}$$

and it is straightforward to restrict them on a line.

A.2.1 Propagators From Line to Circle

When computing quantum correction on a circle, e.g. in the Wilson Loop with circular contour, it is convenient to express propagators as functions of the angular variable.

The circle is a curve parametrized as $x^\mu(\tau) = (0, \cos\tau, \sin\tau)$. Then it's a trigonometric exercise proving that

$$|x_1 - x_2| = \left|2\sin\frac{\tau_1 - \tau_2}{2}\right| \tag{A.23}$$

So the scalar propagator on the circle is straightforward

$$\langle (C_I)_i{}^{\hat{j}}(\tau_1)\ (\bar{C}^J)_{\hat{k}}{}^{l}(\tau_2)\rangle^{(0)} = \delta_I^J \delta_i^l \delta_{\hat{k}}^{\hat{j}}\ \frac{\Gamma(\frac{1}{2}-\varepsilon)}{4\pi^{\frac{3}{2}-\varepsilon}} \frac{1}{\left|2\sin\frac{\tau_1-\tau_2}{2}\right|^{1-2\varepsilon}} \tag{A.24}$$

The fermions coming from the line are always coupled with the spinors $\eta, \bar{\eta}$. On the circle, the conformal transformation give them the following form

$$\eta = \sqrt{2}\delta_I^1 \left(\cos\frac{\tau}{2}, i\sin\frac{\tau}{2}\right)^\alpha \qquad \bar{\eta} = i\sqrt{2}\delta_1^I \begin{pmatrix} \cos\frac{\tau}{2} \\ -i\sin\frac{\tau}{2}\end{pmatrix}_\alpha \tag{A.25}$$

so the fermions coupled to the line are

$$\Psi\bar{\eta} = i\sqrt{2}\left(\cos\frac{\tau}{2}\psi^+ - i\sin\frac{\tau}{2}\psi^-\right) \qquad \eta\bar{\Psi} = \sqrt{2}\left(\cos\frac{\tau}{2}\bar{\psi}_+ + i\sin\frac{\tau}{2}\bar{\psi}_-\right) \tag{A.26}$$

Considering the correlator $\langle(\Psi\bar{\eta})(\eta\bar{\Psi})\rangle$ and expanding as (A.26), after some manipulations we get

$$\langle (\Psi\bar{\eta})_{\hat{i}}{}^{j}(\eta\bar{\Psi})_k{}^{\hat{l}}\rangle = -\delta_{\hat{i}}{}^{\hat{l}}\delta_k{}^{j}\frac{\Gamma(\frac{3}{2}-\varepsilon)}{\pi^{\frac{3}{2}-\varepsilon}}\frac{\left(2\sin\frac{\tau_1-\tau_2}{2}\right)}{\left|2\sin\frac{\tau_1-\tau_2}{2}\right|^{3-2\varepsilon}} \tag{A.27}$$

B

Supersymmetry and superalgebras

B.1 Supersymmetry transformations

B.1.1 In $SU(4)$ notations

The ABJ(M) action in (1.56) is invariant under the following superconformal transformations

$$
\begin{aligned}
\delta C_K &= -\bar\zeta^{IJ,\alpha}\,\varepsilon_{IJKL}\,\bar\psi^L_\alpha \\
\delta \bar C^K &= 2\bar\zeta^{KL,\alpha}\,\psi_{L,\alpha} \\
\delta \bar\psi^{K,\beta} &= 2i\bar\zeta^{KL,\alpha}(\gamma^\mu)_\alpha{}^\beta D_\mu C_L + \frac{4\pi i}{k}\bar\zeta^{KL,\beta}(C_L\bar C^M C_M - C_M \bar C^M C_L) \\
&\quad + \frac{8\pi i}{k}\bar\zeta^{IJ,\beta} C_I \bar C^K C_J + 2i\bar\varepsilon^{KL,\beta} C_L \\
\delta \psi_K^\beta &= -i\bar\zeta^{IJ,\alpha}\varepsilon_{IJKL}(\gamma^\mu)_\alpha{}^\beta D_\mu \bar C^L + \frac{2\pi i}{k}\bar\zeta^{IJ,\beta}\varepsilon_{IJKL}(\bar C^L C_M \bar C^M - \bar C^M C_M \bar C^L) \\
&\quad + \frac{4\pi i}{k}\bar\zeta^{IJ,\beta}\varepsilon_{IJML}\bar C^M C_K \bar C^L - i\bar\varepsilon^{IJ,\beta}\varepsilon_{IJKL}\bar C^L \\
\delta A_\mu &= \frac{4\pi i}{k}\bar\zeta^{IJ,\alpha}(\gamma_\mu)_\alpha{}^\beta\left(C_I\psi_{J\beta} - \frac{1}{2}\varepsilon_{IJKL}\bar\psi^K_\beta \bar C^L\right) \\
\delta \hat A_\mu &= \frac{4\pi i}{k}\bar\zeta^{IJ,\alpha}(\gamma_\mu)_\alpha{}^\beta\left(\psi_{J\beta}C_I - \frac{1}{2}\varepsilon_{IJKL}\bar C^L \bar\psi^K_\beta\right)
\end{aligned}
\tag{B.1}
$$

where the parameters of the transformations are expressed in terms of supersymmetry and superconformal parameters as

$$
\bar\zeta^{IJ}_\alpha = \bar\Theta^{IJ}_\alpha - x^\mu(\gamma_\mu)_\alpha^{\ \beta}\bar\varepsilon^{IJ}_\beta \tag{B.2}
$$

We recall that they satisfy $\bar\zeta^{IJ} = -\bar\zeta^{JI}$, and are subject to the reality conditions $\bar\zeta^{IJ} = (\zeta_{IJ})^*$ with $\zeta_{IJ} = \frac{1}{2}\varepsilon_{IJKL}\bar\zeta^{KL}$.

If we set $\bar{\varepsilon}^{IJ} = 0$ in (B.1) we obtain $\mathcal{N} = 6$ supersymmetry transformations. Expressing them as

$$\delta\Phi = [\bar{\Theta}^{IJ}\bar{Q}_{IJ}, \Phi] = [\Theta_{IJ}Q^{IJ}, \Phi] \tag{B.3}$$

for a generic field Φ, it is easy to realize that the Q^{IJ} supercharges (or equivalently $\bar{Q}_{IJ}$) satisfy the $\mathfrak{osp}(6|4)$ algebra (1.50) under the identification $P_\mu = i\partial_\mu$.

B.1.2 In $SU(3)$ notations

The generic supersymmetry transformation defined in (B.3) can be specialized to the $\mathfrak{su}(1,1|3)$ supercharges $(Q^a, \bar{Q}_a)$ defined in (B.11,B.13). For a generic field $\tilde{\Phi}$ in a given representation of the $\mathfrak{su}(3)$ R-symmetry algebra it reduces to

$$\delta\tilde{\Phi} = \left[\theta_a Q^a + \bar{\theta}^a \bar{Q}_a, \tilde{\Phi}\right] \tag{B.4}$$

under the parameter identification

$$\begin{aligned} &\theta_1 = 2\,\Theta^1_{1(a+1)} \qquad a = 1,2,3 \\ &\bar{\theta}^1 = -2i\,\Theta^2_{34} \qquad \bar{\theta}^2 = -2i\,\Theta^2_{42} \qquad \bar{\theta}^3 = -2i\,\Theta^2_{23} \end{aligned} \tag{B.5}$$

From the variations in (B.1) we can easily read the supersymmetry transformations of the ABJ(M) fundamental fields reorganized in $\mathfrak{su}(3)$ R-symmetry representations (see eqs. (3.12) and (3.13)). Comparing these transformations with the general variation defined in (B.4) we obtain the action of the supercharges on the fields, which takes the following form

- Scalar fields

$$\begin{array}{llll} Q^a Z = -\bar{\chi}^a_1 & \bar{Q}_a Z = 0 & Q^a \bar{Z} = 0 & \bar{Q}_a \bar{Z} = i\chi^1_a \\ Q^a Y_b = \delta^a_b \bar{\psi}_1 & \bar{Q}_a Y_b = -i\varepsilon_{abc}\bar{\chi}^c_2 & Q^a \bar{Y}^b = -\varepsilon^{abc}\chi^2_c & \bar{Q}_a \bar{Y}^b = -i\delta^b_a \psi^1 \end{array} \tag{B.6}$$

- Fermions

$$\bar{Q}_a \psi^1 = 0 \qquad Q^a \psi^1 = -iD_3\bar{Y}^a - \frac{2\pi i}{k}\left(\bar{Y}^a l_B - \hat{l}_B \bar{Y}^a\right) \tag{B.7a}$$

$$Q^a \psi^2 = -iD\bar{Y}^a \qquad \bar{Q}_a \psi^2 = -\frac{4\pi}{k}\varepsilon_{abc}\bar{Y}^b Z \bar{Y}^c \tag{B.7b}$$

$$\bar{Q}_a \chi^1_b = \varepsilon_{abc}\,\bar{D}\bar{Y}^c \qquad Q^a \chi^1_b = i\delta^a_b D_3\bar{Z} + \frac{4\pi i}{k}\left(\bar{Z}\Lambda^a_b - \hat{\Lambda}^a_b \bar{Z}\right) \tag{B.7c}$$

$$Q^a \chi^2_b = i\delta^a_b\, D\bar{Z} \qquad \bar{Q}_a \chi^2_b = -\varepsilon_{abc}D_3\bar{Y}^c - \frac{2\pi}{k}\varepsilon_{acd}\left(\bar{Y}^c\Theta^d_b - \hat{\Theta}^d_b \bar{Y}^c\right) \tag{B.7d}$$

$$Q^a \bar{\psi}_1 = 0 \qquad \bar{Q}_a \bar{\psi}_1 = -D_3 Y_a - \frac{2\pi}{k}\left(Y_a \hat{l}_B - l_B Y_a\right) \tag{B.7e}$$

$$\bar{Q}_a \bar{\psi}_2 = -\bar{D}Y_a \qquad Q^a \bar{\psi}_2 = \frac{4\pi i}{k}\varepsilon^{abc} Y_b \bar{Z} Y_c \tag{B.7f}$$

$$Q^a\bar\chi_1^b = -i\varepsilon^{abc}\, DY_c \quad \bar Q_a\bar\chi_1^b = \delta_b^a D_3 Z + \frac{4\pi}{k}\left(Z\hat\Lambda_b^a - \Lambda_b^a Z\right) \tag{B.7g}$$

$$\bar Q_a\bar\chi_2^b = \delta_a^b\,\bar D Z \qquad Q^a\bar\chi_2^b = i\varepsilon^{abc} D_3 Y_c + \frac{2\pi i}{k}\varepsilon^{acd}\left(Y_c\hat\Theta_d^b - \Theta_d^b Y_c\right) \tag{B.7h}$$

- Gauge fields

$$\begin{aligned}
&Q^aA_3 = -\frac{2\pi i}{k}\left(\bar\psi_1\bar Y^a - \bar\chi_1^a\bar Z + \varepsilon^{abc}Y_b\chi_c^2\right) && \bar Q_aA_3 = \frac{2\pi}{k}\left(Z\chi_a^1 - Y_a\psi^1 - \varepsilon_{abc}\bar\chi_2^b\bar Y^c\right)\\
&Q^aA = 0 && \bar Q_aA = -\frac{4\pi}{k}\left(Y_a\psi^2 - Z\chi_a^2 - \varepsilon_{abc}\bar\chi_1^b\bar Y^c\right)\\
&Q^a\bar A = -\frac{4\pi i}{k}\left(\bar\psi_2\bar Y^a - \bar\chi_2^a\bar Z - \varepsilon^{abc}Y_b\chi_c^1\right) && \bar Q_a\bar A = 0\\
&Q^a\hat A_3 = -\frac{2\pi i}{k}\left(\bar Y^a\bar\psi_1 - \bar Z\bar\chi_1^a + \varepsilon^{abc}\chi_c^2 Y_b\right) && \bar Q_a\hat A_3 = \frac{2\pi}{k}\left(\chi_a^1 Z - \psi^1 Y_a - \varepsilon_{abc}\bar Y^c\bar\chi_2^b\right)\\
&Q^a\hat A = 0 && \bar Q_a\hat A = \frac{4\pi}{k}\left(\psi^2 Y_a - \chi_a^2 Z - \varepsilon_{abc}\bar Y^c\bar\chi_1^b\right)\\
&Q^a\hat{\bar A} = -\frac{4\pi i}{k}\left(\bar Y^a\bar\psi_2 - \bar Z\bar\chi_2^a - \varepsilon^{abc}\chi_c^1 Y_b\right) && \bar Q_a\hat{\bar A} = 0
\end{aligned} \tag{B.8a}$$

where we have defined the bilinear scalar fields

$$\begin{aligned}
\begin{pmatrix}\Lambda_a^b & 0\\ 0 & \hat\Lambda_a^b\end{pmatrix} &= \begin{pmatrix} Y_a\bar Y^b + \frac12\delta_a^b l_B & 0\\ 0 & \bar Y^b Y_a + \frac12\delta_a^b\hat l_B\end{pmatrix}\\
\begin{pmatrix}\Theta_a^b & 0\\ 0 & \hat\Theta_a^b\end{pmatrix} &= \begin{pmatrix} Y_a\bar Y^b - \delta_a^b(Z\bar Z + Y_c\bar Y^c) & 0\\ 0 & \bar Y^b Y_a - \delta_a^b(\bar Z Z + \bar Y^c Y_c)\end{pmatrix}\\
\begin{pmatrix}\ell_B & 0\\ 0 & \hat\ell_B\end{pmatrix} &= \begin{pmatrix} Z\bar Z - Y_c\bar Y^c & 0\\ 0 & \bar Z Z - \bar Y^c Y_c\end{pmatrix}
\end{aligned} \tag{B.9}$$

B.2 $\mathfrak{su}(1,1|3)$ algebra

In Chapter 3, we presented the bosonic part of the $\mathfrak{su}(1,1|3)$ algebra. We now give some details on the fermionic sector. We have seen we have 12 superconformal charges

$$Q_1^{12}, Q_1^{13}, Q_1^{14}, Q_2^{23}, Q_2^{24}, Q_2^{34} \qquad \text{and} \qquad S_1^{12}, S_1^{13}, S_1^{14}, S_2^{23}, S_2^{24}, S_2^{34} \tag{B.10}$$

and they can be reorganized in a SU(3) form

$$\begin{aligned}
&Q^{k-1} \equiv Q_1^{1k} && \bar Q_{k-1} \equiv \frac{i}{2}\,\varepsilon_{klm}\, Q_2^{lm}\\
&S^{k-1} \equiv i\, S_1^{1k} && \bar S_{k-1} \equiv \frac12\,\varepsilon_{klm}\, S_2^{lm} \qquad k,l,m = 2,3,4
\end{aligned} \tag{B.11}$$

and make the shift $Q^{k-1}\to Q^a$, $\bar Q_{k-1}\to\bar Q_a$ with $a = 1,2,3$, and similarly for the superconformal charges.

This set of generators inherits the following hermicity conditions

$$\begin{aligned} (Q^a)^\dagger &= \bar{S}_a & (\bar{Q}_a)^\dagger &= S^a \\ (S^a)^\dagger &= \bar{Q}_a & (\bar{S}_a)^\dagger &= Q^a & a &= 1,2,3 \end{aligned} \tag{B.12}$$

and the following anti-commutation relations

$$\begin{aligned} \{Q^a, \bar{Q}_b\} &= \delta^a_b\, P & \{S^a, \bar{S}_b\} &= \delta^a_b\, K \\ \{Q^a, \bar{S}_b\} &= \delta^a_b\left(D + \frac{1}{3}M\right) - R_b{}^a & \{\bar{Q}_a, S^b\} &= \delta^b_a\left(D - \frac{1}{3}M\right) + R_a{}^b \end{aligned} \tag{B.13}$$

together with the mixed commutation rules

$$\begin{aligned} [D, Q^a] &= \frac{1}{2}Q^a & [K, Q^a] &= S^a & [R_a{}^b, Q^c] &= \delta^c_a Q^b - \frac{1}{3}\delta^b_a Q^c & [M, Q^a] &= \frac{1}{2}Q^a \\ [D, \bar{Q}_a] &= \frac{1}{2}\bar{Q}_a & [K, \bar{Q}_a] &= \bar{S}_a & [R_a{}^b, \bar{Q}_c] &= -\delta^b_c \bar{Q}_a + \frac{1}{3}\delta^b_a \bar{Q}_c & [M, \bar{Q}_a] &= -\frac{1}{2}\bar{Q}_a \\ [D, S^a] &= -\frac{1}{2}S^a & [P, S^a] &= -Q^a & [R_a{}^b, S^c] &= \delta^c_a S^b - \frac{1}{3}\delta^b_a S^c & [M, S^a] &= \frac{1}{2}S^a \\ [D, \bar{S}_a] &= -\frac{1}{2}\bar{S}_a & [P, \bar{S}_a] &= -\bar{Q}_a & [R_a{}^b, \bar{S}_c] &= -\delta^b_c \bar{S}_b + \frac{1}{3}\delta^b_a \bar{S}_c & [M, \bar{S}_a] &= -\frac{1}{2}\bar{S}_a \end{aligned} \tag{B.14}$$

From eq. (1.55) and definitions (3.9) it follows that the action of the $SU(3)$ R-symmetry generators on fields in the (anti-)fundamental representation is

$$[R_a{}^b, \Phi_c] = \frac{1}{3}\delta^b_a \Phi_c - \delta^b_c \Phi_a \qquad [R_a{}^b, \bar{\Phi}^c] = \delta^c_a \bar{\Phi}^b - \frac{1}{3}\delta^b_a \bar{\Phi}^c \tag{B.15}$$

B.2.1 Irreducible representations

In this appendix, we shall briefly review the classification of the multiplet of $\mathfrak{su}(1,1|3)$ presented in [72]. We shall classify the states in terms of the four Dynkin labels $[\Delta, m, j_1, j_2]$ associated to the bosonic subalgebra $\mathfrak{su}(1,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)$. Here Δ stands for the conformal weight, m for the $\mathfrak{u}(1)$ charge and (j_1, j_2) are the eigenvalues corresponding to the two $\mathfrak{su}(3)$ Cartan generators J_1 and J_2. We choose

$$\begin{aligned} J_1 &\equiv \frac{R_2{}^2 - R_1{}^1}{2} = -\frac{2R_1{}^1 + R_3{}^3}{2} \\ J_2 &\equiv \frac{R_3{}^3 - R_2{}^2}{2} = \frac{R_1{}^1 + 2R_3{}^3}{2} \end{aligned} \tag{B.16}$$

where we have exploited the traceless property $R_a{}^a = 0$ to remove the dependence on $R_2{}^2$. The commutations rules (3.10) implies that we can associate an $\mathfrak{sl}(2)$ subalgebra with each Cartan generator. In fact, the two sets of operators

$$\{R_2{}^1, R_1{}^2, J_1\} \equiv \{E_1^-, E_1^+, J_1\} \quad , \quad \{R_3{}^2, R_2{}^3, J_2\} \equiv \{E_2^-, E_2^+, J_2\} \tag{B.17}$$

satisfy the following algebraic relations

$$[E_i^+, E_i^-] = 2J_i \qquad [J_i, E_i^\pm] = \pm E_i^\pm \qquad i = 1,2 \tag{B.18}$$

and define the raising and lowering operators used to construct the representation of $\mathfrak{su}(3)$. In the main text, we have chosen a different $\mathfrak{sl}(2)$ to define the twisted algebra. We have preferred to use the one generated by $\{R_3{}^1, R_1{}^3, -J_1 - J_2\}$, which treats the two Dynkin labels (j_1, j_2) symmetrically. Moreover, the supercharges with this choice of basis possess well-defined Dynkin labels, whose values are displayed in Table B.1. When localized on the line, the ABJ(M) fundamental fields

Generators	$[\Delta, m, j_1, j_2]$
Q^1 $\bar{Q}_1$	$\left[\frac{1}{2}, \frac{1}{2}, -1, 0\right]$ $\left[\frac{1}{2}, -\frac{1}{2}, 1, 0\right]$
Q^2 $\bar{Q}_2$	$\left[\frac{1}{2}, \frac{1}{2}, 1, -1\right]$ $\left[\frac{1}{2}, -\frac{1}{2}, -1, 1\right]$
Q^3 $\bar{Q}_3$	$\left[\frac{1}{2}, \frac{1}{2}, 0, 1\right]$ $\left[\frac{1}{2}, -\frac{1}{2}, 0, -1\right]$
S^1 $\bar{S}_1$	$\left[-\frac{1}{2}, \frac{1}{2}, -1, 0\right]$ $\left[-\frac{1}{2}, -\frac{1}{2}, 1, 0\right]$
S^2 $\bar{S}_2$	$\left[-\frac{1}{2}, \frac{1}{2}, 1, -1\right]$ $\left[-\frac{1}{2}, -\frac{1}{2}, -1, 1\right]$
S^3 $\bar{S}_3$	$\left[-\frac{1}{2}, \frac{1}{2}, 0, 1\right]$ $\left[-\frac{1}{2}, -\frac{1}{2}, 0, -1\right]$

Table B.1: Table of Dynkin labels of fermionic generators. For a generic element v_μ transforming in a weight-μ representation, the Dynkin label corresponding to a generator H_i of the Cartan subalgebra is defined as $j_i(v_\mu) \equiv 2[H_i, v_\mu]$.

also have definite quantum numbers with respect to $\mathfrak{su}(1,1) \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)$. Their values are listed in Table B.2 for the scalar fields and in Table B.3 for the fermionic ones.

Finally we do not consider directly the gauge fields, but their covariant derivatives. Their Dynkin labels are given by

$$D\ [1,3,0,0] \qquad \bar{D}\ [1,-3,0,0] \qquad D_3\ [1,0,0,0] \tag{B.19}$$

Therefore their action on an operator that is an eigenstate $|\Delta, m, j_1, j_2\rangle$ of the Cartan generators simply shifts the the first two quantum numbers. Next we summarize the relevant superconformal multiplets constructed in [72].

The $\mathcal{A}$ Multiplets

We start with the so-called long multiplets, denoted by $\mathcal{A}^{\Delta}_{m;j_1,j_2}$. Their highest weight of the representations, namely the super-conformal primary (SCP), is iden-

Scalar fields	$[\Delta, m, j_1, j_2]$	
Z , $\bar{Z}$	$\left[\frac{1}{2}, \frac{3}{2}, 0, 0\right]$	$\left[\frac{1}{2}, -\frac{3}{2}, 0, 0\right]$
Y_1 , $\bar{Y}^1$	$\left[\frac{1}{2}, -\frac{1}{2}, 1, 0\right]$	$\left[\frac{1}{2}, \frac{1}{2}, -1, 0\right]$
Y_2 , $\bar{Y}^2$	$\left[\frac{1}{2}, -\frac{1}{2}, -1, 1\right]$	$\left[\frac{1}{2}, \frac{1}{2}, 1, -1\right]$
Y_3 , $\bar{Y}^3$	$\left[\frac{1}{2}, -\frac{1}{2}, 0, -1\right]$	$\left[\frac{1}{2}, \frac{1}{2}, 0, 1\right]$

Table B.2: Quantum number assignments to scalar matter fields of the ABJ(M) theory defined in eq. (3.12).

Fermionic fields	$[\Delta, m, j_1, j_2]$	
$(\psi)_1$, $(\psi)_2$	$[1, 3, 0, 0]$	$[1, 0, 0, 0]$
$(\bar{\psi})_1$, $(\bar{\psi})_2$	$[1, 0, 0, 0]$	$[1, -3, 0, 0]$
$(\chi_1)_1$, $(\chi_1)_2$	$[1, 1, 1, 0]$	$[1, -2, 1, 0]$
$(\bar{\chi}^1)_1$, $(\bar{\chi}^1)_2$	$[1, 2, -1, 0]$	$[1, -1, -1, 0]$
$(\chi_2)_1$, $(\chi_2)_2$	$[1, 1, -1, 1]$	$[1, -2, -1, 1]$
$(\bar{\chi}^2)_1$, $(\bar{\chi}^2)_2$	$[1, 2, 1, -1]$	$[1, -1, 1, -1]$
$(\chi_3)_1$, $(\chi_3)_2$	$[1, 1, 0, -1]$	$[1, -2, 0, -1]$
$(\bar{\chi}^3)_1$, $(\bar{\chi}^3)_2$	$[1, 2, 0, 1]$	$[1, -1, 0, 1]$

Table B.3: Quantum number assignments to fermionic matter fields of the ABJ(M) theory defined in eq. (3.12).

tified by requiring that

$$S^a \left|\Delta, m, j_1, j_2\right\rangle^{\mathrm{hw}} = 0 \quad \bar{S}_a \left|\Delta, m, j_1, j_2\right\rangle^{\mathrm{hw}} = 0 \quad E_a^+ \left|\Delta, m, j_1, j_2\right\rangle^{\mathrm{hw}} = 0 \tag{B.20}$$

Then the entire multiplet is bult by acting with the supercharges Q^a and $\bar{Q}_a$. For unitary representations, the Dynkin label of the highest weight are constrained by the following inequalities

$$\Delta \geq \begin{cases} \frac{1}{3}(2j_2 + j_1 - m), & m < \frac{j_2 - j_1}{2} \\ \frac{1}{3}(j_2 + 2j_1 + m), & m \geq \frac{j_2 - j_1}{2} \end{cases} \tag{B.21}$$

At the threshold of the unitary region, these multiplets split into shorter ones because of the recombination phenomenon. For $m < \frac{j_1 - j_2}{2}$ the unitarity bound is for $\Delta = \frac{1}{3}(2j_1 + j_2 - m)$ and one can verify that

$$\mathcal{A}^{-\frac{1}{3}m+\frac{2}{3}j_1+\frac{1}{3}j_2}_{m,j_1,j_2} = \mathcal{B}^{\frac{1}{6},0}_{m,j_1,j_2} \oplus \mathcal{B}^{\frac{1}{6},0}_{m+\frac{1}{2},j_1+1,j_2} \tag{B.22}$$

Equivalently, for $m > \frac{j_1 - j_2}{2}$ one has

$$\mathcal{A}^{\frac{1}{3}m+\frac{1}{3}j_1+\frac{2}{3}j_2}_{m,j_1,j_2} = \mathcal{B}^{0,\frac{1}{6}}_{m,j_1,j_2} \oplus \mathcal{B}^{0,\frac{1}{6}}_{m-\frac{1}{2},j_1,j_2+1} \tag{B.23}$$

For the particular case $m = \frac{j_1 - j_2}{2}$ we have

$$\mathcal{A}^{\frac{j_2+j_1}{2}}_{\frac{j_2-j_1}{2};j_1,j_2} = \mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1,j_2} \oplus \mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2}+\frac{1}{2};j_1,j_2+1} \oplus \mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2}-\frac{1}{2};j_1+1,j_2+1} \oplus \mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{\frac{j_2-j_1}{2};j_1+1,j_2+1}. \tag{B.24}$$

Above the symbols $\mathcal{B}^{\frac{1}{N},\frac{1}{M}}_{m;j_1,j_2}$ stand for a type of short multiplets (see below). The two superscripts denote respectively the fraction of Q and $\bar{Q}$ charges with respect to the total number of charges $(Q + \bar{Q})$, which annihilates the super-conformal primary.

The $\mathcal{B}$ Multiplets

Let us now have a closer look to short multiplets. They are obtained by imposing that the highest weight is annihilated by some of the Q and $\bar{Q}$ charges (*shortening condition*). First we consider the case

$$Q^a \, |\Delta, m, j_1, j_2\rangle^{\mathrm{hw}} = 0 \tag{B.25}$$

from which we get three possible short supermultiplets

$$a = 3 \qquad \Delta = \frac{1}{3}(j_1 + 2j_2 - m) \qquad \mathcal{B}^{\frac{1}{6},0}_{m;j_1,j_2} \tag{B.26}$$

$$a = 3,2 \qquad \Delta = \frac{1}{3}(j_1 - m), \quad j_2 = 0 \qquad \mathcal{B}^{\frac{1}{3},0}_{m;j_1,0} \tag{B.27}$$

$$a = 3,2,1 \qquad \Delta = -\frac{1}{3}m, \quad j_1 = j_2 = 0 \qquad \mathcal{B}^{\frac{1}{2},0}_{m;0,0} \tag{B.28}$$

according to the number of charges obeying the condition (B.26). Obviously we can also consider the conjugate shortening condition

$$\bar{Q}_a \, |\Delta, m, j_1, j_2\rangle^{\mathrm{hw}} = 0 \tag{B.29}$$

which yields short multiplets conjugate to the ones considered above

$$a = 1 \qquad \Delta = \frac{1}{3}(j_2 + 2j_1 + m) \qquad \mathcal{B}^{0,\frac{1}{6}}_{m;j_1,j_2} \tag{B.30}$$

$$a = 1, 2 \qquad \Delta = \frac{1}{3}(j_2 + m), \quad j_1 = 0 \qquad \mathcal{B}^{0,\frac{1}{3}}_{m;0,j_2} \tag{B.31}$$

$$a = 1, 2, 3 \qquad \Delta = \frac{1}{3}m, \quad j_1 = j_2 = 0 \qquad \mathcal{B}^{0,\frac{1}{2}}_{m;0,0} \tag{B.32}$$

Finally we may have mixed multiplets where the highest weight is annihilated both by Q^a and $\bar{Q}_a$. Those include

$$\mathcal{B}^{\frac{1}{6},\frac{1}{6}}_{m,j_1,j_2} \qquad \Delta = \frac{j_1 + j_2}{2} \qquad m = \frac{j_1 - j_2}{2} \tag{B.33}$$

$$\mathcal{B}^{\frac{1}{3},\frac{1}{6}}_{m,0,j_2} \qquad \Delta = \frac{j_2}{2} \qquad m = -\frac{j_2}{2} \qquad j_1 = 0 \tag{B.34}$$

$$\mathcal{B}^{\frac{1}{6},\frac{1}{3}}_{m,j_1,0} \qquad \Delta = \frac{j_1}{2} \qquad m = \frac{j_1}{2} \qquad j_2 = 0 \tag{B.35}$$

C

Integrals appearing in loop computations

In this appendix, we collect some results of the integrals appearing in loop computations.

C.1 Computing the bosonic integrals

In the computations of operators insertions correlation functions on the circle we stepped in computing an integral of this type:

$$\int_0^\sigma \frac{d\tau}{\left(4\sin\frac{\tau}{2}\sin\frac{\sigma-\tau}{2}\right)^{1-2\varepsilon}}. \tag{C.1}$$

Using the trigonometric sum-to-product rule

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \tag{C.2}$$

we can rewrite the previous integral as

$$\begin{aligned}\int_0^\sigma \frac{d\tau}{\left(2\left(\cos\left(\tau-\frac{\sigma}{2}\right)-\cos\frac{\sigma}{2}\right)\right)^{1-2\varepsilon}} =& \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} \frac{d\tau}{\left(2\left(\cos\tau-\cos\frac{\sigma}{2}\right)\right)^{1-2\varepsilon}} \\ =& 2^{2\varepsilon}\int_0^{\frac{\sigma}{2}} \frac{d\tau}{\left(\cos\tau-\cos\frac{\sigma}{2}\right)^{1-2\varepsilon}}\end{aligned} \tag{C.3}$$

Rewrite the cos using

$$\cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \tag{C.4}$$

so

$$(C.3) = 2^{2\varepsilon}\int_0^{\frac{\sigma}{2}} d\tau \left(\frac{1-\tan^2\frac{\tau}{2}}{1+\tan^2\frac{\tau}{2}} - \frac{1-\tan^2\frac{\sigma}{4}}{1+\tan^2\frac{\sigma}{4}}\right)^{2\varepsilon-1} \tag{C.5}$$

and after the change of variable $\tau = 2\arctan z$ it changes as

$$2^{2\varepsilon}\int_0^{\tan\frac{\sigma}{4}} 2\frac{dz}{1+z^2}\left(\frac{1-z^2}{1+z^2}-\frac{1-\tan^2\frac{\sigma}{4}}{1+\tan^2\frac{\sigma}{4}}\right)^{2\varepsilon-1} \tag{C.6}$$

This becomes

$$\frac{2^{4\varepsilon}}{\left(1+\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}}\int_0^{\tan\frac{\sigma}{4}} dz\,\left(1+z^2\right)^{-2\varepsilon}\left(\tan^2\frac{\sigma}{4}-z^2\right)^{2\varepsilon-1} \tag{C.7}$$

Changing variable again using $z = w\tan\frac{\sigma}{4}$ we get

$$\frac{2^{4\varepsilon}\left(\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}\tan\frac{\sigma}{4}}{\left(1+\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}}\int_0^1 dw\left(1+w^2\tan^2\frac{\sigma}{4}\right)^{-2\varepsilon}\left(1-w^2\right)^{2\varepsilon-1} \tag{C.8}$$

and this results in

$$\frac{2^{4\varepsilon-1}\sqrt{\pi}\left(\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}\tan\frac{\sigma}{4}}{\left(1+\tan^2\frac{\sigma}{4}\right)^{2\varepsilon-1}}\frac{\Gamma\left(2\varepsilon\right)}{\Gamma\left(\frac{1}{2}+2\varepsilon\right)}\,{}_2F_1\left(\frac{1}{2},2\varepsilon;\frac{1}{2}+2\varepsilon;-\tan^2\frac{\sigma}{4}\right) \tag{C.9}$$

In computing the integral

$$\int_{-\pi}^{\pi}\frac{d\tau}{\left[4\sin\left(\frac{\sigma-\tau}{2}\right)\sin\left(\frac{\tau}{2}\right)\right]^{1-2\varepsilon}} \tag{C.10}$$

we basically follow the same steps. First, use sum-to-product rule (C.2) to get

$$\begin{aligned}\int_{-\pi}^{\pi}\frac{d\tau}{\left(2\left(\cos\left(\tau-\frac{\sigma}{2}\right)-\cos\frac{\sigma}{2}\right)\right)^{1-2\varepsilon}} &= \int_{-\pi}^{\pi}\frac{d\tau}{\left(2\left(\cos\tau-\cos\frac{\sigma}{2}\right)\right)^{1-2\varepsilon}}\\ &= 2^{2\varepsilon}\int_0^{\pi}\frac{d\tau}{\left(\cos\tau-\cos\frac{\sigma}{2}\right)^{1-2\varepsilon}}\end{aligned} \tag{C.11}$$

the second equality follows from the fact that we are integrating over the whole period. Using (C.4) and the change of variables $\tau = 2\arctan(z)$ followed by $w = z\tan\left(\frac{\sigma}{4}\right)$ eventually we get

$$\frac{2^{4\varepsilon}\left(\tan\left(\frac{\sigma}{4}\right)\right)^{4\varepsilon-1}}{\left(1+\tan^2\left(\frac{\sigma}{4}\right)\right)^{2\varepsilon-1}}\int_0^{+\infty} dw\left(1+w^2\tan^2\left(\frac{\sigma}{4}\right)\right)^{-2\varepsilon}\left(1+w^2\right)^{2\varepsilon-1} \tag{C.12}$$

that gives

$$\begin{aligned}\frac{2^{4\varepsilon-1}\sqrt{\pi}\left(\tan\left(\frac{\sigma}{4}\right)^{4\varepsilon-1}\right)}{\left(1+\tan^2\left(\frac{\sigma}{4}\right)\right)^{2\varepsilon-1}}\frac{\Gamma(2\varepsilon)}{\Gamma\left(\frac{1}{2}+2\varepsilon\right)}&\left[{}_2F_1\left(\frac{1}{2},2\varepsilon;\frac{1}{2}+2\varepsilon;-\tan^2\left(\frac{\sigma}{4}\right)\right)\right.\\ &\left.+(-1)^{1-2\varepsilon}\,{}_2F_1\left(\frac{1}{2},2\varepsilon;\frac{1}{2}+2\varepsilon;-\cot^2\left(\frac{\sigma}{4}\right)\right)\left(\tan\left(\frac{\sigma}{4}\right)\right)^{-4\varepsilon}\right]\end{aligned} \tag{C.13}$$

C.2 Solve fermionic intergrals

In the second order expansion of the Wilson Loop perturbative computation we encounter an integral coming from the fermionic propagator

$$I=\int_a^b d\tau_1\int_c^d d\tau_2\left(\sin\left(\frac{\tau_1-\tau_2}{2}\right)\right)^{2\varepsilon-2} \tag{C.14}$$

Using the following identity

$$\sin\left(\frac{\tau_1-\tau_2}{2}\right)^{2\varepsilon-2}=-\frac{2}{\varepsilon(2\varepsilon-1)}\partial_{\tau_2}\partial_{\tau_1}\left(\sin\left(\frac{\tau_1-\tau_2}{2}\right)^{2\varepsilon}\right)+\frac{2\varepsilon}{2\varepsilon-1}\sin\left(\frac{\tau_1-\tau_2}{2}\right)^{2\varepsilon} \tag{C.15}$$

then it becomes a direct integration. For an example, setting $a=0,\ b=\sigma,\ c=0,\ d=\tau_1$ as in the integral (4.133)

$$\begin{aligned}I&=\int_0^\sigma d\tau_1\int_0^{\tau_1}d\tau_2\left[-\frac{2}{\varepsilon(2\varepsilon-1)}\partial_{\tau_2}\partial_{\tau_1}\left(\sin\left(\frac{\tau_1-\tau_2}{2}\right)^{2\varepsilon}\right)+o(\varepsilon)\right]\\&=-\frac{2}{\varepsilon(2\varepsilon-1)}\int_0^\sigma d\tau_1\int_0^{\tau_1}d\tau_2\partial_{\tau_2}\partial_{\tau_1}\left(\sin\left(\frac{\tau_1-\tau_2}{2}\right)^{2\varepsilon}\right)+o(\varepsilon)\\&=\frac{2}{\varepsilon(2\varepsilon-1)}\sin\left(\frac{\sigma}{2}\right)^{2\varepsilon}+o(\varepsilon).\end{aligned} \tag{C.16}$$

C.3 Two-loop integrals

In this section, we list the integrals corresponding to the two-loop diagrams in figures 3.2(a)-3.2(l), dressed by their color factors.

Diagram 3.2(a) contains the two-loop correction to the scalar propagator. This has been computed in [60] and reads

$$\begin{aligned}\mathcal{C}(N_1,N_2)=&\frac{N_1N_2}{k^2}\left(N_1^2+N_2^2-4N_1N_2+2\right)\left(\frac{\pi}{3\varepsilon}+2\pi+O\left(\varepsilon\right)\right)\\&+\frac{N_1N_2}{k^2}\left(N_1^2+N_2^2-2\right)\left(-\frac{4\pi}{3\varepsilon}+\pi\left(\pi^2-8\right)+O\left(\varepsilon\right)\right)\\&+\frac{N_1N_2}{k^2}\left(N_1N_2-1\right)\left(-\frac{8\pi}{3\varepsilon}+4\pi(\pi^2-20\pi)+O\left(\varepsilon\right)\right)\end{aligned} \tag{C.17}$$

To compute the contributions of the other diagrams it is sufficient to rely on Feynman rules listed in appendix A.2, together with the product of polarization vectors. Explicitly, we find

$$\begin{aligned}(3.2(b))=-s^2\,\frac{\Gamma^6\left(\frac{1}{2}-\varepsilon\right)}{32\,\pi^{7-6\varepsilon}}\,\frac{N_1^2N_2^2}{k^2}\int d^dx d^dy&\frac{x^\mu y^\nu}{(x^2)^{\frac{3}{2}-\varepsilon}(y^2)^{\frac{3}{2}-\varepsilon}\left((x-s)^2\right)^{\frac{1}{2}-\varepsilon}\left((y-s)^2\right)^{\frac{1}{2}-\varepsilon}}\\&\times\left[\frac{\delta_{\mu\nu}}{\left[(x-y)^2\right]^{1-2\varepsilon}}-\partial_\mu\partial_\nu\frac{\left[(x-y)^2\right]^{2\varepsilon}}{4\varepsilon(1+2\varepsilon)}\right]\end{aligned} \tag{C.18}$$

$$
\begin{aligned}
(3.2(c)) = s^2\, \frac{\Gamma^4\left(\frac{1}{2}-\varepsilon\right)\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}{128\,\pi^{7-6\varepsilon}} \frac{N_1N_2}{k^2}((N_1-N_2)^2-2N_1N_2+2)\times \\
\int \frac{d^dx d^dy}{(x^2)^{\frac{1}{2}-\varepsilon}(y^2)^{\frac{1}{2}-\varepsilon}\left((x-y)^2\right)^{2-2\varepsilon}\left((x-s)^2\right)^{\frac{1}{2}-\varepsilon}\left((y-s)^2\right)^{\frac{1}{2}-\varepsilon}}
\end{aligned}
\tag{C.19}
$$

$$
\begin{aligned}
(3.2(d)) = s^2\, \frac{\Gamma^6\left(\frac{1}{2}-\varepsilon\right)\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}{256\,\pi^{10-8\varepsilon}}\ \frac{N_1N_2}{k^2}\ (N_1-N_2)^2\ \varepsilon_{\mu\nu\eta}\varepsilon_{\rho\sigma\tau}\times \\
\int d^dxd^dyd^dzd^dw\, \frac{(x-y)^\eta(z-w)^\tau}{\left((x-y)^2\right)^{\frac{3}{2}-\varepsilon}\left((z-w)^2\right)^{\frac{3}{2}-\varepsilon}\left((x-s)^2\right)^{\frac{1}{2}-\varepsilon}\left((y-s)^2\right)^{\frac{1}{2}-\varepsilon}(z^2)^{\frac{1}{2}-\varepsilon}(w^2)^{\frac{1}{2}-\varepsilon}} \\
\times\ \partial^\mu\partial^\rho \frac{1}{\left((x-z)^2\right)^{\frac{1}{2}-\varepsilon}}\ \partial^\nu\partial^\sigma \frac{1}{\left((y-w)^2\right)^{\frac{1}{2}-\varepsilon}}
\end{aligned}
\tag{C.20}
$$

$$
\begin{aligned}
(3.2(e)) = -s^2\, \frac{\Gamma^6\left(\frac{1}{2}-\varepsilon\right)\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}{128\,\pi^{10-8\varepsilon}}\ \frac{N_1N_2}{k^2}\ (N_1N_2-1)\ \varepsilon_{\mu\nu\eta}\varepsilon_{\rho\sigma\tau}\times \\
\int d^dxd^dyd^dzd^dw\, \frac{(x-y)^\eta(z-w)^\tau}{\left((x-y)^2\right)^{\frac{3}{2}-\varepsilon}\left((z-w)^2\right)^{\frac{3}{2}-\varepsilon}\left((x-s)^2\right)^{\frac{1}{2}-\varepsilon}\left((w-s)^2\right)^{\frac{1}{2}-\varepsilon}(y^2)^{\frac{1}{2}-\varepsilon}(z^2)^{\frac{1}{2}-\varepsilon}} \\
\times\ \partial^\mu\partial^\rho \frac{1}{\left((x-z)^2\right)^{\frac{1}{2}-\varepsilon}}\ \partial^\nu\partial^\sigma \frac{1}{\left((y-w)^2\right)^{\frac{1}{2}-\varepsilon}}
\end{aligned}
\tag{C.21}
$$

$$
\begin{aligned}
(3.2(f)) = s^2\, \frac{\Gamma^4\left(\frac{1}{2}-\varepsilon\right)\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}{16\,\pi^{7-6\varepsilon}} \frac{N_1N_2}{k^2}(N_1N_2-1)\times \\
\int \frac{d^dx d^dy}{(x^2)^{\frac{1}{2}-\varepsilon}(y^2)^{\frac{1}{2}-\varepsilon}\left((x-y)^2\right)^{2-2\varepsilon}\left((x-s)^2\right)^{\frac{1}{2}-\varepsilon}\left((y-s)^2\right)^{\frac{1}{2}-\varepsilon}}
\end{aligned}
\tag{C.22}
$$

We note that in the large N_1, N_2 approximation we obtain $(3.2(f)) = -4(3.2(c))$, in agreement with the results in [91].

$$
\begin{aligned}
(3.2(g)) = -s^2\, \frac{\Gamma^5\left(\frac{1}{2}-\varepsilon\right)\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}{128\,\pi^{\frac{17}{2}-7\varepsilon}}\ \frac{N_1N_2}{k^2}(N_1^2+N_2^2-4N_1N_2+2)\ \varepsilon_{\mu\rho\sigma}\varepsilon_{\mu\nu\eta}\times \\
\int d^dxd^dyd^dz \frac{(x-z)^\sigma}{\left((x-z)^2\right)^{\frac{3}{2}-\varepsilon}}\ \frac{(x-y)^\eta}{\left((x-y)^2\right)^{\frac{3}{2}-\varepsilon}}\times \\
\partial^\rho \frac{1}{\left((y-z)^2\right)^{\frac{1}{2}-\varepsilon}}\ \partial^\nu \frac{1}{\left((y-s)^2\right)^{\frac{1}{2}-\varepsilon}}\ \frac{1}{(x^2)^{\frac{1}{2}-\varepsilon}(z^2)^{\frac{1}{2}-\varepsilon}\left((x-s)^2\right)^{\frac{1}{2}-\varepsilon}}
\end{aligned}
\tag{C.23}
$$

$$
(3.2(h)) = 0 \qquad\qquad (3.2(i)) = 0 \tag{C.24}
$$

$$
\begin{aligned}
(3.2(\mathrm{j})) = s^2\, & \frac{\Gamma^5\left(\frac{1}{2}-\varepsilon\right)\Gamma^3\left(\frac{3}{2}-\varepsilon\right)}{128\,\pi^{10-8\varepsilon}}\,\frac{N_1N_2}{k^2}(N_1^2+N_2^2-2)\;\varepsilon_{\rho\nu\tau}\varepsilon_{\rho\eta\sigma}\varepsilon_{\nu\mu\varphi}\varepsilon_{\tau\chi\xi}\,\times \\
& \int d^dx d^dy d^dz d^dw \frac{(x-z)^\varphi (y-z)^\xi (w-z)^\sigma}{((x-z)^2)^{\frac{3}{2}-\varepsilon}\,((y-z)^2)^{\frac{3}{2}-\varepsilon}\,((w-z)^2)^{\frac{3}{2}-\varepsilon}} \\
& \times\ \partial^\eta \frac{1}{((w-s)^2)^{\frac{1}{2}-\varepsilon}}\,\partial^\chi \frac{1}{((x-y)^2)^{\frac{1}{2}-\varepsilon}}\,\partial^\mu \frac{1}{((x-s)^2)^{\frac{1}{2}-\varepsilon}}\,\frac{1}{(y^2)^{\frac{1}{2}-\varepsilon}(w^2)^{\frac{1}{2}-\varepsilon}}
\end{aligned}
\tag{C.25}
$$

$$
\begin{aligned}
(3.2(\mathrm{k})) = s^2\, & \frac{\Gamma^6\left(\frac{1}{2}-\varepsilon\right)\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}{256\,\pi^{10-8\varepsilon}}\,\frac{N_1N_2}{k^2}(N_1N_2-2)\;\varepsilon_{\mu\nu\varepsilon}\varepsilon_{\rho\sigma\tau}\,\times \\
& \int d^dx d^dy d^dz d^dw \frac{(x-y)^\varepsilon (z-w)^\tau}{((x-y)^2)^{\frac{3}{2}-\varepsilon}\,((z-w)^2)^{\frac{3}{2}-\varepsilon}}\,\frac{1}{((w-s)^2)^{\frac{1}{2}-\varepsilon}} \\
& \times\ \partial^\rho \frac{1}{((x-z)^2)^{\frac{1}{2}-\varepsilon}}\,\partial^\nu \frac{1}{((y-z)^2)^{\frac{1}{2}-\varepsilon}}\,\partial^\sigma \frac{1}{(w^2)^{\frac{1}{2}-\varepsilon}} \\
& \qquad \times \left[\partial^\mu \frac{1}{((x-s)^2)^{\frac{1}{2}-\varepsilon}}\,\frac{1}{(y^2)^{\frac{1}{2}-\varepsilon}} - \partial^\mu \frac{1}{(x^2)^{\frac{1}{2}-\varepsilon}}\,\frac{1}{((y-s)^2)^{\frac{1}{2}-\varepsilon}}\right]
\end{aligned}
\tag{C.26}
$$

$$
\begin{aligned}
(3.2(\mathrm{l})) = -s^2\, & \frac{\Gamma^4\left(\frac{1}{2}-\varepsilon\right)\Gamma^2\left(\frac{3}{2}-\varepsilon\right)}{32\,\pi^{7-6\varepsilon}}\,\frac{N_1N_2}{k^2}(N_1-N_2)^2 \\
& \times \int \frac{d^dx d^dy}{((x-s)^2)^{1-2\varepsilon}\,((x-y)^2)^{2-2\varepsilon}\,(y^2)^{1-2\varepsilon}}
\end{aligned}
\tag{C.27}
$$

D

Supermatrices

We have seen that the fermionic loop needs a representation in terms of a super-connection on the superalgebra of the supergroup $U(N_1|N_2)$. This appendix is devoted to presenting all the properties of the supermatrices we will use in the Wilson Loop computations.

D.1 Basics on supergroups

A *supergroup* G is a $\mathbb{Z}_2$-graded direct sum of two sets

$$G = G_B \oplus G_F \tag{D.1}$$

where the B in G_B means "bosonic" and the F "fermionic"; indeed, we can label the element in the two sets by their parity, i.e. $b \in G_B$ is parity even and $f \in G_F$ is parity odd. The composition works as follow

$$b \cdot b \in G_B \qquad b \cdot f \in G_F \qquad f \cdot f \in G_B \tag{D.2}$$

Obviously, the neutral element $e \in G_B$.

The associated algebra $\mathfrak{g}$ has the same direct sum structure

$$\mathfrak{g} = \mathfrak{g}_e \oplus \mathfrak{g}_o \tag{D.3}$$

which we labeled the two parts as "even" and "odd". Also the algebra elements can be parity even or odd, and we will denote it as $|X|$ such that

$$|X| = 0 \; for \; X \in \mathfrak{g}_e \qquad |X| = 1 \; for \; X \in \mathfrak{g}_o \tag{D.4}$$

The usual commutator is then lifted to a supercommutator, defined as

$$[X, Y\} = XY - (-1)^{|X||Y|} YX \tag{D.5}$$

that reduce to the usual commutator if at least one element is even, while it becomes the anticommutator when both elements are odd.

D.2 Representation on supermatrices

Our interest is on the supergroup $U(N_1|N_2)$. We can represent its algebra on the space of supermatrices $\mathfrak{sgl}(N_1|N_2)$ that can be decomposed as

$$\mathfrak{sgl} = \mathfrak{sgl}_e \oplus \mathfrak{sgl}_o \tag{D.6}$$

Supermatrices can be seen as composed by blocks, whose blocks posses a certain parity under $\mathbb{Z}_2$, in the following way

$$X = \begin{pmatrix} even & odd \\ odd & even \end{pmatrix} \qquad \Rightarrow \qquad |X| = 0 \tag{D.7}$$

$$Y = \begin{pmatrix} odd & even \\ even & odd \end{pmatrix} \qquad \Rightarrow \qquad |Y| = 1 \tag{D.8}$$

The supermatrix product is the same as the ordinary matrix product. On the other hand, the scalar multiplication is little trickier. On this space, even scalars have parity. The actions of the scalars on the supermatrices follow these rules

$$\alpha \cdot X = \begin{pmatrix} \alpha X_1 & \alpha X_2 \\ (-1)^{|\alpha|}\alpha X_3 & (-1)^{|\alpha|}\alpha X_4 \end{pmatrix} \tag{D.9}$$

when the scalar acts from the left, while

$$X \cdot \alpha = \begin{pmatrix} X_1\alpha & X_2(-1)^{|\alpha|}\alpha \\ X_3\alpha & X_4(-1)^{|\alpha|}\alpha \end{pmatrix} \tag{D.10}$$

We can obtain these rules just by noting that scalars can be written as supermatrix too, depending on their parity

$$\alpha \to \begin{pmatrix} \alpha & 0 \\ 0 & (-1)^{|\alpha|}\alpha \end{pmatrix} \tag{D.11}$$

and using the supermatrix product. This property is useful to define the supercharge action on the supermatrices of the 1/2-BPS Wilson Loop in ABJ(M).

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