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A simplified behavioral approach to inversion-based control of linear systems

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Për Arbenin, Zamiren dhe Martinen
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Abstract

For linear time-invariant systems, we introduce a simplified behavioral approach which is based on piecewise infinitely differentiable functions. Compared to the function space used in the standard behavioral approach, our setting allows to simplify mathematical machinery while at the same time preserving the richness of signal's features required in many practical applications (e.g. mechatronics one). We employ the simplified behavioral approach to derive two main contributions to the field of inversion-based control. As a first contribution, we introduce a novel solution to the stable input-output inversion problem for square nonminimum-phase systems. Differently from state-of-the-art solutions, our solution can be applied to nondecoupable systems too. As a second contribution, we formally prove the equivalence among the two most common inversion architectures: the closed-loop and plant inversion architecture. This equivalence dictates that the two architectures deliver the same performances for any disturbance and mis-modeling affecting the controlled plant.

Contents

Chapter 1

Introduction

The focus of this thesis is on the *inversion-based control* of linear time-invariant systems. For a minimum-phase system, an inversion-based controller amounts to the inverse of the system's matrix transfer function (*standard inverse* procedure). However, it is well known that, for nonminimum-phase systems (i.e. systems having at least one zero with positive real part) the standard inverse is unstable and the corresponding output unbounded. Although several *approximated* inversion procedures have been proposed to obtain a stable inverse for the class of nonminimum-phase systems [27], [49], [52], in this work we are interested in *exact* inversion procedures.

In [15] and [25], in a state-space setting, an *exact* stable (noncausal) solution to the inversion-based control problem was shown to exist for linear systems *decoupable* by static state feedback (cf. Chapter 4). The resulting inversion approach is known as the *stable inversion* procedure.

1.1 Motivation

The main goal of this thesis, is to present a new solution to the stable inversion problem which can also be applied to nondecoupable systems. As we are going to show, in order to conveniently approach this problem, it is beneficial to rely on a theory of linear time-invariant systems which is centered on input-output trajectories. Indeed, addressing a stable-inversion problem entails *more* than simply looking for an inverse input. In particular, the function space of involved signals must be wisely chosen and conditions for guaranteeing the existence of an inverse input must be studied (i.e. a study on continuity properties of input-output pairs is required).

In the late '80s, as an attempt to rigorously approach the analysis of linear time-invariant systems, in [64], J. C. Willems introduced the *behavioral approach* to systems and control. The core idea of the behavioral approach is to view a dynamical system as a collection of admissible trajectories, the *behavior set*. The main consequence of such viewpoint is that system's structure and properties are also defined in terms of features exhibited by this set [65].

In order to define the behavior set, in most cases, one needs to specify an equation together with a function space. All elements in the given function space which satisfy the chosen equation constitute the behavior set. In [65], Polderman and Willems considered behaviors induced by linear ordinary differential equations and, in order to allow for discontinuous signals, they proposed as functions space $\mathcal{L}_1^{\mathtt{loc}}$, the set of locally integrable functions. Formally, $\boldsymbol{u} \in \mathcal{L}_1^{\mathtt{loc}}$ if

$$\int_a^b ||\boldsymbol{u}(t)||dt < \infty \tag{1.1}$$

for all $a, b \in \mathbb{R}$. *Weak solutions* are then introduced by replacing the ordinary differential equation with a suitable integral equation which must hold *almost everywhere* on $\mathbb{R}$ [65]. We refer to this resulting setting as the $\mathcal{L}_1^{\mathtt{loc}}$ behavioral theory (or *standard behavioral theory*).

In this thesis, instead of working with the more general space of locally integrable functions, we restrict our interest to the simpler space of piecewise C^∞ functions, C_{p}^∞ (cf. Section 2.2 in Chapter 2). There are two main reasons for such a choice. Firstly, working in C_{p}^∞ allows for the introduction of a straightforward theory of weak solutions which does not require advanced topics of functional analysis. Indeed, since C_{p}^∞ functions are Riemann-integrable and because we define weak solutions in terms of an integral equation that must be satisfied *for all time instants* (see Definition 9); we are able to simplify the derivation of several results, which, in the standard behavioral theory, are either obtained via advanced (and rather abstract) mathematical tools (see

Remark 2.4.14 in [65]) or can not be derived at all. In order to justify this, in Chapter 3, 4 and 5 we present several applications of our simplified behavioral approach to inversion-based control.

The second reason for working with C_{p}^{∞} functions is that C_{p}^{∞} is a sufficiently rich function space for many applications, e.g. electromechanical ones ([12]). In other words, when compared with C_{p}^{∞}, the additional features that $\mathcal{L}_1^{\mathrm{loc}}$ allows are not so relevant in real-world applications (think about functions whose limit at a certain point does not exist or functions with vertical asymptotes).

We refer to the resulting behavioral approach based on the space of C_{p}^{∞} functions as the C_{p}^{∞} behavioral approach (or *simplified behavioral approach*).

1.2 Thesis structure and contributions

In Chapter 2 we introduce the space of C_{p}^{∞} functions and the simplified behavioral approach for linear time-invariant systems. In introducing our behavioral approach, we take into account both input-output and input-state-output representations. Then, we prove that under natural assumptions, input-output and input-state-output representations are equivalent, i.e. their behavior sets are the same.

In Chapter 3, we take advantage of the simplified behavioral approach and we derive a new straightforward relationship between jumps of the input signal (and its derivatives) with jumps of the output signal (and its derivatives). As an application, we propose to use our findings in solving an initial conditions problem.

In Chapter 4 we present the new solution to the stable input-output inversion problem. Although our solution relies on an input-output representation of the system, on several occasions we still exploit an input-state-output representation (for instance, in order to characterize the continuity properties of input-output pairs).

Chapter 5 presents another application of the simplified behavioral approach. Specifically, we prove the equivalence among two of the most common feedforward-feedback inversion-based architectures employed in practical applications.

Finally, Appendices A, B conclude the thesis. In Appendix A, we report the notation and basic results about polynomial matrices and matrix fraction descriptions. In

Appendix B, we present the Beghelli-Guidorzi input-output form. The background material presented in Appendix A is used in Chapters 2 and 4. Instead, Appendix B is only required in the proof of Theorem 2 in Chapter 2.

1.2.1 List of publications

The content of the thesis is based on the following publications.

- Chapter 3:
  - J. Kavaja and A. Piazzi. Input-output jumps of scalar linear systems. *IFAC-PapersOnLine*, 52(17):13–18, 2019.
- Chapter 4:
  - J. Kavaja, A. Minari, and A. Piazzi. Stable input-output inversion for nondecoupable nonminimum-phase linear systems. In *2018 European Control Conference (ECC)*, pages 2855–2860. IEEE, 2018.
  - J. Kavaja and A. Piazzi. On the structure of the multivariable free response. **Submitted to** *30th Mediterranean Conference on Control and Automation*.
- Chapter 5:
  - J. Kavaja and A. Piazzi. On the equivalence of model inversion architectures for control applications. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 5173–5179. IEEE, 2020.
  - J. Kavaja and A. Piazzi. On the equivalence of inversion-based control architectures. **Submitted to** *Transactions on Automatic Control*.

1.3 General notation

The following notation will be employed throughout this thesis. Further notation, when needed, will be added at the beginning of each chapter.

$\mathbb{N}_{>0}$ denotes the set of natural numbers greater than 0.

Scalars and real-valued functions are denoted by lower-case letters. Vectors and vector-valued functions are denoted by bold lower-case letters, whereas matrices are denoted by capital letters with a few exceptions carefully remarked. Given a matrix $C \in \mathbb{R}^{m\times n}$, its entries are denoted by c_{ij} or $(C)_{ij}$, $i = 1,\ldots,m$, $j = 1,\ldots,n$, so that $C \equiv [c_{ij}]$ or $C \equiv [(C)_{ij}]$. The i-th row and the j-th column of C are denoted by $\boldsymbol{c}^i \in \mathbb{R}^{1\times n}$ and $\boldsymbol{c}_j \in \mathbb{R}^{m\times 1} \equiv \mathbb{R}^m$ respectively.

The set of real coefficients polynomials in the indeterminate s is denoted by $\mathbb{R}[s]$. The degree of the null polynomial is -1. The notation $\mathbb{R}^{m\times n}[s]$ denotes polynomial matrices, i.e. $m \times n$ matrices whose entries are elements of $\mathbb{R}[s]$. Laurent polynomials and rational functions in the indeterminate s, with real coefficients, are denoted by $\mathbb{R}[s,s^{-1}]$ and $\mathbb{R}(s)$ respectively. Their matrix counterparts are denoted by $\mathbb{R}^{m\times n}[s,s^{-1}]$ and $\mathbb{R}^{m\times n}(s)$ respectively.

A function $\boldsymbol{f} \equiv [f_1, f_2,\ldots, f_m]^\top : \mathbb{R} \to \mathbb{R}^m$ belongs to $C^n(\mathbb{R},\mathbb{R}^m)$ if all its components f_i are continuous with continuous derivatives until the n-th order over $\mathbb{R}$. When the number of components of $\boldsymbol{f}$ is clear from the context, we write C^n to denote $C^n(\mathbb{R},\mathbb{R}^m)$. If $\boldsymbol{f} \in C^n$ then we say $\boldsymbol{f}$ has *continuity order* n. If for all components of $\boldsymbol{f}$ there exist derivatives of any order on $D \subseteq \mathbb{R}$ then $\boldsymbol{f}$ belongs to $C^\infty(D,\mathbb{R}^m)$.

We denote the n-th order derivative of $\boldsymbol{f}$ by $\boldsymbol{f}^{(n)} := [f_1^{(n)}, f_2^{(n)},\ldots, f_m^{(n)}]^\top$ or $D^n\boldsymbol{f} := [D^n f_1, D^n f_2,\ldots, D^n f_m]^\top$, where D stands for the *derivative operator*. The notation $\boldsymbol{f}(t^+)$ denotes the *right-hand limit* of $\boldsymbol{f}$ at t, namely: $\boldsymbol{f}(t^+) = \lim_{v\to t^+} \boldsymbol{f}(v)$. Similarly, we denote the *left-hand limit* of $\boldsymbol{f}$ at t by $\boldsymbol{f}(t^-)$.

Chapter 2

The simplified behavior of linear time-invariant systems

2.1 Introduction

The essence of the behavioral approach to systems and control theory lies in viewing a dynamical system as a set of admissible trajectories [65]. It is well known that the set of possible trajectories for a variety of physical systems can be conveniently represented as the solution set of a differential equation of form:

$$R(D)\boldsymbol{w} = 0, \tag{2.1}$$

where $R(s) \in \mathbb{R}^{g\times q}[s]$ is a polynomial matrix, $D \equiv \frac{d}{dt}$ is the derivative operator and $\boldsymbol{w} \in \mathbb{R}^q$ is the *manifest variable*, i.e. it is the physical variable which one wants to model and is the reason why (2.1) is derived. Representations of this form are called *kernel representations*. However, when modeling from first principles, it is more common to end up with a differential equation of form:

$$R(D)\boldsymbol{w} = M(D)\boldsymbol{\ell} \tag{2.2}$$

where $M \in \mathbb{R}^{g\times d}[s]$ and $\boldsymbol{\ell} \in \mathbb{R}^d$ is the so-called *latent variable*. At first, when the modeling process starts, latent variables are not meant to be modeled, nevertheless, they still come into play either because they are needed to explain the physical process or because they allow for a more convenient representation of the laws governing $\boldsymbol{w}$ [65]. A typical example of latent variable representation is the *input-state-output* one:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned}, \tag{2.3}$$

where $\boldsymbol{u} \in \mathbb{R}^m$, $\boldsymbol{y} \in \mathbb{R}^p$ and $\boldsymbol{x} \in \mathbb{R}^n$ are the input, output and state variable respectively and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. In this context, it is assumed that (2.3) aims at describing the relationship between $\boldsymbol{u}$ and $\boldsymbol{y}$ (so that the manifest variable is $\boldsymbol{w} = \begin{bmatrix}\boldsymbol{u}^\top & \boldsymbol{y}^\top\end{bmatrix}^\top$) while $\boldsymbol{x}$ is only considered in order to allow the resulting representation to be a first-order differential equation. Hence, the state variable is viewed as a latent variable.

It is then natural to ask whether (2.3) admits a representation of form (2.1) and, more specifically, an *input-output* representation:

$$P(D)\boldsymbol{y} = Q(D)\boldsymbol{u} \tag{2.4}$$

where $P(s) \in \mathbb{R}^{p\times p}[s]$ is assumed invertible, $Q(s) \in \mathbb{R}^{p\times m}$ and $P^{-1}(s)Q(s)$ is strictly proper.

This problem entails two sub-problems [56]. First, one has to decide what are equivalent representations and, then, one must describe how to go from one representation to the other. As for the first problem, several notions of equivalence have been proposed in the literature. Among these, we recall the definitions proposed by Rosenbrock [53] and Wolovich [66] which are aimed at more general representations of form:

$$\begin{aligned} P(D)\boldsymbol{z} &= Q(D)\boldsymbol{u} \\ \boldsymbol{y} &= R(D)\boldsymbol{z} + W(D)\boldsymbol{u} \end{aligned}. \tag{2.5}$$

Remark that, when $P(s) = sI - A$, $Q(s) = B$, $R(s) = C$ and $W(s) = 0$, representation (2.5) reduces to (2.3). Whereas, when $W(s) = 0$ and $R(s) = I$, it reduces to an input-output representation of form (2.4). Another definition of equivalence, only for input-state-output and input-output representations, is also reported in Guidorzi [23]. Nevertheless, in the behavioral framework, there is no ambiguity in deciding when two representations are equivalent. The only sensible definition is to consider two representations equivalent if they have same behavior. Hence, in what follows, we focus on this definition of equivalence.

In [65], by relying on a procedure for eliminating latent variables (Theorem 6.2.6 in [65]), representations (2.3) and (2.4) are shown to be equivalent in the single-input single-output case *only*. Here, we extend this result to the multi-input multi-output case. However, for the sake of simplicity of presentation, we make other simplifying assumptions. Namely, we assume that in (2.3), (C,A) is observable and (A,B) is controllable with $\operatorname{rank} C = p$. Furthermore, we assume that in (2.4), pair $P(s),Q(s)$ is left-coprime (see Appendix A). Rather than using the elimination procedure described in [65], we exploit the structure of observable state-space representations and we show how to derive an input-output representation starting from an input-state-output one and vice versa. An advantage of this approach is that, as a by-product, we obtain a recipe for constructing the state variable starting from an input-output representation. We remark that such a recipe is only sketched in [65] but is not formally proven. Our proof is inspired by [22], [23], however, differently form these works, we employ a behavioral setting and we allow for weak solutions of the involved differential equations. We also point-out that a similar proof has also been sketched by Willems (see Theorem 5.1 and 6.2 in [63]), however several details of that proof have been omitted.

Chapter organization: This chapter is organized as follows. In Section 2.2 we introduce the space of C_{p}^{∞} functions. This function space is employed in Section 2.3 to introduce the notion of weak solution and that of behavior for input-output equations. In doing so, we present three equivalent definitions of behavior set. As we show, according to the intended usage, one definition could be more suitable than the others. Next, in Section 2.3 we extend the same notions to input-state-output representations and we prove the equivalence between input-output and input-state-output representations. Finally, in Section 2.5 we conclude the chapter.

Notation: Given a strictly proper matrix transfer function $H(s) = P^{-1}(s)Q(s)$ where $P(s) = \sum_{i=0}^{n_P} P_i s^i \in \mathbb{R}^{p\times p}[s]$, $Q(s) = \sum_{i=0}^{n_Q} Q_i s^i \in \mathbb{R}^{p\times m}[s]$, with $n_P := \max_i\{n_P^i\}$ and n_P^i the i-th *row degree* of $P(s)$ (see Appendix A), we define the *asterisk* operator

$*$ by:

$$* : \mathbb{R}^{p\times(p+m)}[s] \to \mathbb{R}^{p\times(p+m)}[s]$$
$$[P(s)\ Q(s)] \mapsto [P^*(s)\ Q^*(s)] := s^{n_P}[P(\frac{1}{s})\ Q(\frac{1}{s})],$$

and the *bar* operator $\bar{}$ by:

$$\bar{} : \mathbb{R}^{p\times(p+m)}[s] \to \mathbb{R}^{p\times(p+m)}[s]$$
$$[P(s)\ Q(s)] \mapsto [\bar{P}(s)\ \bar{Q}(s)] := \mathrm{diag}\{s^{n_P^1}, s^{n_P^2}, \ldots, s^{n_P^p}\}[P(\frac{1}{s})\ Q(\frac{1}{s})].$$

2.2 The space of piecewise C^∞ functions: C^∞_{p}

The space of piecewise C^∞ functions was originally introduced in [12] for scalar-valued functions. In this section, we extend its definition to vector-valued functions.

Definition 1 (Sparse set)**.** *A set $S \subset \mathbb{R}$ is said to be* sparse *if, for every $a, b \in \mathbb{R}$, $S \cap [a, b]$ has finite cardinality or it is the empty set.*

Definition 2. *The set of* piecewise C^∞ functions*, denoted C^∞_{p}, consists of all functions $\boldsymbol{f}$ for which there exists a sparse set S such that $\boldsymbol{f} \in C^\infty(\mathbb{R} \setminus S, \mathbb{R}^m)$ and the limits $\boldsymbol{f}^{(n)}(t^-)$ and $\boldsymbol{f}^{(n)}(t^+)$ exist and are finite for any $n \in \mathbb{N}$ and $t \in S$.*

When $\boldsymbol{f} \in C^\infty_{\mathrm{p}}$ is defined in $t \in S$, conventionally $\boldsymbol{f}(t) := \boldsymbol{f}(t^+)$.

Observe that, by construction, C^∞_{p} is a closed space with respect to derivation. In other words, the derivative of a C^∞_{p} function belongs to C^∞_{p}.

Definition 3. *$C^{-1}(\mathbb{R}, \mathbb{R}^m) := C^\infty_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m)$ denotes the set of piecewise C^∞ functions defined over the whole set of reals.*

Remark that if $\boldsymbol{f} \in C^{-1}(\mathbb{R}, \mathbb{R}^m)$ then $\boldsymbol{f}$ is right-continuous over $\mathbb{R}$, i.e. $\boldsymbol{f}(t) = \boldsymbol{f}(t^+)$.

Definition 4. *The zero-order discontinuity set of $\boldsymbol{f}$ is $S^{(0)}_{\boldsymbol{f}} := \{t \in \mathbb{R} : \boldsymbol{f}$ is not defined in t, i.e. at least one component of $\boldsymbol{f}$ is not defined in t, or $\boldsymbol{f}(t^-) \neq \boldsymbol{f}(t^+)\}$.*
Given $n \in \mathbb{N}_{>0}$, the n-th order discontinuity set of $\boldsymbol{f}$ is $S^{(n)}_{\boldsymbol{f}} := \{t \in \mathbb{R} : \boldsymbol{f}^{(n)}$ does not exist in t, i.e. at least one component of $\boldsymbol{f}^{(n)}$ does not exists in $t\}$.

The following are two straightforward results about discontinuity sets.

Lemma 1. *Let $\boldsymbol{f} \in C^{-1}(\mathbb{R},\mathbb{R}^m)$ and $n \in \mathbb{N}$. Then $\boldsymbol{f} \in C^n(\mathbb{R},\mathbb{R}^m)$ if and only if $S^{(n)}_{\boldsymbol{f}} = \emptyset$.*

Lemma 2. *Let $\boldsymbol{f} \in C^{-1}(\mathbb{R},\mathbb{R}^m)$ and $L \in \mathbb{R}^{n\times m}$ be a matrix with* $\mathrm{rank}(L) = m$*. Then $S^{(0)}_{\boldsymbol{f}} = S^{(0)}_{L\boldsymbol{f}}$.*

Proof: Taking into account that $\boldsymbol{f} \in C^\infty_{\mathrm{p}}(\mathbb{R},\mathbb{R}^m)$ (cf. Definition 3) we have $S^{(0)}_{\boldsymbol{f}} = \{t \in \mathbb{R} : \ \boldsymbol{f}(t^-) \neq \boldsymbol{f}(t^+)\}$ and $S^{(0)}_{L\boldsymbol{f}} = \{t \in \mathbb{R} : \ L\boldsymbol{f}(t^-) \neq L\boldsymbol{f}(t^+)\}$. Relation $L\boldsymbol{f}(t^-) \neq L\boldsymbol{f}(t^+)$ is equivalent to $L(\boldsymbol{f}(t^-) - \boldsymbol{f}(t^+)) \neq 0$, i.e. $\boldsymbol{f}(t^-) - \boldsymbol{f}(t^+) \notin \ker L$. But $\ker L = \{\mathbf{0}\}$ so that $\boldsymbol{f}(t^-) - \boldsymbol{f}(t^+) \neq 0$. □

Functions in C^∞_{p} are Riemann integrable (see Theorem 6.10 in [54]), furthermore, integrating a function in C^∞_{p} results in another C^∞_{p} function (see Theorem 6.20 in [54]). In other words, C^∞_{p} is a closed space with respect to integration. This motivates the following definition.

Definition 5. *Let $\boldsymbol{f} \in C^\infty_{\mathrm{p}}$, we denote by $\int \boldsymbol{f} \equiv \int^1 \boldsymbol{f}$ the function $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^m$, $t \mapsto \boldsymbol{F}(t) = \int_0^t \boldsymbol{f}(\xi)d\xi$ where $\int$ stands for the* Riemann integral operator. *Given $k \in \mathbb{Z}$, we set $\int^0 \boldsymbol{f} := \boldsymbol{f}$ and define $\int^k \boldsymbol{f}$ by the recursion $\int^k \boldsymbol{f} := \int(\int^{k-1} \boldsymbol{f})$ if $k \geq 1$ and $\int^k \boldsymbol{f} := D^{-k}\boldsymbol{f}$ if $k \leq -1$.*

We now present three technical results, for a proof of these see [12].

Lemma 3. *Let $\boldsymbol{f} \in C^p$ with $p \in \mathbb{N} \cup \{-1\}$ and $k \in \mathbb{N}$. Then*

$$\int^k \boldsymbol{f} \in C^{p+k}.$$

Lemma 4. *Let $\boldsymbol{f} \in C^\infty_{\mathrm{p}}$ and $p,k \in \mathbb{N}$. Then $D^k\left(\int^p \boldsymbol{f}\right)$ is defined on $\mathbb{R}$ if $p > k$ and on $\mathbb{R} \setminus S^{(k-p)}_{\boldsymbol{f}}$ if $p \leq k$. Furthermore*

$$D^k\left(\int^p \boldsymbol{f}\right) = \int^{p-k} \boldsymbol{f}.$$

Lemma 5. *Let $\boldsymbol{f} \in C^\infty_{\mathrm{p}} \cap C^0$ then*

$$\int D\boldsymbol{f}(t) = \boldsymbol{f}(t) - \boldsymbol{f}(0), \quad t \in \mathbb{R}.$$

Next, we introduce constant-coefficient operators. These will allow for compact notations of differential and integral equations.

Definition 6. *A* constant-coefficient differential (integral) operator *is a function*

$$\begin{aligned} A : C^\infty_{\mathrm{p}} &\to C^\infty_{\mathrm{p}} \\ \boldsymbol{f} &\mapsto \sum_{i=0}^{n} A_i D^i \boldsymbol{f}, \quad n \in \mathbb{N},\ A_i \in \mathbb{R}^{p\times m} \\ \Big(\boldsymbol{f} &\mapsto \sum_{i=0}^{n} A_i \int^i \boldsymbol{f}, \quad n \in \mathbb{N},\ A_i \in \mathbb{R}^{p\times m}\Big). \end{aligned}$$

The notation $\sum_{i=0}^n A_i D^i$ $\left(\sum_{i=0}^n A_i \int^i\right)$ *is used to denote A, i.e.* $A = \sum_{i=0}^n A_i D^i$ $\left(A = \sum_{i=0}^n A_i \int^i\right)$.

We say that the polynomial matrix associated to the constant-coefficient differential (or integral) operator $A = \sum_{i=0}^n A_i D^i$ $\left(A = \sum_{i=0}^n A_i \int^i\right)$ is $P_A(s) = \sum_{i=0}^n A_i s^i \in \mathbb{R}^{p\times m}[s]$. Similarly, given a polynomial matrix $A(s) = \sum_{i=0}^n A_i s^i$, the constant-coefficient differential (or integral) operator associated to $A(s)$ is $A(D) = \sum_{i=0}^n A_i D^i$ $\left(A(\int) = \sum_{i=0}^n A_i \int^i\right)$.

Definition 7. *A* constant-coefficient differential-integral operator *is a function*

$$\begin{aligned} A : C^\infty_{\mathrm{p}} &\to C^\infty_{\mathrm{p}} \\ \boldsymbol{f} &\mapsto \sum_{i=-m_A}^{n_A} A_i \int^i \boldsymbol{f}, \quad m_A, n_A \in \mathbb{N},\ A_i \in \mathbb{R}^{p\times m}. \end{aligned}$$

The notation $\sum_{i=-m_A}^{n_A} A_i \int^i$ *is used to denote A, i.e.* $A = \sum_{i=-m_A}^{n_A} A_i \int^i$.

It is natural to associate a Laurent polynomial (matrix) to these operators.

Definition 8. *The Laurent polynomial (matrix) associated to the constant-coefficient differential-integral operator* $A = \sum_{i=-m_A}^{n_A} A_i \int^i$ *is*

$$P_A(s) := \sum_{i=-m_A}^{n_A} A_i s^i \in \mathbb{R}^{p\times m}[s, s^{-1}].$$

Given a Laurent polynomial $A(s) = \sum_{i=-m_A}^{n_A} A_i s^i \in \mathbb{R}^{p\times m}[s, s^{-1}]$*, the constant-coefficient differential-integral operator associated to* $A(s)$ *is* $A(\int) = \sum_{i=-m_A}^{n_A} A_i \int^i$.

A useful result about the composition of constant-coefficient differential-integral operators is the following.

Proposition 1. *Let* $A = \sum_{i=-m_A}^{n_A} A_i \int^i$ *with* $A_i \in \mathbb{R}^{p\times p}$, $B = \sum_{i=-m_B}^{n_B} B_i \int^i$ *with* $B_i \in \mathbb{R}^{p\times m}$ *and* $\boldsymbol{f} \in C^\infty_\mathrm{p} \cap C^{m_B-1}$. *Then*

$$A(B\boldsymbol{f})(t) = (P_A \cdot P_B)(\int)\boldsymbol{f}(t) - \boldsymbol{w}(t), \quad t \in \mathbb{R}\setminus S_{\boldsymbol{f}}^{(m_A+m_B)},$$

where $\boldsymbol{w} \in \mathbb{R}^p[s]$ *is such that* $\deg \boldsymbol{w} \leq n_A - 1$. *Furthermore, if* $m_B = 0$ *then* $\boldsymbol{w} = \boldsymbol{0}$.

Proof. The function $A(B\boldsymbol{f})(t) = \sum_{i=-m_A}^{n_A} A_i \int^i \left(\sum_{j=-m_B}^{n_B} B_j \int^j \boldsymbol{f}\right)(t)$ for $t \in \mathbb{R}\setminus S_{\boldsymbol{f}}^{(m_A+m_B)}$ can be expanded as:

$$\begin{aligned} A(B\boldsymbol{f})(t) = \sum_{i=1}^{n_A}\sum_{j=1}^{n_B} A_i B_j \int^{i+j} \boldsymbol{f}(t) \; + \underbrace{\sum_{i=1}^{n_A} A_i \int^i \Big(\sum_{j=-m_B}^{0} B_j \int^j \boldsymbol{f}\Big)(t)}_{(a)} \\ + \underbrace{\sum_{i=-m_A}^{0}\sum_{j=1}^{n_B} A_i B_j \int^{i+j} \boldsymbol{f}(t)}_{(b)} + \sum_{i=-m_A}^{0}\sum_{j=-m_B}^{0} A_i B_j \int^{i+j} \boldsymbol{f}(t), \end{aligned} \tag{2.6}$$

where, term (b) follows from Lemma 4, while, by virtue of Lemma 5 and assumption $\boldsymbol{f} \in C^\infty_\mathrm{p} \cap C^{m_B-1}$, term (a) can be expressed as:

$$\sum_{i=1}^{n_A}\sum_{j=-m_B}^{0} A_i B_j \int^{i+j} \boldsymbol{f}(t) + \boldsymbol{w}(t), \quad \deg \boldsymbol{w} \leq n_A - 1. \tag{2.7}$$

Note that, if $m_B = 0$ then $\boldsymbol{w}(t) = \boldsymbol{0}$. Substituting expression (2.7) in (2.6) yields:

$$A(B\boldsymbol{f})(t) = \underbrace{\sum_{i=-m_A}^{n_A}\sum_{j=m_B}^{n_B} A_i B_j \int^{i+j}}_{(P_A\cdot P_B)(\int)} \boldsymbol{f}(t) + \boldsymbol{w}(t), \quad t \in \mathbb{R}\setminus S_{\boldsymbol{f}}^{(m_A+m_B)}, \; \deg(\boldsymbol{w}) \leq n_A - 1. \tag{2.8}$$

□

2.3 Behavior of input-output representations

We consider the linear time-invariant system Σ described by the differential equation:

$$P(D)\boldsymbol{y}(t) = Q(D)\boldsymbol{u}(t), \tag{2.9}$$

where $\boldsymbol{u}, \boldsymbol{y}$ denote the system's input and output respectively and $P(D), Q(D)$ are constant-coefficient differential operators associated to the polynomial matrices $P(s) = \sum_{i=0}^{n_P} P_i s^i \in \mathbb{R}^{p\times p}[s]$, $Q(s) = \sum_{i=0}^{n_Q} Q_i s^i \in \mathbb{R}^{p\times m}[s]$ with $\det P(s) \neq 0$ and $P(s)$, $Q(s)$ left coprime (cf. Appendix A). We say that (2.9) is the differential equation associated to $\big(P(s), Q(s)\big)$. The matrix transfer function of Σ is

$$H(s) = P^{-1}(s)Q(s) \in \mathbb{R}^{p\times m}(s) \tag{2.10}$$

which we assume to be strictly proper so that $n_P > n_Q$.

A *strong solution* (or classical solution) of (2.9) is a pair $(\boldsymbol{u}, \boldsymbol{y}) \in C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) \times C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^p)$ such that all involved derivatives exist. However, in order to allow for discontinuities in the input and output of Σ, this concept of solution must be generalized. Inspired by [65], we adopt the following definition of *weak solution*.

Definition 9. *A pair $(\boldsymbol{u}, \boldsymbol{y}) \in C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) \times C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^p)$ is a weak solution of the differential equation (2.9) if there exists $\boldsymbol{g}(t) \in \mathbb{R}^p[t]$, $\deg \boldsymbol{g} \leq n_P - 1$ such that:*

$$\sum_{i=0}^{n_P} P_i \int^{n_P - i} \boldsymbol{y}(t) = \sum_{i=0}^{n_Q} Q_i \int^{n_P - i} \boldsymbol{u}(t) + \boldsymbol{g}(t), \quad t \in \mathbb{R} \tag{2.11}$$

Remark 1. *In our definition of weak solution, the integral equation (2.11) must be satisfied for all $t \in \mathbb{R}$. This is different from the weak solution definition reported in [65] where the integral equation could be false over a sparse set (see the discussion following Example 2.3.5 in [65]).*

It should be noted that every strong solution is a weak solution (in order to see this, simply integrate (2.9) n_P times). However, the converse is generally not true. Indeed, differently from (2.9), equation (2.11) does not impose any additional requirements on the continuity order of $\boldsymbol{u}$ (actually, as we will see in Chapter 3, equation

(2.11) introduces some constraints among the continuity orders of the input and of the output).

By applying the asterisk operator $*$ on $[P(s)\ Q(s)]$ (see the notation paragraph at the beginning of this chapter) a more compact way to express the integral equation (2.11) can be obtained. Note that $P^*(s) = \sum_{i=0}^{n_P} P_i s^{n_P - i}$ and $Q^*(s) = \sum_{i=0}^{n_Q} Q_i s^{n_P - i}$ so that (2.11) can be written as:

$$P^*(\textstyle\int)\boldsymbol{y}(t) = Q^*(\textstyle\int)\boldsymbol{u}(t) + \boldsymbol{g}(t), \quad t \in \mathbb{R}. \tag{2.12}$$

Property 1. *Let* $(\boldsymbol{u}, \boldsymbol{y})$ *be a solution of (2.11) with* $\boldsymbol{g}(t) = [g_1(t) \dots g_p(t)]^\top$,

$$g_i(t) = \sum_{k=0}^{n_P - 1} g_{i,k} t^k, \quad i = 1, \dots, p.$$

Then, $g_i(t)$ *is such that* $g_{i,k} = 0,\ k = 0, 1, \dots, n_P - n_P^i - 1$ *where* n_P^i *is the i-th row degree of* $P(s)$ *(see Appendix A).*

Proof. Let the i-th row of (2.11) be such that $n_P^i < n_P$. Then, this row can be expressed as:

$$\sum_{k=0}^{n_P} \boldsymbol{p}_k^i \int^{n_P - k} \boldsymbol{y}(t) = \sum_{k=0}^{n_Q} \boldsymbol{q}_k^i \int^{n_P - k} \boldsymbol{u}(t) + \sum_{k=0}^{n_P - 1} g_{i,k} t^k, \tag{2.13}$$

where $\boldsymbol{p}_k^i \in \mathbb{R}^{1 \times p}$, $\boldsymbol{q}_k^i \in \mathbb{R}^{1 \times m}$ denote the i-th row of P_k and Q_k respectively. By definition of row degree, $\boldsymbol{p}_k^i = \boldsymbol{0}$, $k = n_P, n_P - 1, \dots, n_P^i + 1$. Furthermore, since $P^{-1}(s)Q(s)$ is strictly proper, it follows that $\boldsymbol{q}_k^i = \boldsymbol{0}$, $k = n_P, n_P - 1, \dots, n_P^i$. Then, (2.13) becomes:

$$\sum_{k=0}^{n_P^i} \boldsymbol{p}_k^i \int^{n_P - k} \boldsymbol{y}(t) = \sum_{k=0}^{n_P^i - 1} \boldsymbol{q}_k^i \int^{n_P - k} \boldsymbol{u}(t) + \sum_{k=0}^{n_P - 1} g_{i,k} t^k, \tag{2.14}$$

Evaluating the previous expression at $t = 0$ yields $g_{i,0} = 0$. By Lemma 3 $\int^{n_P - n_P^i} \boldsymbol{y} \in C^{n_P - n_P^i - 1}$ and $\int^{n_P - n_P^i + 1} \boldsymbol{u} \in C^{n_P - n_P^i}$. Hence, if $n_P - n_P^i > 1$, by differentiating (2.14) and evaluating the resulting expression at $t = 0$, yields $g_{i,1} = 0$. Iterating this reasoning, thereby differentiating a total of $n_P - n_P^i - 1$ times, results in $g_{i,k} = 0$, $k = 0, 1, \dots, n_P - n_P^i - 1$. $\square$

Definition 10. *The* behavior *of* Σ *is the set:*

$$\mathscr{B} := \{(\boldsymbol{u},\boldsymbol{y}) \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^{m}) \times C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^{p}) \mid (\boldsymbol{u},\boldsymbol{y}) \text{ is a weak solution of (2.9) } \}. \quad (2.15)$$

Remark 2. *We identify* Σ *with its behavior set. Instead, we simply consider the differential equation (2.9) as a representation of* Σ*. In other words, once the representation (2.9) is given, the system is uniquely specified by the set of weak solutions of (2.9). However, the same system can have more than one input-output differential equation representing it.*

While for single-input single-output systems the differential equation (2.9) is a unique input-output representation (up to a scaling factor) of the behavior set (and therefore of Σ), for multi-input multi-output systems the same behavior can be represented by (apparently) infinite different differential equations. To see this, consider a second linear time-invariant system Σ_U described by the differential equation

$$Z(D)\boldsymbol{y}(t) = S(D)\boldsymbol{u}(t) \quad (2.16)$$

where $Z(D)$, $S(D)$ are constant-coefficient differential operators associated to $Z(s) = U(s)P(s) \in \mathbb{R}^{p\times p}[s]$ and $S(s) = U(s)Q(s) \in \mathbb{R}^{p\times m}[s]$ respectively where $U(s)$ is assumed to be invertible. Observe that $Z^{-1}(s)S(s) = P^{-1}(s)Q(s) = H(s)$ so that, by the hypothesis of strict properness of $H(s)$, it follows $n_Z > n_S$. Then, we denote by $\mathscr{B}_U$ the behavior set of Σ_U, i.e.:

$$\begin{aligned}\mathscr{B}_U := \{(\boldsymbol{u},\boldsymbol{y}) \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^{m}) \times C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^{p}) \mid \exists \boldsymbol{g}_U(t) \in \mathbb{R}^{p}[t],\ \deg \boldsymbol{g}_U \le n_Z - 1 :\\ Z^{*}\Big(\int\Big)\boldsymbol{y}(t) = S^{*}\Big(\int\Big)\boldsymbol{u}(t) + \boldsymbol{g}_U(t), \quad t \in \mathbb{R}\},\end{aligned} \quad (2.17)$$

where $Z^{*}(s) = s^{n_Z}U(\frac{1}{s})P(\frac{1}{s})$ and $S^{*}(s) = s^{n_Z}U(\frac{1}{s})S(\frac{1}{s})$.

Theorem 1. *Let* $U(s) \in \mathbb{R}^{p\times p}[s]$ *be any unimodular matrix then* $\mathscr{B} = \mathscr{B}_U$*, i.e.* $\Sigma = \Sigma_U$*.*

Theorem 1 means that the behavior of the differential equation associated to the pair $\big(P(s),Q(s)\big)$ is the same as that of the differential equation associated to the pair $\big(U(s)P(s),U(s)Q(s)\big)$ where $U(s)$ is a unimodular matrix.

Proof. We prove the result by mutual inclusion.
($\mathscr{B} \subset \mathscr{B}_U$):
Let $(\boldsymbol{u},\boldsymbol{y}) \in \mathscr{B}$. First observe that $Z^*(s) = K(s)P^*(s)$, $S^*(s) = K(s)Q^*(s)$ where $K(s) = s^{n_Z - n_P}U(\frac{1}{s})$. The constant-coefficient differential-integral operator associated to $K(s)$ is $K(\int) = \sum_{i=-m_K}^{n_K} K_i \int^i$, where, if $n_Z - n_P > 0$ then $n_K \leq n_Z - n_P$ otherwise if $n_Z - n_P \leq 0$ then $n_K = 0$ and $K_i = 0$, $i = n_Z - n_P + 1, \ldots, 0$. It follows that

$$\deg\Big(K(\int)\boldsymbol{g}\Big) \leq n_Z - 1, \quad \deg \boldsymbol{g} \leq n_P - 1. \tag{2.18}$$

Next, by noting that $m_{P^*}, m_{Q^*} = 0$, $\boldsymbol{u},\boldsymbol{y} \in C_{\mathrm{p}}^{\infty} \cap C^{-1}$, $n_K - 1 \leq n_Z - 1$ we can employ Proposition 1 to obtain:

$$K(\int)\Big(P^*(\int)\boldsymbol{y}\Big)(t) = Z^*(\int)\boldsymbol{y}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{y}}^{(m_K)}, \tag{2.19}$$

$$K(\int)\Big(Q^*(\int)\boldsymbol{u}\Big)(t) = S^*(\int)\boldsymbol{u}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{u}}^{(m_K)}. \tag{2.20}$$

Since by assumption $(\boldsymbol{u},\boldsymbol{y})$ satisfies (2.12), it follows from (2.18)-(2.20):

$$Z^*(\int)\boldsymbol{y}(t) = S^*(\int)\boldsymbol{u}(t) + \boldsymbol{g}_U(t),\ t \in \mathbb{R} \setminus (S_{\boldsymbol{y}}^{(m_K)} \cup S_{\boldsymbol{u}}^{(m_K)}), \quad \deg \boldsymbol{g}_U(t) \leq n_Z - 1, \tag{2.21}$$

where $\boldsymbol{g}_U(t) := K(\int)\boldsymbol{g}(t)$. We now show that (2.21) actually holds for any $t \in \mathbb{R}$. This can be done through a continuity argument. Take $t_0 \in S_{\boldsymbol{y}}^{(m_K)} \cup S_{\boldsymbol{u}}^{(m_K)}$ then

$$Z^*(\int)\boldsymbol{y}(t_0) = Z^*(\int)\boldsymbol{y}(t_0^+) = S^*(\int)\boldsymbol{u}(t_0^+) + \boldsymbol{g}_U(t_0^+) = S^*(\int)\boldsymbol{u}(t_0) + \boldsymbol{g}_U(t_0).$$

($\mathscr{B}_U \subset \mathscr{B}$):
This part of the proof is analogous to the previous one and is therefore only sketched. Indeed, since $P^*(s) = K^{-1}(s)Z^*(s)$, $Q^*(s) = K^{-1}(s)S^*(s)$, instead of working with the operator associated to $K(s)$, this time we deal with the operator $W(\int) = \sum_{i=m_W}^{n_W} W_i \int^i$ associated to $W(s) := K^{-1}(s) = s^{n_P - n_Z}U^{-1}(\frac{1}{s})$. It is important to note that, since $U(s)$ is unimodular, the inverse of $U(s)$ is still a polynomial matrix which implies that we can still come up with a bound on n_W as done in the previous part for n_K. Then, using Proposition 1, in the same way as done above, allows to conclude the proof. □

Remark 3. *The previous result is the analogous of Theorem 2.5.4 presented in [65]. However, remark that the proof we present here is explicit, i.e. our definition of weak solution, coupled with the properties of* C_{p}^{∞}*, allow us to* explicitly *show that the integral equation defining weak solutions for one differential equation is satisfied by any pair in the behavior set of the other differential equation. This is opposite to [65] where the proof is presented only for the* C^{∞}*-part of the behavior set and then extended to the rest of the behavior via convergence arguments.*

In Definition 9, we introduced what it means for a pair $(\boldsymbol{u},\boldsymbol{y}) \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^m) \times C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^p)$ to be a weak solution of the differential equation (2.9). In practice, the rationale behind such definition can be considered to be the following. Given a strong (classical) solution of (2.9), we integrate expression (2.9) n_P times. It is clear, that such an operation results in the integral expression (2.11) which we then interpret as an equation generalizing (2.9). The reason for such a choice is twofold. Firstly, a strong solution is still a solution of (2.9) (i.e. the set of strong solutions is preserved), and secondly, working with an integral equation is advantageous as it allows to overcome the smoothness constraints imposed by (2.9) on the input $\boldsymbol{u}$ thereby allowing for solutions that are not continuous.

As we described above, the result of integrating equation (2.9) a total of n_P times, is captured by the asterisk $*$ operator applied on $[P(s)\ Q(s)]$. However, we should note that there is another way to end-up with an integral equation that preserves the set of strong solutions. This variant is captured by the bar $^-$ operator applied on $[P(s)\ Q(s)]$ (see the notation paragraph at the beginning of this chapter). Indeed, as we are going to see next, we can consider each of the scalar equations in (2.9) separately and integrate each of these a number of times equal to the highest-order derivative that appears in each of them, which, in general, is lower than n_P. Not surprisingly, this new definition of weak solution is equivalent to the one we introduced above.

Recall that

$$\bar{P}(s) = \begin{bmatrix} s^{n_P^1} & & & \\ & s^{n_P^2} & & \\ & & \ddots & \\ & & & s^{n_P^p} \end{bmatrix} P(\frac{1}{s}); \quad \bar{Q}(s) = \begin{bmatrix} s^{n_P^1} & & & \\ & s^{n_P^2} & & \\ & & \ddots & \\ & & & s^{n_P^p} \end{bmatrix} Q(\frac{1}{s}), \quad (2.22)$$

where $n_P^1, n_P^2, \ldots, n_P^p$ denote the row degrees of $P(s)$, $n_P = \max\{n_P^1, n_P^2, \ldots, n_P^p\}$.

Property 2. *Consider the set*

$$\bar{\mathscr{B}} := \{(\boldsymbol{u}, \boldsymbol{y}) \in C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m) \times C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^p) \mid \exists \bar{\boldsymbol{g}}(t) \in \mathbb{R}^p[t],\ \deg \bar{\boldsymbol{g}}_i(t) \le n_P^i - 1,\ i = 1, \ldots, p : \bar{P}(\int)\boldsymbol{y}(t) = \bar{Q}(\int)\boldsymbol{u}(t) + \bar{\boldsymbol{g}}(t), \quad t \in \mathbb{R}\}, \quad (2.23)$$

then $\mathscr{B} = \bar{\mathscr{B}}$.

Proof. ($\mathscr{B} \subset \bar{\mathscr{B}}$): Let $(\boldsymbol{u}, \boldsymbol{y}) \in \mathscr{B}$. Observe that $\bar{P}(s) = K(s)P^*(s)$ where $K(s) = \mathrm{diag}\{\frac{1}{s^{n_P - n_P^1}}, \frac{1}{s^{n_P - n_P^2}}, \ldots, \frac{1}{s^{n_P - n_P^p}}\}$. The constant-coefficient operator associated to $K(s)$ is $K(\int) = \sum_{i=-m_K}^{0} K_i \int^i$ where $n_K = 0$ because of the definition $n_P = \max_i\{n_P^i\}$. Our goal is to evaluate the expression:

$$K(\int)\big(P^*(\int)\boldsymbol{y}\big)(t) = K(\int)\big(Q^*(\int)\boldsymbol{u} + \boldsymbol{g}\big)(t) \quad (2.24)$$

In order to do so, we consider each term in (2.24) separately and by using Proposition 1, we obtain:

$$K(\int)P^*(\int)\boldsymbol{y}(t) = \bar{P}(\int)\boldsymbol{y}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{y}}^{(m_K)}, \quad (2.25)$$

$$K(\int)Q^*(\int)\boldsymbol{u}(t) = \bar{Q}(\int)\boldsymbol{u}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{u}}^{(m_K)}. \quad (2.26)$$

Next, remark that

$$K(\int)\boldsymbol{g}(t) = \begin{bmatrix} D^{n_P - n_P^1}\boldsymbol{g}_1 & D^{n_P - n_P^2}\boldsymbol{g}_2 & \ldots & D^{n_P - n_P^p}\boldsymbol{g}_p \end{bmatrix}^{\top}(t), \quad (2.27)$$

where, since by hypothesis $\deg \boldsymbol{g}(t) \leq n_P - 1$, it follows $\deg \big(K(\int)\boldsymbol{g}\big)_i(t) \leq n_P^i - 1$, $i = 1,\dots,p$. Setting $\bar{\boldsymbol{g}}(t) := K(\int)\boldsymbol{g}(t) \in \mathbb{R}^p[s]$, and substituting (2.25) - (2.26) in (2.24) we obtain:

$$\bar{P}(\int)\boldsymbol{y}(t) = \bar{Q}(\int)\boldsymbol{u}(t) + \bar{\boldsymbol{g}}(t), \quad t \in \mathbb{R} \setminus (S_{\boldsymbol{y}}^{(m_K)} \cup S_{\boldsymbol{u}}^{(m_K)}),\ \deg \bar{\boldsymbol{g}}_i \leq n_P^i - 1,\ i = 1,\dots,p. \tag{2.28}$$

Finally, by a continuity argument we can show that equation (2.28) is true for any $t \in \mathbb{R}$. Take $t_0 \in (S_{\boldsymbol{y}}^{m_K} \cup S_{\boldsymbol{u}}^{m_K})$ then:

$$\bar{P}(\int)\boldsymbol{y}(t_0) = \bar{P}(\int)\boldsymbol{y}(t_0^+) = \bar{Q}(\int)\boldsymbol{u}(t_0^+) + \bar{\boldsymbol{g}}(t_0^+) = \bar{Q}(\int)\boldsymbol{u}(t_0) + \bar{\boldsymbol{g}}(t_0).$$

($\bar{\mathscr{B}} \subset \mathscr{B}$): Let $(\boldsymbol{u},\boldsymbol{y}) \in \bar{\mathscr{B}}$. Clearly, $P^*(s) = W(s)\bar{P}(s)$, $Q^*(s) = W(s)\bar{Q}(s)$ where $W(s) := K^{-1}(s) = \mathrm{diag}\{s^{n_P - n_P^1}, s^{n_P - n_P^2}, \dots, s^{n_P - n_P^p}\}$. The integral operator associated to $W(s)$ is $W(\int) = \sum_{i=0}^{n_W} W_i \int^i$. Then, since only integral operators appear, the expression:

$$W(\int)\big(\bar{P}(\int)\boldsymbol{y}\big)(t) = W(\int)\big(\bar{Q}(\int)\boldsymbol{u} + \bar{\boldsymbol{g}}\big), \tag{2.29}$$

is certainly equivalent to:

$$P^*(\int)\boldsymbol{y}(t) = Q^*(\int)\boldsymbol{u} + \boldsymbol{g}(t), \quad t \in \mathbb{R},\ \deg \boldsymbol{g} \leq n_P - 1. \tag{2.30}$$

□

We apply the previous result in the proof of the next property.

Property 3. *Let* $(\boldsymbol{u},\boldsymbol{y}) \in \mathscr{B}$ *then* $\boldsymbol{y} \in C^0$.

Proof. Recall that there always exists a unimodular matrix $U(s) \in \mathbb{R}^{p \times p}[s]$ such that $P_{\mathtt{R}}(s) = U(s)P(s)$ is row reduced (see Definition 29 and Theorem 10 in Appendix A). Hence, it follows from Theorem 1 that the behavior of (2.9) is the same as that of

$$P_{\mathtt{R}}(D)\boldsymbol{y}(t) = Q_{\mathtt{R}}(D)\boldsymbol{u}(t)$$

where $Q_{\mathtt{R}}(s) = U(s)Q(s)$. For simplicity of notation we will next drop the subscript R and we will refer to equation (2.9) assuming that $P(s)$ is row reduced.

Using Property 2, $(\boldsymbol{u},\boldsymbol{y}) \in C^\infty_{\mathrm{p}}(\mathbb{R},\mathbb{R}^m) \times C^\infty_{\mathrm{p}}(\mathbb{R},\mathbb{R}^p)$ is a weak solution of (2.9) if there exists $\bar{\boldsymbol{g}} \in \mathbb{R}^p[s]$, with $\deg \bar{g}_i \le n^i_P - 1$ such that:

$$\bar{P}(\int)\boldsymbol{y}(t) = \bar{Q}(\int)\boldsymbol{u}(t) + \bar{\boldsymbol{g}}(t), \quad t \in \mathbb{R}, \tag{2.31}$$

where expressions for $\bar{P}(s)$ and $\bar{Q}(s)$ are given in (2.22). Let us set $\bar{P}(s) = \sum_{i=0}^{n_P} \bar{P}_i s^{n_P - i}$, $\bar{Q}(s) = \sum_{i=0}^{n_P-1} \bar{Q}_i s^{n_P - i}$. Observe that

$$P(s) = \begin{bmatrix} s^{n_P^1} & & & \\ & s^{n_P^2} & & \\ & & \ddots & \\ & & & s^{n_P^p} \end{bmatrix} P_{\mathtt{hr}} + P_1(s)$$

where $P_{\mathtt{hr}}$ is the *row degree coefficient matrix* of $P(s)$ (see Appendix A) and $P_1(s)$ a suitable polynomial matrix whose degree of the i-th row is less than n^i_P. Hence, it follows that $\bar{P}_{n_P} = P_{\mathtt{hr}}$ and, since $P(s)$ is row reduced, $\bar{P}_{n_P}$ is invertible. This allows to write (2.31) as:

$$\boldsymbol{y}(t) = -\sum_{i=0}^{n_P-1} P_{\mathtt{hr}}^{-1}\bar{P}_i \int^{n_P-i} \boldsymbol{y}(t) + \sum_{i=0}^{n_P-1} P_{\mathtt{hr}}^{-1}\bar{Q}_i \int^{n_P-i} \boldsymbol{u}(t) + \bar{\boldsymbol{g}}(t), \quad t \in \mathbb{R}, \tag{2.32}$$

from which, by virtue of Lemma 3, the thesis follows. □

By comparing definitions of $\mathscr{B}$ and $\bar{\mathscr{B}}$ we see that, in defining weak solutions, what really matters is integrating each i-th row at least n^i_P times. Integrating more than n^i_P times will not alter the behavior set. Based on this observation, we introduce a third equivalent definition of weak solution for the differential equation (2.9). Let

$$K(s) := \mathrm{diag}\{s^k, \ldots, s^k\},$$
$$^kP^*(s) := K(s)P^*(s), \quad ^kQ^*(s) := K(s)Q^*(s).$$

where $k \in \mathbb{N}_{>0}$. Then, the following property follows.

Property 4. *Consider the set*

$$^k\mathscr{B} := \{(\boldsymbol{u},\boldsymbol{y}) \in C^\infty_{\mathrm{p}}(\mathbb{R},\mathbb{R}^m) \times C^\infty_{\mathrm{p}}(\mathbb{R},\mathbb{R}^p) \mid \exists\, {}^k\boldsymbol{g}(t) \in \mathbb{R}^p[t],\ \deg {}^k\boldsymbol{g} \le n_P + k - 1 : \\ {}^kP^*(\int)\boldsymbol{y}(t) = {}^kQ^*(\int)\boldsymbol{u}(t) + {}^k\boldsymbol{g}(t), \quad t \in \mathbb{R}\}, \tag{2.33}$$

then $\mathscr{B} = {}^k\mathscr{B}$.

Proof. Once again we perform the proof by mutual inclusion.

($\mathscr{B} \subset {}^k\mathscr{B}$): Since $(\boldsymbol{u},\boldsymbol{y}) \in \mathscr{B}$ then $P^*(\int)\boldsymbol{y}(t) = Q^*(\int)\boldsymbol{u}(t) + \boldsymbol{g}(t)$, $\deg \boldsymbol{g} \leq n_P - 1$, $t \in \mathbb{R}$. Clearly,

$$\begin{aligned}
&K(\int)\big(P^*(\int)\boldsymbol{y}\big) = {}^kP^*(\int)\boldsymbol{y},\\
&K(\int)\big(Q^*(\int)\boldsymbol{u}\big) = {}^kQ^*(\int)\boldsymbol{u},\\
&K(\int)\boldsymbol{g} = {}^k\boldsymbol{g}, \quad \deg {}^k\boldsymbol{g} \leq n_P + k + 1,
\end{aligned}$$

hence, ${}^kP^*(\int)\boldsymbol{y}(t) = {}^kQ^*(\int)\boldsymbol{u}(t) + {}^k\boldsymbol{g}(t)$, $t \in \mathbb{R}$, i.e. $(\boldsymbol{u},\boldsymbol{y}) \in {}^k\mathscr{B}$.

(${}^k\mathscr{B} \subset \mathscr{B}$): Recall that, if $(\boldsymbol{u},\boldsymbol{y}) \in {}^k\mathscr{B}$ then ${}^kP^*(\int)\boldsymbol{y}(t) = {}^kQ^*(\int)\boldsymbol{u}(t) + {}^k\boldsymbol{g}(t)$, $\deg {}^k\boldsymbol{g} \leq n_P + k - 1$, $t \in \mathbb{R}$. It follows from Proposition 1 that:

$$\begin{aligned}
&K^{-1}(\int)\big({}^kP^*(\int)\boldsymbol{y}\big)(t) = P^*(\int)\boldsymbol{y}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{y}}^{(k)},\\
&K^{-1}(\int)\big({}^kQ^*(\int)\boldsymbol{u}\big)(t) = Q^*(\int)\boldsymbol{u}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{u}}^{(k)},\\
&K^{-1}(\int){}^k\boldsymbol{g} = \boldsymbol{g}, \quad \deg \boldsymbol{g} \leq n_P - 1.
\end{aligned} \tag{2.34}$$

Hence, $P^*(\int)\boldsymbol{y}(t) = Q^*(\int)\boldsymbol{u}(t) + \boldsymbol{g}(t)$, $t \in \mathbb{R} \setminus \big(S_{\boldsymbol{y}}^{(k)} \cup S_{\boldsymbol{u}}^{(k)}\big)$. By means of the now usual continuity argument it is easy to show that the last relation is true for any $t \in \mathbb{R}$. □

We will employ this property in the next section.

2.4 Behavior of input-state-output representations

It is well known that we can always associate to the matrix transfer function $H(s)$ of Σ an *input-state-output* representation:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \tag{2.35}$$

$$\boldsymbol{y} = C\boldsymbol{x}, \tag{2.36}$$

such that $H(s) = C(sI - A)^{-1}B$ and where $\boldsymbol{x} \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^n)$ denotes the state of the system, $\boldsymbol{u} \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^m)$, $\boldsymbol{y} \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R}^p)$ are the input and output respectively, $A \in$

$\mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$. We assume that the pair (C,A) is observable and (A,B) is controllable with $\operatorname{rank} C = p$.

Remark that we introduced the input-state-output representation (2.35)-(2.36) by relying on the equality between transfer matrices $P^{-1}(s)Q(s) = C(sI-A)^{-1}B$. Clearly, this establishes a link between the input-output representation (2.9) of Σ and (2.35)-(2.36). However, it is at this point unclear whether the system represented by (2.35)-(2.36) is actually Σ itself. In order to address this problem, we should at first introduce what is the behavior of an input-state-output representation. This is done in what follows.

Since equation (2.36) is a static map, i.e. it does not contain any derivatives, we will for the moment focus only on (2.35). This differential equation can be written in the form:

$$(DI-A)\boldsymbol{x} = B\boldsymbol{u}. \tag{2.37}$$

Clearly, since $sI-A$ is invertible, and because $(sI-A)^{-1}B$ is strictly proper, we can interpret (2.35) as a differential equation with input $\boldsymbol{u}$ and output $\boldsymbol{x}$. Therefore, according to Definition 9 a pair $(\boldsymbol{u},\boldsymbol{x}) \in C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m) \times C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^n)$ is a weak solution of (2.35) if there exists a constant $\boldsymbol{c} \in \mathbb{R}^n$ such that

$$\boldsymbol{x}(t) = A\int \boldsymbol{x}(t) + B\int \boldsymbol{u}(t) + \boldsymbol{c}, \quad t \in \mathbb{R}. \tag{2.38}$$

Clearly, our definition of behavior set applies to (2.35)-(2.36). Hence, we are next going to introduce what is the behavior of this model. Before doing so, we should first note that we view $(\boldsymbol{u},\boldsymbol{y})$ as the *raison d'être* of (2.35)-(2.36) while we simply interpret $\boldsymbol{x}$ as an auxiliary variable. What we mean by this is that we are not directly interested in $\boldsymbol{x}$, nevertheless this variable is still taken into account as it allows us to describe the relation between $(\boldsymbol{u},\boldsymbol{y})$ in a convenient way, namely with a first order differential equation. For this reason, we call $(\boldsymbol{u},\boldsymbol{y})$ the *manifest variables* while we refer to $\boldsymbol{x}$ as the *latent variable*. This distinction between variables justifies the following definition of behavior.

Definition 11. *The* manifest behavior $\mathscr{B}_{\mathtt{i(s)o}}$ *of the system described by equations*

(2.35)-(2.36) is the set

$$\mathscr{B}_{\mathtt{i(s)o}} := \{(\boldsymbol{u},\boldsymbol{y}) \in C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m) \times C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^p) \mid \exists \boldsymbol{x} \in C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^n), \boldsymbol{c} \in \mathbb{R}^n : \\ (\boldsymbol{u},\boldsymbol{x}) \text{ is a solution of (2.38) and } \boldsymbol{y}(t) = C\boldsymbol{x}(t), \quad t \in \mathbb{R}\} \tag{2.39}$$

The next Theorem shows that $\mathscr{B} = \mathscr{B}_{\mathtt{i(s)o}}$.

Theorem 2. *Consider the input-output representation (2.9) where $P^{-1}(s)Q(s)$ is strictly proper and $P(s), Q(s)$ are left coprime matrices. Let the input-state-output representation (2.35)-(2.36) be a minimal realization of $P^{-1}(s)Q(s)$ with* $\operatorname{rank} C = p$. *Then, $\mathscr{B} = \mathscr{B}_{\mathtt{i(s)o}}$.*

Proof. First, consider the input-state-output representation. Since (C,A) is observable and $\operatorname{rank} C = p$, there exists a change of coordinates $\boldsymbol{x} = T\boldsymbol{w}$ such that the state-space representation (2.35)-(2.36) is transformed into the observability form

$$\dot{\boldsymbol{w}} = A_o\boldsymbol{w} + B_o\boldsymbol{u}, \tag{2.40}$$

$$\boldsymbol{y} = C_o\boldsymbol{w}, \tag{2.41}$$

where the expressions for matrices A_o, C_o are given in (B.9)-(B.12) and $B_o = T^{-1}B$ has no specific structure. Recall that we denote by $(\nu_1, \nu_2, \dots, \nu_p)$ the ordered set of Kronecker invariants of the pair (C,A), ν_{ij}, $i, j = 1, \dots, p$ are defined in (B.6) and $a_{ij,k}$, $k = 1, \dots, \nu_{ij}$ are the characteristic parameters of (A,C) (see Appendix B and in particular (B.5)).

It is clear that the manifest behavior of (2.40)-(2.41) is equal to $\mathscr{B}_{\mathtt{i(s)o}}$ (i.e. a change of coordinates does not affect the behavior set). Hence, we will now express $\mathscr{B}_{\mathtt{i(s)o}}$ in terms of representation (2.40)-(2.41). Let ν denote the maximum observability index of pair (A,C), i.e. $\nu := \max\{\nu_1, \nu_2, \dots, \nu_p\}$ and consider the integral equation:

$$\int^{\nu-1} \boldsymbol{w}(t) = A_o \int^{\nu} \boldsymbol{w}(t) + B_o \int^{\nu} \boldsymbol{u}(t) + \boldsymbol{\gamma} t^{\nu-1}, \quad t \in \mathbb{R}. \tag{2.42}$$

By virtue of Property 4 we can express $\mathscr{B}_{\mathtt{i(s)o}}$ as:

$$\mathscr{B}_{\mathtt{i(s)o}} = \{(\boldsymbol{u},\boldsymbol{y}) \in C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m) \times C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^p) \mid \exists \boldsymbol{w} \in C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^n), \boldsymbol{\gamma} \in \mathbb{R}^n : \\ (\boldsymbol{u},\boldsymbol{w}) \text{ is a solution of (2.42) and } \boldsymbol{y}(t) = C_o\boldsymbol{w}(t), \quad t \in \mathbb{R}\}. \tag{2.43}$$

We are now going to leverage the special structure of (C_o, A_o) in order to observe the state, i.e. our goal is to deduce an expression of $\boldsymbol{w}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{y}$ only. The i-th row of (2.41), $i = 1, \dots, p$, yields $y_i = w_{v_1+\dots+v_{i-1}+1}$ from which we deduce $y_i \in C^0$ and

$$\int^{v-1} w_{v_1+\dots+v_{i-1}+1} = \int^{v-1} y_i. \tag{2.44}$$

Next, we consider the $v_1 + \dots + v_{i-1} + 1$ row of the integral equation (2.42). Remark that we can compactly refer to this row as (see also (B.13)): $\boldsymbol{c}_o^i \int^{v-1} \boldsymbol{w}(t) = \boldsymbol{c}_o^i \big(A_o \int^v \boldsymbol{w}(t) + B_o \int^v \boldsymbol{u}(t) + \boldsymbol{\gamma} t^{v-1}\big)$. This yields:

$$\int^{v-1} w_{v_1+\dots+v_{i-1}+1}(t) = \int^{v} w_{v_1+\dots+v_{i-1}+2}(t) + \boldsymbol{c}_o^i B_o \int^{v} \boldsymbol{u}(t) + \boldsymbol{c}_o^i \boldsymbol{\gamma} t^{v-1}. \tag{2.45}$$

We substitute (2.44) in (2.45) and, since $v \geq v_i$ and $y_i \in C^0$, by differentiating the resulting expression we get:

$$\int^{v-1} w_{v_1+\dots+v_{i-1}+2}(t) = \int^{v-2} y_i(t) - \boldsymbol{c}_o^i B_o \int^{v-1} \boldsymbol{u}(t) - \boldsymbol{c}_o^i \boldsymbol{\gamma}(v-1) t^{v-2}. \tag{2.46}$$

By iterating this reasoning we end up with the following expression for each of the components of $\boldsymbol{w}$:

$$\begin{aligned}\int^{v-1} w_{v_1+\dots+v_{i-1}+k}(t) = \int^{v-k} y_i(t) - \boldsymbol{c}_o^i B_o \int^{v-k+1} \boldsymbol{u}(t) - \boldsymbol{c}_o^i A_o B_o \int^{v-k+2} \boldsymbol{u}(t) - \dots \\ - \boldsymbol{c}_o^i A_o^{k-2} B_o \int^{v-1} \boldsymbol{u}(t) - \sum_{j=0}^{k-2} \boldsymbol{c}_o^i A_o^j \boldsymbol{\gamma} \Big[\prod_{l=1}^{k-j-1} (v-l) \Big] t^{v-k+j},\end{aligned} \tag{2.47}$$

where $i = 1, \dots, p$, $v_0 := 0$, $k = 1, \dots, v_i$.

It is easy to verify that substituting expressions (2.47), and its integral, for $i = 1, \dots, p$, $k = 1, \dots, v_i - 1$ (remark that we excluded the case $k = v_i$) back into the $(v_1 + \dots + v_{i-1} + k)$-th row of (2.42) results into a set of identities. However, performing the same substitution into the p remaining rows of (2.42), i.e. the rows located at positions $v_1+\cdots+v_i$, $i=1,\dots,p$ results into:

$$\int^{v-v_i} y_i - \sum_{j=0}^{v_i-2} \boldsymbol{c}_o^i A_o^j B_o \int^{v-v_i+j+1} \boldsymbol{u} - \sum_{j=0}^{v_i-2} \boldsymbol{c}_o^i A_o^j \boldsymbol{\gamma} \Big[\prod_{l=1}^{k-j-1}(v-l)\Big] t^{v-k+j} =$$

$$\boldsymbol{a}_o^{v_1+\cdots+v_i} \underbrace{\begin{bmatrix} \int^v y_1 \\ \int^{v-1} y_1 - \boldsymbol{c}_o^1 B \int^v \boldsymbol{u} - \boldsymbol{c}_o^1 \boldsymbol{\gamma} t^{v-1} \\ \vdots \end{bmatrix}}_{\boldsymbol{w}} + \boldsymbol{c}_o^i A^{v_i-1} B_o \int^v \boldsymbol{u} + \boldsymbol{c}_o^i A^{v_i-1} \boldsymbol{\gamma} t^{v-1}, \tag{2.48}$$

where $\boldsymbol{a}_o^{v_1+\cdots+v_i}$ denotes the $v_1+\cdots+v_i$ row of A_o, i.e. $\boldsymbol{a}_o^{v_1+\cdots+v_i} = [\boldsymbol{a}_{ij}] \in \mathbb{R}^{1\times n}$, $j=1,\dots,p$ with

$$\boldsymbol{a}_{ij} = \begin{bmatrix} a_{ij,1} & a_{ij,2} & \dots & a_{ij,v_{ij}} & 0_{v_j-v_{ij}} \end{bmatrix} \in \mathbb{R}^{1\times v_j}. \tag{2.49}$$

It is now only a matter of appropriately re-ordering terms before recognizing that equation (2.48), $i=1,\dots,p$, is the $i-th$ row of:

$$P_{\mathrm{g}}^*(\int)\boldsymbol{y}(t) = Q_{\mathrm{g}}^*(\int)\boldsymbol{u}(t) + \boldsymbol{g}(t), \quad \deg \boldsymbol{g} \le v-1, \tag{2.50}$$

with $P_{\mathrm{g}}^*(s) = s^v P_{\mathrm{g}}(\frac{1}{s})$, $Q_{\mathrm{g}}^*(s) = s^v Q_{\mathrm{g}}(\frac{1}{s})$ where $P_{\mathrm{g}}(s)$ is given by (B.19)-(B.21) and $Q_{\mathrm{g}}(s)$ is described in (B.22)-(B.27). This means $(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s))$ is a pair of matrices in the canonical Beghelli-Guidorzi input-output form. Further, recall that (2.50) is the definition of weak solution (cf. Definition 9 and (2.12)) of the input-output equation

$$P_{\mathrm{g}}(D)\boldsymbol{y} = Q_{\mathrm{g}}(D)\boldsymbol{u}. \tag{2.51}$$

Then, it follows from (2.50) that $\mathscr{B}_{\mathtt{i(s)o}}$ coincides with the behavior of the differential equation associated to $(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s))$

It remains to be shown that $(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s))$ is the Beghelli-Guidorzi canonical form of $P(s), Q(s)$. To do so, remark that, since $P_{\mathrm{g}}(s)$ is row reduced, then $\deg\det P_{\mathrm{g}} = \sum_{i=1}^p v_i = n$. Furthermore, clearly, $H(s) = C(sI-A)^{-1}B = P_{\mathrm{g}}^{-1}(s)Q_{\mathrm{g}}(s)$. Hence, by Theorem 13 we deduce $(P_{\mathrm{g}}, Q_{\mathrm{g}})$ is left coprime. Then, since $H(s) = P_{\mathrm{g}}^{-1}(s)Q_{\mathrm{g}}(s) = P^{-1}(s)Q(s)$, by Theorem 14 there exits a unimodular matrix $U(s) \in \mathbb{R}^{p\times p}(s)$ such that $P_{\mathrm{g}}(s) = U(s)P(s)$, $Q_{\mathrm{g}}(s) = U(s)Q(s)$. Hence, $(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s))$ is the Beghelli-Guidorzi form of $P(s), Q(s)$. Therefore, by virtue of Theorem 1 we deduce that $\mathscr{B} = \mathscr{B}_{\mathtt{i(s)o}}$. □

Remark 4. *The construction of the input-output representation (2.51) starting from the state-space model (2.40)-(2.41) has been presented originally in [22]. There, it was assumed that all involved signals are strong solutions of the differential equation (2.40)-(2.41). Here, we extended the proof to the more general case in which weak solutions of (2.40)-(2.41) are considered.*

Remark 5. *The components of polynomial $\boldsymbol{g}(t)$ in (2.50) are of form*

$$g_i(t) = \sum_{k=\nu-\nu_i}^{\nu-1} g_{i,k} \Big[\prod_{j=1}^{\nu-k-1} (\nu - j) \Big] t^k, \tag{2.52}$$

where $g_{i,k} \in \mathbb{R}$ are determined by the entries of the product

$$M\boldsymbol{\gamma} = G = \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \\ \vdots \\ \boldsymbol{g}_p \end{bmatrix}, \quad \boldsymbol{g}_i = \begin{bmatrix} g_{i,\nu-1} \\ g_{i,\nu-2} \\ \vdots \\ g_{i,\nu-\nu_i} \end{bmatrix}, \tag{2.53}$$

where M is the invertible matrix given in (B.25)-(B.27). Note that, since M is structurally invertible, relations (2.52),(2.53) also allow to recover $\boldsymbol{\gamma}$ given a polynomial $\boldsymbol{g}(t) = [g_1(t)\ g_2(t) \dots g_p(t)]^\top$ where $g_i(t)$ are of form (2.52) (see also Property 1).

Theorem 2 means that, under the given assumptions, input-output and input-state-output representations are equivalent, i.e. they have the same behavior set. This means that we can switch between one or the other representation at will. We will exploit this result in Chapter 4.

2.5 Conclusions

In this chapter, we have introduced the space of vector-valued C_p^∞ functions. Then, we defined weak solutions together with the associated concept of C_p^∞ behavior for input-output representations. Furthermore, we showed that the behavior set admits three equivalent definitions. Although being equivalent, according to the intended usage, working with one definition could be more or less advantageous if compared with the others. In this regard, we showed that the representation of the behavior set given in Property 2 is especially useful when one needs an expression of $\boldsymbol{y}$, which, for instance, is needed when studying continuity properties of $(\boldsymbol{u}, \boldsymbol{y})$.

In the last part of the chapter, we extended the notion of C_{p}^{∞} behavior to input-state-output representations. In Theorem 2, we showed that, under natural assumptions, input-output and input-state-output representations are equivalent, i.e. they have the same behavior set. This result, will play a role in the derivation of the stable-inversion procedure described in Chapter 4.

Chapter 3

Input-output jumps for scalar systems

3.1 Introduction

Classical control theory relies on linear constant-coefficient differential equations and Laplace transforms for describing and analyzing the dynamics of control systems (cf. [9, 40, 16]). However, in explaining the transient response, classical treatments encounter difficulties or pitfalls in the case of non-zero (discontinuous) initial conditions [34]. In particular, it is well known that, when initial conditions at time 0^+ are available, the use of straightforward Laplace properties, such as e.g. the transform of the (usual) derivative function, allow to easily find the transform of the total response thereby allowing to obtain the explicit expression of the (output) response over $\mathbb{R}_{\geq 0}$. Nevertheless, initial conditions are usually known only at time 0^- (also called *pre-initial conditions*) and this causes a difficulty.

In certain cases, this difficulty is worked around by means of physical insights. For instance, this is the case of certain mechanical and electrical systems, where conditions at time 0^+ (also called *post-initial conditions*) can be derived from the pre-initial conditions. In other cases, an alternative tentative approach achieving the same goal consists in obtaining the post-initial conditions through the so-called *technique* *of impulse matching* as presented with an example in [57]. Nevertheless, the mainstream way to determine the total response is to directly use the pre-initial conditions (cf. e.g. [9]). This is achieved by using the $\mathscr{L}_-$ definition of the Laplace transform and by relying on generalized derivatives rather than 'usual' derivatives (see [28] and [34]). However, the use of generalized derivatives as done in this context appears somewhat unsatisfactory or convolute [21, 35, 1].

Here, by relying on the C_p^∞ behavioral approach introduced in Chapter 2, we derive a simple relation between the initial conditions at time 0^+ with those at time 0^-. We remark that working in C_p^∞ is of crucial importance for achieving this result. Indeed, the assumption $f \in C_p^\infty(\mathbb{R}, \mathbb{R})$ guarantees that right and left limits exists and are bounded everywhere on the time axis for f itself and for any of its derivatives $D^i f$, $\forall i \geq 1$. This, in turn, guarantees that initial conditions do exists and are bounded for any possible system trajectory. This situation is opposite to the more general $\mathcal{L}_1^{\mathrm{loc}}$ behavioral theory [65] whose function space of locally integrable functions does neither guarantee the existence nor the boundedness of left and right limits thereby preventing the development of the approach herein described.

Even though our results are valid for multivariable systems, for the sake of simplicity, we present our study for *scalar systems* only. Specifically, considering a system whose order and relative degree are n and $r = n - m$ respectively, we employ the C_p^∞ behavioral approach in order to relate the jump discontinuities of the input and its derivatives up to the order $m-1$ to the jump discontinuities of the output and its derivatives up to the order $n-1$ (cf. Proposition 2 and Corollary 1). Indeed, as we saw in the previous chapter, when jump discontinuities occur, the differential equation (2.9) cannot be satisfied in the usual sense over the entire time axis because the input and the output cannot be differentiated up to the required orders. To overcome this obstruction, in Chapter 2, we introduced the concept of weak solution (see Definition 9) where the integral equation (2.11) replaces the differential equation (2.9). In this way the *behavior* of the system is defined as the set of all input-output (signal) pairs that are weak solutions of the differential equation, i.e. the input-output pairs actually satisfy (in the usual sense) the corresponding integral equation over the entire time axis (cf. Definition 10).

Here, we deduce the input-output jump relations (3.4)-(3.5) by taking left and right limits on the *k*th derivative of the integral equation characterizing the weak solution at each given time instant, not only at the conventional origin time 0. These relations are also simplified in (3.11) by introducing a lower triangular Toeplitz matrix (defined by Markov parameters) that directly relates the output jumps to the input ones. The found relations permit to easily compute the output conditions at time 0^+ from the pre-initial conditions (involving both the input and the output at time 0^-) and the input conditions at time 0^+. As a possible application, we employ the found relations in order to solve the *initial conditions problem* (cf. Problem 1) in a behavioral setting, i.e. to determine the total response starting from an arbitrary system evolution.

Chapter organization: The rest of the chapter is organized as follows. In Section 2 we specialize some of the results introduced in Chapter 2 to the case of single-input single-output systems. Furthermore, we introduce some additional key notions of behavioral theory, such as e.g. the input-output representation (cf. Theorem 3), which will be useful in solving Problem 1. The third section is focused on the main result, i.e. the input-output jump relations which are expressed in vector form by Proposition 2 and Corollary 1. The found relations are applied to solve the initial conditions problem (i.e. Problem 1) in Section 4. A result used in this solution is Corollary 2. It provides the initial conditions of the free response at time 0^+ from the knowledge of the pre-initial conditions. An example of solution of Problem 1 is reported in Section 5.

The content of this chapter is based on [31].

Notation: The Laplace transform of f is denoted by $\mathscr{L}[f(t)]$ or $F(s)$.

3.2 Preliminaries

As in the previous chapter, we consider the linear time-invariant system Σ described by the differential equation (2.9). However, since we are here interested in single-input single-output systems only, in order to favor readability, we express (2.9) in a more suitable notation for scalar systems.

Let Σ be the continuous-time scalar linear system with input $u \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R})$ and output $y \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R})$ described by the differential equation

$$a(D)y(t) = b(D)u(t), \tag{3.1}$$

where $a(D)$, $b(D)$ are the constant-coefficients differential operators associated to the coprime polynomials $a(s) = \sum_{i=0}^{n} a_i s^i$, $b(s) = \sum_{i=0}^{m} b_i s^i \in \mathbb{R}[s]$ with $m \leq n$. The transfer function of Σ is

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}.$$

According to Definition 9, a pair $(u,y) \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R})^2$ is a weak solution of

$$\sum_{i=0}^{n} a_i D^i y(t) = \sum_{i=0}^{m} b_i D^i u(t) \tag{3.2}$$

if there exists a polynomial $g \in \mathbb{R}[s]$ with $\deg g \leq n-1$ for which the integral equation

$$\sum_{i=0}^{n} a_i \int^{n-i} y(t) = \sum_{i=0}^{m} b_i \int^{n-i} u(t) + g(t) \tag{3.3}$$

is satisfied for all $t \in \mathbb{R}$. Following Definition 10 the behavior of Σ is

$$\mathscr{B} := \{\, (u,y) \in C_{\mathrm{p}}^{\infty}(\mathbb{R},\mathbb{R})^2 : (u,y) \text{ is a weak solution of (3.2)} \,\}.$$

A property on the continuity order of the output is the following.

Property 5. *Let $(u,y) \in \mathscr{B}$, then $y \in C^{r-1}$.*

The poles of Σ are the roots (with multiplicity) of $a(s)$. The associated concept of *pole modes* can be then introduced.

Definition 12. *[Pole modes of Σ] Given a real (complex) pole $p \in \mathbb{R}$ ($p = \sigma \pm j\omega \in \mathbb{C}$) with multiplicity μ, the associated modes are*

$$e^{pt}, te^{pt}, \ldots, t^{\mu-1}e^{pt} \quad \big(e^{\sigma t}\cos(\omega t), e^{\sigma t}\sin(\omega t), \ldots \\ , t^{\mu-1}e^{\sigma t}\cos(\omega t), t^{\mu-1}e^{\sigma t}\sin(\omega t)\big).$$

All the pole modes of Σ are denoted by $m_i(t)$, $i = 1, \ldots, n$.

Denote by $h(t)$ the analytical extension over $\mathbb{R}$ of $\mathscr{L}^{-1}[H(s)]$. Then, an explicit (input-output) representation of the system behavior is the following.

Theorem 3. *[Behavior's input-output representation] Define the following set*

$$\mathscr{B}_{\mathtt{i/o}} := \{(u,y) \in C_{\mathtt{p}}^{\infty}(\mathbb{R})^2 : y(t) = \int_0^t h(t-v)u(v)dv + \sum_{i=1}^{n} f_i m_i(t),\ t \in \mathbb{R},\ f_i \in \mathbb{R}\}.$$

Then $\mathscr{B}_{\mathtt{i/o}} = \mathscr{B}$.

For a proof of Proposition 5 and Theorem 3 see [12].

3.3 Input-Output jump relations

Form a causal viewpoint, jump discontinuities on the input and its derivatives cause jump discontinuities on the output and its derivatives. A set of algebraic relations between them is presented in the following result.

Proposition 2. *[Input-output jump relations] Let be given any* $(u,y) \in \mathscr{B}$. *Then, at any time* $t \in \mathbb{R}$ *the possible input and output jump discontinuities (up to the* $(m-1)$*th and* $(n-1)$*th derivative order respectively) satisfy the following relations:*

$$y^{(i)}(t^+) = y^{(i)}(t^-), \quad i = 0,1,\ldots,r-1 \quad \textit{(void if } r = 0\textit{)} \tag{3.4}$$

$$\begin{bmatrix} a_n & 0 & \dots & 0 \\ a_{n-1} & a_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{r+1} & \dots & a_{n-1} & a_n \end{bmatrix} \begin{bmatrix} y^{(r)}\big(t^+\big) - y^{(r)}\big(t^-\big) \\ y^{(r+1)}\big(t^+\big) - y^{(r+1)}\big(t^-\big) \\ \vdots \\ y^{(n-1)}\big(t^+\big) - y^{(n-1)}\big(t^-\big) \end{bmatrix} =$$

$$\begin{bmatrix} b_m & 0 & \dots & 0 \\ b_{m-1} & b_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ b_1 & \dots & b_{m-1} & b_m \end{bmatrix} \begin{bmatrix} u\big(t^+\big) - u\big(t^-\big) \\ u^{(1)}\big(t^+\big) - u^{(1)}\big(t^-\big) \\ \vdots \\ u^{(m-1)}\big(t^+\big) - u^{(m-1)}\big(t^-\big) \end{bmatrix}$$

(void if $m = 0$). (3.5)

Proof. Consider any pair $(u, y) \in \mathscr{B}$ and choose any time $t \in \mathbb{R}$. Then, relations (3.4) derive straightforwardly from Proposition 5. (The output has always continuity order $r-1$.) Pair (u, y) satisfies the integral equation (3.3) which can be written as

$$\sum_{i=0}^{n} a_i \int^{n-i} y(v) = \sum_{i=0}^{n} b_i \int^{n-i} u(v) + g(v), \quad v \in \mathbb{R} \tag{3.6}$$

having set $b_i = 0$, $i = m+1, \dots, n$ for the case $r \geq 1$.

Consider any $k \in \mathbb{N}$, $0 \leq k \leq n-1$ and take the kth derivative of (3.6). By virtue of Lemma 4 there exists a neighborhood of t, $I_k(t)$, for which this derivative can be expressed as

$$\begin{aligned} &\sum_{i=0}^{n} a_i \int^{n-i-k} y(v) = \sum_{i=0}^{n} b_i \int^{n-i-k} u(v) + D^k g(v), \\ &v \in I_k(t) \setminus \{t\}. \end{aligned} \tag{3.7}$$

By some algebraic manipulations and by taking into account that an integral operator having a negative exponent is actually a derivative operator (cf. Definition 5), (3.7) is written as

$$\sum_{i=0}^{k} a_{n-i} D^{k-i} y(v) - \sum_{i=0}^{k} b_{n-i} D^{k-i} u(v) = - \sum_{i=0}^{n-(k+1)} a_i \int^{n-i-k} y(v) + \sum_{i=0}^{n-(k+1)} b_i \int^{n-i-k} u(v) + D^k g(v), \quad v \in I_k(t) \setminus \{t\}. \tag{3.8}$$

By Lemma 3, the right-hand side of (3.8) is a sum of continuous functions. Hence, take the right and left limits of (3.8) at t to obtain

$$\begin{cases} \sum_{i=0}^{k} a_{n-i} D^{k-i} y(t^+) - \sum_{i=0}^{k} b_{n-i} D^{k-i} u(t^+) = c \\ \sum_{i=0}^{k} a_{n-i} D^{k-i} y(t^-) - \sum_{i=0}^{k} b_{n-i} D^{k-i} u(t^-) = c \end{cases} \tag{3.9}$$

with $c \in \mathbb{R}$,

$$c = - \sum_{i=0}^{n-(k+1)} a_i \int^{n-i-k} y(t) + \sum_{i=0}^{n-(k+1)} b_i \int^{n-i-k} u(t) + D^k g(t).$$

Form the difference of the equations in (3.9) we eventually have

$$\sum_{i=0}^{k} a_{n-i} \left(D^{k-i} y(t^+) - D^{k-i} y(t^-) \right) = \sum_{i=0}^{k} b_{n-i} \left(D^{k-i} u(t^+) - D^{k-i} u(t^-) \right), \quad 0 \le k \le n-1. \tag{3.10}$$

When $r = 0$, note that the above relations (3.10) are the scalar version of the vector relation in (3.5).

Now, consider the case $r \geq 1$. Relations (3.10) are still valid, in particular on the index subset $r \leq k \leq n-1$. By taking into account that $D^i y(t^+) - D^i y(t^-) = 0$ and $b_{n-i} = 0$, $i = 0, \ldots, r-1$ (cf. (3.4) and the assumption on (3.6) respectively), these relations can be simplified as follows:

$$\sum_{i=0}^{k-r} a_{n-i} \left(D^{k-i} y(t^+) - D^{k-i} y(t^-) \right) = \sum_{i=r}^{k} b_{n-i} \left(D^{k-i} u(t^+) - D^{k-i} u(t^-) \right), \; r \le k \le n-1.$$

The above scalar relations can be then rewritten in vector form to obtain the input-output jump relations (3.5) and this concludes the proof. □

Remark 6. *The set of relations (3.4) and the vector equation (3.5) form a set of n scalar linear relations. When $r = 0$ the set (3.4) is empty whereas when $m = 0$ equation (3.5) is absent.*

Let us introduce the Markov parameters of Σ, h_i, $i \in \mathbb{N}$ for which $H(s) = \sum_{i=0}^{\infty} h_i s^{-i}$ (see [10]). The next result is a useful technical lemma. (Without loss of generality we assume $a_n = 1$ in the following.)

Lemma 6. *The first $n+1$ Markov parameters of Σ can be recursively obtained by means of the following relations:*

$$
\begin{array}{l}
h_i = 0, \quad 0 \leq i \leq r-1, \quad \textit{(void if } r = 0\textit{)}; \\
\begin{cases} h_r = b_m \\ h_{r+i} = b_{m-i} - \sum_{j=1}^{i} a_{n-j} h_{r+i-j}, \quad 1 \leq i \leq m \end{cases}
\end{array} .
$$

For brevity a proof is omitted. (It can be found in [10] when $r = 1$.)

Corollary 1. *Let be given any (u, y) in $\mathscr{B}$ and any time $t \in \mathbb{R}$. Then, the input-output jump relations (3.5) can also be expressed as*

$$
\begin{bmatrix} y^{(r)}(t^+) - y^{(r)}(t^-) \\ y^{(r+1)}(t^+) - y^{(r+1)}(t^-) \\ \vdots \\ y^{(n-1)}(t^+) - y^{(n-1))}(t^-) \end{bmatrix} =
\begin{bmatrix} h_r & 0 & \dots & 0 \\ h_{r+1} & h_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{n-1} & \dots & h_{r+1} & h_r \end{bmatrix}
\begin{bmatrix} u(t^+) - u(t^-) \\ u^{(1)}(t^+) - u^{(1)}(t^-) \\ \vdots \\ u^{(m-1)}(t^+) - u^{(m-1)}(t^-) \end{bmatrix} . \tag{3.11}
$$

A proof of the above corollary is omitted for brevity.

Remark 7. *It is worth stressing that the input-output jump relations presented in Proposition 2 and Corollary 1 hold at any instant of the time axis not only at the origin time 0. From this viewpoint, the time 0 has nothing special. Indeed, jump discontinuities in the input-output evolution can happen at any time instant for which the found relations still hold. Moreover, if at a given time t the input has not discontinuities up to the derivative of order* $m-1$*, i.e.* $u^{(i)}(t^-) = u^{(i)}(t^+)$*,* $i = 0,1,\dots,m-1$ *then by the input-output relations (3.4)-(3.5) it follows in turn that the output too has no discontinuities up to the derivative of order* $n-1$*, i.e.* $y^{(i)}(t^-) = y^{(i)}(t^+)$*,* $i = 0,1,\dots,n-1$*. When this happens for some neighborhood N of t the input* $u \in C^{m-1}(N,\mathbb{R})$ *and* $y \in C^{n-1}(N,\mathbb{R})$ *(cf. Proposition 4 in [12]).*

3.4 Application to the initial conditions problem

As an application of the found input-output jump relations (3.4)-(3.5) we consider the following initial conditions problem.

Problem 1. *[The initial conditions problem] Let be given any input-output pair* $(u_0, y_0) \in \mathscr{B}$ *whose signals* u_0*,* y_0 *are known for* $t < 0$ *and suppose that, at time* $t = 0$*, a new input* $u_1(t)$*,* $t \geq 0$ *is applied to* Σ*. Find the corresponding output* $y_1(t)$*,* $t \geq 0$.

We propose the following solution to this problem. Let the signals $u_0|u_1$, $y_0|y_1$ be defined as

$$u_0|u_1(t) := \begin{cases} u_0(t) \text{ if } t < 0 \\ u_1(t) \text{ if } t \geq 0 \end{cases}, \; y_0|y_1(t) := \begin{cases} y_0(t) \text{ if } t < 0 \\ y_1(t) \text{ if } t \geq 0 \end{cases}$$

so that evidently $(u_0|u_1, y_0|y_1) \in \mathscr{B}$. By Theorem 3 there exist real coefficients f_i, $i = 1,\dots,n$ for which

$$y_0|y_1(t) = \int_0^t h(t-v)u_0|u_1(v)dv + \sum_{i=1}^{n} f_i m_i(t), \quad t \in \mathbb{R}. \tag{3.12}$$

Define

$$y_{1\mathtt{d}}(t) := \int_0^t h(t-v)u_1(v)dv, t \geq 0 \tag{3.13}$$

and

$$y_{1\mathtt{e}}(t) := \sum_{i=1}^{n} f_i m_i(t),\ t \in \mathbb{R} \tag{3.14}$$

so that (3.12) implies

$$y_1(t) = y_{1\mathtt{d}}(t) + y_{1\mathtt{e}}(t),\ t \geq 0, \tag{3.15}$$

i.e. the *total response* of Σ is the sum of the *forced response* $y_{1\mathtt{d}}(t)$ and the *free* (or *natural*) *response* $y_{1\mathtt{e}}(t)$ (cf. [33]). The forced response is then determined by the convolution of $h(t)$ and $u_1(t)$ (or equivalently by $\mathscr{L}^{-1}[H(s)U_1(s)]$ with $U_1(s) := \mathscr{L}[u_1(t)]$) whereas the free response can be determined by means of the input-output jump relations (3.4)-(3.5). Indeed, relation (3.12) holds for any $u_1 \in C_{\mathtt{p}}^{\infty}(\mathbb{R}_{\geq 0})$, hence if $u_1(t) = 0$, $t \geq 0$ it follows that $(u_0|0, y_0|y_{1\mathtt{e}}) \in \mathscr{B}$. By applying Proposition 2 and Corollary 1 to pair $(u_0|0, y_0|y_{1\mathtt{e}})$ the following straightforward result is obtained.

Corollary 2. *Let us consider the assumptions of Problem 1. Then, the free response $y_{1\mathtt{e}}(t)$ (3.14) has initial conditions at time 0^+ given by the following relations:*

$$y_{1\mathtt{e}}^{(i)}(0^+) = y_0^{(i)}(0^-),\ i = 0, 1, \ldots, r-1 \quad \textit{(void if } r = 0\textit{)} \tag{3.16}$$

$$\begin{bmatrix} y_{1\mathtt{e}}^{(r)}(0^+) \\ y_{1\mathtt{e}}^{(r+1)}(0^+) \\ \vdots \\ y_{1\mathtt{e}}^{(n-1)}(0^+) \end{bmatrix} = \begin{bmatrix} y_0^{(r)}(0^-) \\ y_0^{(r+1)}(0^-) \\ \vdots \\ y_0^{(n-1)}(0^-) \end{bmatrix} - \begin{bmatrix} h_r & 0 & \ldots & 0 \\ h_{r+1} & h_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{n-1} & \ldots & h_{r+1} & h_r \end{bmatrix} \begin{bmatrix} u_0(0^-) \\ u_0^{(1)}(0^-) \\ \vdots \\ u_0^{(m-1)}(0^-) \end{bmatrix} \quad \textit{(void if } m = 0\textit{)} \tag{3.17}$$

Proof. Relations (3.16) are just those in (3.4) applied to the output $y_0|y_{1\mathtt{e}}$ at time 0. On the other hand, the vector equality (3.17) follows from that in (3.11) by taking into account that the initial conditions of the input $u_0|0$ at time 0^+ are all zeros. □

A way to determine the free response is to compute the coefficients f_i's appearing in (3.14). Take the derivatives of $y_{1\mathrm{e}}(t)$ up to the order $n-1$ and evaluate them at time 0^+ to obtain the following algebraic linear equation in the unknowns f_i's:

$$\begin{bmatrix} m_1(0^+) & \dots & m_n(0^+) \\ m_1^{(1)}(0^+) & \dots & m_n^{(1)}(0^+) \\ \vdots & & \vdots \\ m_1^{(n-1)}(0^+) & \dots & m_n^{(n-1)}(0^+) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} y_{1\mathrm{e}}(0^+) \\ y_{1\mathrm{e}}^{(1)}(0^+) \\ \vdots \\ y_{1\mathrm{e}}^{(n-1)}(0^+) \end{bmatrix}. \tag{3.18}$$

The right-hand side of (3.18) is computed by means of Corollary 2. Then, the f_i's can be uniquely determined because the coefficient matrix in (3.18), denoted by M in the following, is always nonsingular. For example, if all the poles are simple and real (i.e. the roots of $a(s)$ are $p_i \in \mathbb{R}$, $i = 1, \dots, n$ and $p_i \neq p_j$ if $i \neq j$) M becomes the classic Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_n \\ \vdots & \vdots & & \vdots \\ p_1^{n-1} & p_2^{n-1} & \dots & p_n^{n-1} \end{bmatrix}$$

whose determinant is $\prod_{1\leq i<j\leq n}(p_i - p_j) \neq 0$. When there are poles (real or complex) with multiplicities, M becomes a generalized Vandermonde matrix which is still nonsingular (cf. [29]).

Another way to find the free response (3.14) is the Laplace transform method. The pair $(u_0|0, y_0|y_{1\mathrm{e}})$ satisfies the integral equation (3.3). Take the the nth derivative of this integral equation on $[0, +\infty)$ to obtain

$$\sum_{i=0}^{n} a_i D^i y_{1\mathrm{e}}(t) = 0$$

and by applying the Laplace transform

$$\sum_{i=0}^{n} a_i s^i Y_{1\mathrm{e}}(s) - \sum_{i=1}^{n} a_i \sum_{j=0}^{i-1} y_{1\mathrm{e}}^{(i-1-j)}(0^+) s^j = 0$$

and eventually

$$Y_{1\mathtt{e}}(s) = \frac{\sum_{i=1}^{n}\sum_{j=0}^{i-1} a_i y_{1\mathtt{e}}^{(i-1-j)}(0^+) s^j}{a(s)}. \tag{3.19}$$

The free response is then given by

$$y_{1\mathtt{e}}(t) = \mathscr{L}^{-1}[Y_{1\mathtt{e}}(s)].$$

The explicit computation of this inverse Laplace transform can be routinely performed by partial fraction decomposition and subsequent application of Laplace table correspondences.

3.5 An example

Consider a system Σ having transfer function (cf. [47])

$$H(s) = -4\,\frac{(s-1)(s+1)}{(s+2)(s^2+s+2)}.$$

Its order and relative degree are $n = 3$ and $r = 1$ respectively. The following instance of the *initial conditions problem* is set (cf. Problem 1): Let $u_0(t) = sin(t)$ and $y_0(t) = 4\sqrt{\frac{2}{5}}sin\big(t - atan(3)\big)$, $t \in \mathbb{R}$ for which $(u_0, y_0) \in \mathscr{B}$ and suppose at time $t = 0$ a new input $u_1(t) = 1$, $t \geq 0$ is applied. Find the corresponding output $y_1(t)$, $t \geq 0$.

A solution to this problem can be given as follows (cf. Section 4). The output y_1 is the total response for which $y_1(t) = y_{1\mathtt{d}}(t) + y_{1\mathtt{e}}(t)$, $t \geq 0$ (cf. (3.15)). The computation of the forced response $y_{1\mathtt{d}}$ does not involve the initial conditions. In a customary way, it can be done by means of the convolution integral (3.13) or by a Laplace procedure. With the latter $y_{1\mathtt{d}}(t) = \mathscr{L}^{-1}[H(s)U_1(s)]$ with $U_1(s) = s^{-1}$ so that by partial fraction decomposition and inverse Laplace transform:

$$y_{1\mathtt{d}}(t) = 1 + \frac{3}{2}e^{-2t} - \frac{5}{2}e^{-\frac{1}{2}t}\cos(\frac{\sqrt{7}}{2}t) - \frac{9\sqrt{7}}{14}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{7}}{2}t). \tag{3.20}$$

The free response is $y_{1\mathtt{e}}(t) = f_1 m_1(t) + f_2 m_2(t) + f_3 m_3(t)$ (cf. (3.14) and Definition 12 on pole modes of Σ) and can be determined by means of Corollary 2. The pre-initial conditions, i.e. the initial conditions of signals u_0 and y_0 at 0^- are the following: $u_0(0^-) = 0$, $u_0^{(1)}(0^-) = 1$ and $y_0(0^-) = -\frac{12}{5}$, $y_0^{(1)}(0^-) = \frac{4}{5}$, $y_0^{(2)}(0^-) = \frac{12}{5}$. The transfer function can be expressed as $H(s) = \frac{-4s^2+4}{s^3+3s^2+4s+4}$ and by applying Lemma 6 the Markov parameters $h_0 = 0$, $h_1 = -4$, and $h_2 = 12$ are determined. Therefore, from (3.16) and (3.17) we obtain the free response initial conditions at 0^+:

$$y_{1\mathsf{e}}(0^+) = y_0(0^-) = -\frac{12}{5}, \tag{3.21}$$

and

$$\begin{bmatrix} y_{1\mathsf{e}}^{(1)}(0^+) \\ y_{1\mathsf{e}}^{(2)}(0^+) \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{12}{5} \end{bmatrix} - \begin{bmatrix} -4 & 0 \\ 12 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{32}{5} \end{bmatrix}. \tag{3.22}$$

The poles of Σ are -2 and $-\frac{1}{2} \pm j\frac{\sqrt{7}}{2}$ with associated modes $m_1(t) = e^{-2t}$, $m_2(t) = e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right)$, and $m_3(t) = e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right)$. The modes are used in defining the Vandermonde matrix M in (3.18) so as to compute the coefficients f_i's of the free response:

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -\frac{1}{2} & \frac{\sqrt{7}}{2} \\ 4 & -\frac{3}{2} & -\frac{\sqrt{7}}{2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ \frac{4}{5} \\ \frac{32}{5} \end{bmatrix} \implies \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -3 \\ \frac{1}{\sqrt{7}} \end{bmatrix};$$

hence,

$$y_{1\mathsf{e}}(t) = \frac{3}{5}e^{-2t} - 3e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) + \frac{1}{\sqrt{7}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right). \tag{3.23}$$

Alternatively, still using the found initial conditions at 0^+ in (3.21)-(3.22), the free response (3.23) can be directly determined (cf. (3.19)) by the inverse Laplace transform of

$$Y_{1\mathsf{e}}(s) = \frac{-\frac{12}{5}s^2 - \frac{32}{5}s - \frac{4}{5}}{s^3 + 3s^2 + 4s + 4}.$$

Eventually, the sum of the forced and free responses (3.20) and (3.23) gives the sought total response ($t \geq 0$):

$$y_1(t) = 1 + \frac{21}{10}e^{-2t} - \frac{11}{2}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{\sqrt{7}}{2}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right).$$

The following figures illustrate all the involved signals. Figure 3.1 plots the input $u_0|u_1(t)$. The forced and free responses are plotted in Figure 3.2 and Figure 3.3 plots the resulting output $y_0|y_1(t)$.

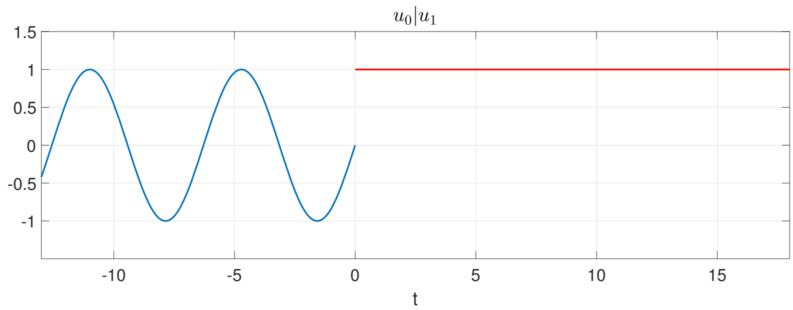

Figure 3.1: Plot of the input signal $u_0|u_1$ in the time interval $[-13, 18]$. The blue and red curves are the plots of $u_0(t) = \sin(t)$, $t \in [-13, 0)$ and $u_1(t) = 1$, $t \in [0, 18]$ respectively.

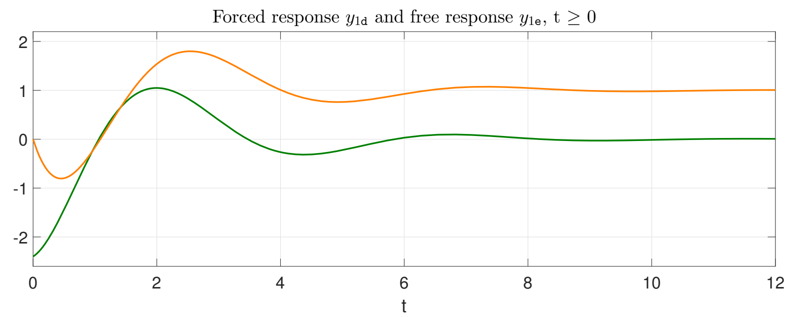

Figure 3.2: Plot of the forced response $y_{1\mathtt{d}}$ (in orange) and free response $y_{1\mathtt{e}}$ (in green). The sum of these signals yields y_1.

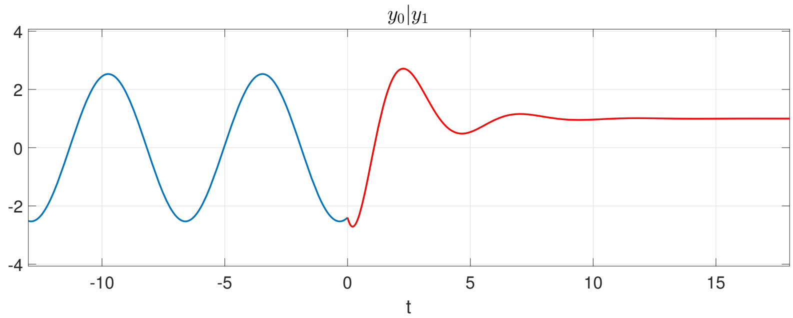

Figure 3.3: Plot of the output signal $y_0|y_1$ in the time interval $[-13, 18]$. The blue curve is $y_0(t)$, $t \in [-13, 0)$ whereas the red one is $y_1(t)$, $t \in [0, 18]$.

3.6 Conclusions

The jump discontinuities of the input vector $(u(t), u^{(1)}(t), \dots, u^{(m-1)}(t))$ cause jump discontinuities on the output vector $(y(t), y^{(1)}(t), \dots, y^{(n-1)}(t))$ at any given time $t \in \mathbb{R}$. Straightforward algebraic relations between them have been established in (3.4) and (3.5) (or (3.11)) for linear time-invariant scalar systems. Significantly, these relations have been obtained by means of a simplified behavioral approach that avoids generalized derivatives and ad hoc assumptions. With ease, the found input-output jump relations have direct application to solve the initial conditions problem. The present findings may be a useful complement in the classic control systems education.

Chapter 4

Stable input-output inversion

4.1 Introduction

Performances in the control and regulation of dynamic systems can be improved by the adoption of feedforward control techniques [59]. Among these, the *input-output inversion* technique (or inversion-based control), allows to choose a desired output thereby computing the control input via an inversion procedure. For minimum-phase systems, the procedure uses directly the inverse system to obtain a so-called standard inversion [58, 24]. However, this inversion fails in the nonminimum-phase case because it leads to an unbounded inverse input regardless of the boundedness of the desired output.

A breakthrough leading to a bounded (noncausal) inverse input for nonminimum-phase systems was presented in [11, 15] and [25] for the nonlinear and linear cases respectively. In these works, the (stable) input-output inversion relies on the construction of the *normal form* in a state-space setting [26]. Then, a bounded noncausal solution of the zero dynamics driven by the desired output can be determined by a convolution integral (linear case) or by Picard iterations (nonlinear case). Eventually, a bounded noncausal inverse input is determined. However, the normal form can only be determined when the corresponding *decoupling matrix* [26] is nonsingular. Hence, only for (input-output) decoupable systems these procedures are effective [17].

Here, we formalize the stable input-output inversion problem in the C^{∞}_{p} behavioral setting presented in Chapter 2. Focusing on multivariable nonminimum-phase linear systems, we present a new solution to this problem. Our solution, relies only on an input-output representation of the behavior set. In particular, we use the inverse of the system matrix transfer function and we split the zero dynamics matrix transfer function into stable and unstable parts. Using such an approach, which does not rely on a state space representation, is advantageous as it allows to come up with a solution that is applicable to nondecoupable systems, i.e. systems that cannot be decoupled by state feedback (cf. (4.9), Definition 15 and Theorem 4).

Even though our solution does not require an input-state-output representation per se, it turns out that such a representation simplifies the derivation of several other results that lead to our input-output inversion formula. For instance, this is used for establishing relations between continuity orders of inputs and outputs (Proposition 4) or for characterizing the input's zero dynamics (Property 7). Alternating between input-output and input-state-output representations is made possible by Theorem 2 in Chapter 2 which guarantees that the behavior set of the two representations is the same.

It should be pointed out that the C^{∞}_{p} behavioral approach we present in this thesis is paramount in achieving sound results for the inversion-based control problem (Problem 2). On the contrary, the more general behavioral theory originally introduced in [65], which relies on the space $\mathcal{L}_{1}^{\mathrm{loc}}$, appears not to be suitable for addressing the problem we present in this chapter. Indeed, in the $\mathcal{L}_{1}^{\mathrm{loc}}$ behavioral theory, the integral equation (2.11) does not need to hold everywhere on $\mathbb{R}$ (see the discussion in [65] following Example 2.3.5). It follows from this fact that the output corresponding to a certain fixed $\boldsymbol{u}$ is not unique. This implies that, trying to solve the input-output inversion problem in the $\mathcal{L}_{1}^{\mathrm{loc}}$ behavioral theory would result in an ill-posed problem as the output generated by an input $\boldsymbol{u}$ is not unique.

Chapter organization: In Section 4.2 we introduce the considered system and, by means of an input-state-output representation, we establish some relationships between continuity orders of the input and the output (cf. Proposition 4 and Theorem 5). Subsequently, in Section 4.3, we use matrix fraction descriptions in order to characterize the matrix transfer function and its inverse. Especially relevant are the results on the input's zero dynamics (Property 7) and the input's particular solution (Proposition 6). In a behavioral setting, we pose and solve the stable input-output inversion problem in Section 4.4 (cf. Problem 2 and Theorem 8). The partial fraction expansion of the zero dynamics matrix transfer function (cf. (4.24), (4.25)) leads to the inversion formula (4.29) which is a direct generalization of an analogous formula for scalar systems [41, 12]. Finally, Section 4.5 presents an example of feedforward regulation for a nondecoupable system.

The content of this chapter is based on [30].

Notation:

We say $\boldsymbol{f}$ is *causal* if $\boldsymbol{f}(t) = \mathbf{0}$, $t < 0$. The notation $\mathscr{L}[\cdot]$ and $\mathscr{L}^{-1}[\cdot]$ denotes the Laplace and inverse Laplace transform respectively. We denote the unit step function by $1(t)$: $1(t) = 0$ if $t < 0$ and $1(t) = 1$ if $t \geq 0$. The empty set is $\emptyset$. Given a scalar $i \in \mathbb{R}$ we define $\boldsymbol{i} := [i, i \ldots, i]^\top \in \mathbb{R}^m$.

4.2 Preliminaries

Let Σ be the (*square*) linear time-invariant continuous-time system satisfying

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{4.1}$$

$$\boldsymbol{y} = C\boldsymbol{x} \tag{4.2}$$

where $\boldsymbol{x} \in C^\infty_{\mathtt{p}}(\mathbb{R}, \mathbb{R}^n)$ is the state, $\boldsymbol{u}, \boldsymbol{y} \in C^\infty_{\mathtt{p}}(\mathbb{R}, \mathbb{R}^m)$ are the input and the output respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$. The following assumptions are made: 1) Σ is controllable and observable; 2) Σ is nonminimum-phase and the zero dynamics [26] is hyperbolic (i.e. there are no zeros on the imaginary axis of $\mathbb{C}$); 3) Σ is invertible, i.e. there exists the inverse of its matrix transfer function $H(s) := C(sI - A)^{-1}B$. Remark that since $H(s)$ is invertible, it follows $\mathrm{rank}(B) = \mathrm{rank}(C) = m$ (observe that by assumption Σ is square). Solutions of the state equation (4.1) are introduced as weak solutions (see Section 2.4). Then, the manifest Behavior $\mathscr{B}_{\mathtt{i(s)o}}$ of Σ is introduced in Definition 11. Under the current assumptions, a useful characterization of weak solutions is the following.

Proposition 3. *Given an input* $\boldsymbol{u} \in C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m)$*, a function* $\boldsymbol{x} \in C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^n)$ *is a weak solution of (4.1) if and only if the following conditions hold:*

$$a) \quad \boldsymbol{x} \in C^0(\mathbb{R}, \mathbb{R}^n), \tag{4.3}$$

$$b) \quad S_{\boldsymbol{x}}^{(1)} = S_{\boldsymbol{u}}^{(0)}, \tag{4.4}$$

$$c) \quad \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{u}}^{(0)}. \tag{4.5}$$

Proof. ($\Longrightarrow$):

It follows from (2.38) and Lemma 3 that $\boldsymbol{x}$ is continuous, hence condition (4.3) holds. Remark that: $D\boldsymbol{x}(t)$ is defined on $t \in \mathbb{R} \setminus S_{\boldsymbol{x}}^{(1)}$, $D\big(A \int \boldsymbol{x}\big)(t) = A\boldsymbol{x}(t)$ for all $t \in \mathbb{R}$ and $D\big(B \int \boldsymbol{u}\big)(t) = B\boldsymbol{u}(t)$ is only defined on $\mathbb{R} \setminus S_{B\boldsymbol{u}}^{(0)}$. Since $\operatorname{rank} B = m$, by Lemma 2, we get $S_{B\boldsymbol{u}}^{(0)} = S_{\boldsymbol{u}}^{(0)}$. Hence, by taking into account that two functions are equal only if they have same domain, by differentiating (2.38) we deduce (4.4) and (4.5).

($\Longleftarrow$):

Since by hypothesis $\boldsymbol{x} \in C^0(\mathbb{R}, \mathbb{R}^n)$, integration of (4.5) yields (2.38) where $\boldsymbol{c} = \boldsymbol{x}(0)$. □

Let $P(s), Q(s) \in \mathbb{R}^{m \times m}[s]$ be suitable left coprime matrices such that $H(s) = P^{-1}(s)Q(s)$ and consider the input-output representation (see Section 2.3):

$$P(D)\boldsymbol{y}(t) = Q(D)\boldsymbol{u}(t), \; t \in \mathbb{R}. \tag{4.6}$$

Solutions of (4.6) are introduced in Definition 9 and the corresponding behavior set $\mathscr{B}$ is introduced in Definition 10. Furthermore, it follows from Theorem 2 that $\mathscr{B}_{\mathtt{i(s)o}} = \mathscr{B}$. This means that system Σ admits both representations (4.1)-(4.2) and (4.6). In other words, we can conveniently switch between such representations. We will exploit this fact in the rest of this chapter and address our problems in one or the other representations according to the advantages offered by each representation.

Definition 13 (Relative degree)**.** *Under the current assumptions,* Σ *has (vector) relative degree* $\boldsymbol{r} = [r_1 \; r_2 \; \cdots \; r_m]^T$*, with* $r_i := \min\{j : \boldsymbol{c}^i A^{j-1} B \neq \boldsymbol{0}, j = 1, \ldots, n\}$*,* $i = 1, \ldots, m$*.*

Remark 8. *It is the observability assumption together with* $\operatorname{rank} B = m$ *that guarantees the existence of* r_i*,* $i = 1, \ldots, m$*.*

The i-th component of the relative degree is the minimum derivation order necessary for the input $\boldsymbol{u}$ to appear in a derivative of the i-th output's component. Indeed, from (4.2) and (4.5) it follows that for $i = 1, \dots, m$:

$$y_i^{(r_i)}(t) = \boldsymbol{c}^i A^{r_i} \boldsymbol{x}(t) + \boldsymbol{c}^i A^{r_i - 1} B \boldsymbol{u}(t), \quad t \in \mathbb{R} \backslash S_{\boldsymbol{u}}^{(0)} \tag{4.7}$$

with $\boldsymbol{c}^i A^{r_i-1} B \neq \mathbf{0}$.

Definition 14 (Vector derivative)**.** *Given $\boldsymbol{f} \in C_{\mathrm{p}}^{\infty}(\mathbb{R}, \mathbb{R}^m)$ and a vector $\boldsymbol{k} \in \mathbb{N}^m$, the $\boldsymbol{k}$-th order derivative of $\boldsymbol{f}$ is denoted by $\boldsymbol{f}^{(\boldsymbol{k})}$ and is defined as $\boldsymbol{f}^{(\boldsymbol{k})} := (f_1^{(k_1)}, \dots, f_m^{(k_m)})$.*

With the help of the above vector derivative notation, the m scalar equation (4.7) can be joined into the following vector equation:

$$\boldsymbol{y}^{(\boldsymbol{r})}(t) = \Psi \boldsymbol{x}(t) + \Gamma \boldsymbol{u}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{u}}^{(0)}, \tag{4.8}$$

where:

$$\Psi := \begin{bmatrix} \boldsymbol{c}^1 A^{r_1} \\ \boldsymbol{c}^2 A^{r_2} \\ \vdots \\ \boldsymbol{c}^m A^{r_m} \end{bmatrix} \in \mathbb{R}^{m \times n}, \; \Gamma := \begin{bmatrix} \boldsymbol{c}^1 A^{r_1 - 1} B \\ \boldsymbol{c}^2 A^{r_2 - 1} B \\ \vdots \\ \boldsymbol{c}^m A^{r_m - 1} B \end{bmatrix} \in \mathbb{R}^{m \times m}. \tag{4.9}$$

The introduced matrix Γ, called the *decoupling matrix*, has a significant role in the control of square multivariable systems. Indeed, Γ must be nonsingular when: 1) the construction of the normal form of Σ [39] is required such as e.g. in solving the stable input-output inversion problem in a state-space setting [15, 25]; 2) input-output decoupling by static state feedback is sought [17].

In particular a systems is said to be decoupable according the following definition.

Definition 15 (Decoupable systems)**.** *Σ is said to be (input-output) decoupable (by static state feedback) if there exist constant matrices $F_x \in \mathbb{R}^{m \times n}$ and $F_v \in \mathbb{R}^{m \times m}$ such that $\boldsymbol{u} = F_x \boldsymbol{x} + F_v \boldsymbol{v}$, (with $\boldsymbol{v} \in \mathbb{R}^m$ being the new input vector), determines a closed-loop matrix transfer function $C(sI - A - BF_x)^{-1} BF_v$ that is diagonal.*

Theorem 4 ([17])**.** *Σ is decoupable if and only if the decoupling matrix Γ is nonsingular.*

Remark 9. *Our approach can allow that Γ be singular. Hence, stable inversion (cf. Theorem 8) is extended to nondecoupable systems.*

The relative degree concept dictates a first result on the output continuity orders.

Lemma 7. *Let $(\boldsymbol{u},\boldsymbol{y}) \in \mathscr{B}$. Then $y_i \in C^{r_i-1}$, $i = 1,\dots,m$.*

Proof: From (4.2) we have $y_i(t) = \boldsymbol{c}^i\boldsymbol{x}(t)$, $t \in \mathbb{R}$ and $y_i^{(k)}(t) = \boldsymbol{c}^i A^k \boldsymbol{x}(t)$, $t \in \mathbb{R}\backslash S_{\boldsymbol{u}}^{(0)}$, $k = 1,\dots,r_i - 1$ (cf. Definition 13 and Proposition (3)). By mathematical induction we now prove that $y_i \in C^k$, $k = 0,1,\dots,r_i - 1$. The base case of $k = 0$ is already proved because $\boldsymbol{x} \in C^0(\mathbb{R},\mathbb{R}^n)$ (see also Proposition 3). For the induction step assume $y_i \in C^k$ with $k \le r_i - 2$. Hence $y_i^{(k+1)}(t) = \boldsymbol{c}^i A^{k+1}\boldsymbol{x}(t)$, $t \in \mathbb{R}\backslash S_{\boldsymbol{u}}^{(0)}$. Let $t_1 \in S_{\boldsymbol{u}}^{(0)}$, it follows that $y_i^{(k+1)}(t_1^-) = y_i^{(k+1)}(t_1^+)$ because the state $\boldsymbol{x}(t)$ is continuous. Hence $y_i^{(k+1)}$ exists and is continuous in t_1 because $y_i^{(k)}$ is continuous over $\mathbb{R}$. Therefore $y_i \in C^{k+1}$. □

A relation between the continuity orders of the input and output signals can be expressed as follows.

Proposition 4. *Let $(\boldsymbol{u},\boldsymbol{y}) \in \mathscr{B}$ and $p \in \mathbb{N}\cup\{-1\}$. If $\boldsymbol{u} \in C^p(\mathbb{R},\mathbb{R}^m)$ then $y_i \in C^{p+r_i}(\mathbb{R},\mathbb{R})$, $i = 1,\dots,m$.*

Proof: Note that the case $p = -1$ is already proved in Lemma 7. Then, by induction we will prove that $y_i \in C^{k+r_i}$, $k = 0,1,\dots,p$. First, the base case of $k = 0$ is considered. Since $\boldsymbol{u} \in C_{\mathrm{p}}^\infty(\mathbb{R},\mathbb{R}^m)\cap C^p$ we have $S_{\boldsymbol{u}}^{(p)} = \emptyset$ and consequently $S_{\boldsymbol{u}}^{(0)} = \emptyset$ (cf. Lemma 1). From (4.7) we conclude that $y_i \in C^{r_i}$. For the induction step assume that $y_i \in C^{r_i+k}$ with $k \le p-1$ and $p \ge 1$. We will show that $y_i \in C^{r_i+k+1}$. Indeed, take the derivative of order $k+1$ of relation (4.7) and obtain $y_i^{(r_i+k+1)}(t) = \boldsymbol{c}^i A^{r_i+k+1}\boldsymbol{x}(t) + \boldsymbol{c}^i\sum_{j=0}^{k+1} A^{r_i+k-j}B\boldsymbol{u}^{(j)}(t)$. This relation holds for all $t \in \mathbb{R}$ because $k+1 \le p$ and $\boldsymbol{u} \in C_{\mathrm{p}}^\infty(\mathbb{R},\mathbb{R}^m)$. Hence $y_i \in C^{k+1+r_i}$. □

Definition 16 (Smoothness degree)**.** *A signal $\boldsymbol{f} \in C_{\mathrm{p}}^\infty$ with $\boldsymbol{f} = (f_1,\dots,f_m)$ is said to have smoothness degree -1 if $\boldsymbol{f} \notin C^0(\mathbb{R},\mathbb{R}^m)$. Signal $\boldsymbol{f}$ has smoothness degree $p \in \mathbb{N}$ if $\boldsymbol{f} \in C^p(\mathbb{R},\mathbb{R}^m)$ and $\boldsymbol{f} \notin C^{p+1}(\mathbb{R},\mathbb{R}^m)$.*

For decoupable systems, a stronger result on input-output continuity orders is the following.

Theorem 5. *Let Σ be input-output decoupable. Consider $(\boldsymbol{u},\boldsymbol{y}) \in \mathscr{B}$ and $p \in \mathbb{N} \cup \{-1\}$. Then $\boldsymbol{u}$ has smoothness degree p if and only if $\boldsymbol{y}^{(\boldsymbol{r})}$ has smoothness degree p.*

Proof. $[\Longrightarrow]$: If $\boldsymbol{u}$ has smoothness degree p then, in particular, $\boldsymbol{u} \in C^p(\mathbb{R},\mathbb{R}^m)$. Hence, by Proposition 4 it follows that $\boldsymbol{y}^{(\boldsymbol{r})} \in C^p(\mathbb{R},\mathbb{R}^m)$. Hence, it remains to be shown that $\boldsymbol{y}^{(\boldsymbol{r})} \notin C^{p+1}(\mathbb{R},\mathbb{R}^m)$. To do so, we first consider $p=-1$. Since $\boldsymbol{u} \in C^\infty_{\mathrm{p}}(\mathbb{R},\mathbb{R}^m)$, then $\boldsymbol{u}$ has smoothness degree -1 if $\exists \bar{t} : \boldsymbol{u}(\bar{t}^-) \neq \boldsymbol{u}(\bar{t}^+)$. Then, taking into account that $\det\Gamma \neq 0$ we deduce from (4.8) that $\boldsymbol{y}^{(\boldsymbol{r})}(\bar{t}^-) \neq \boldsymbol{y}^{(\boldsymbol{r})}(\bar{t}^+)$, i.e. $\boldsymbol{y}^{(\boldsymbol{r})}$ has smoothness degree -1. Now consider $p \geq 0$. Computing $\boldsymbol{y}^{(\boldsymbol{r}+\boldsymbol{p}+\boldsymbol{1})}$ from (4.8) (recall that by hypothesis $\boldsymbol{u} \in C^p(\mathbb{R},\mathbb{R}^m)$) yields:

$$\boldsymbol{y}^{(\boldsymbol{r}+\boldsymbol{p}+\boldsymbol{1})}(t) = \Psi A^{p+1}\boldsymbol{x}(t) + \sum_{j=0}^{p} \Psi A^{p-j} B u^{(j)}(t) + \Gamma \boldsymbol{u}^{(p+1)}(t), \quad t \in \mathbb{R} \setminus S_{\boldsymbol{u}}^{(p+1)}. \tag{4.10}$$

Take $\bar{t} \in S_{\boldsymbol{u}}^{(p+1)}$. Then, it follows from the previous relation that $\boldsymbol{y}^{(\boldsymbol{r}+\boldsymbol{p}+\boldsymbol{1})}(t^-) \neq \boldsymbol{y}^{(\boldsymbol{r}+\boldsymbol{p}+\boldsymbol{1})}(t^+)$, i.e. $\boldsymbol{y}^{(r)} \notin C^{p+1}(\mathbb{R},\mathbb{R}^m)$.

$[\Longleftarrow]$: If $\boldsymbol{y}^{(\boldsymbol{r})}$ has smoothness degree p, then, in particular, $\boldsymbol{y}^{(\boldsymbol{r})} \notin C^{p+1}(\mathbb{R},\mathbb{R}^m)$. Then, by negation of Proposition 4 we deduce $\boldsymbol{u} \notin C^{p+1}(\mathbb{R},\mathbb{R}^m)$. Hence, it remains to be shown that $\boldsymbol{u} \in C^p(\mathbb{R},\mathbb{R}^m)$ for $p \geq 0$. We prove this fact by induction. If $p=0$ then from (4.8) it follows $\boldsymbol{u}(t^-) = \boldsymbol{u}(t^+)$, $\forall t \in \mathbb{R}$, i.e. $\boldsymbol{u} \in C^0(\mathbb{R},\mathbb{R}^m)$. For the induction step assume that $\boldsymbol{u} \in C^p(\mathbb{R},\mathbb{R}^m)$, $p \geq 0$. This means that (4.10) holds. Furthermore, since (by hypothesis) $\boldsymbol{y}^{(\boldsymbol{r}+\boldsymbol{p}+\boldsymbol{1})}$ is continuous, it follows from (4.10) that $\boldsymbol{u}^{(p+1)}(t^-) = \boldsymbol{u}^{(p+1)}(t^+)$, $\forall t \in \mathbb{R}$. Since $\boldsymbol{u} \in C^p(\mathbb{R},\mathbb{R}^m)$ this means $u \in C^{p+1}(\mathbb{R},\mathbb{R}^m)$. □

4.3 Input-output properties

The poles of Σ are introduced as the roots of the *pole polynomial* according to the following definition.

Definition 17 (pole polynomial [2])**.** *The* pole polynomial $p_H(s)$ *of* Σ *is defined as the monic least common denominator of all nonzero minors of* $H(s)$.

Under the current assumptions $p_H(s) = det(sI-A)$ and also $p_H(s) = c\det P(s)$ with a suitable $c \neq 0$.

Definition 18 (minimal pole polynomial [2])**.** *The* minimal pole polynomial $p'_H(s)$ *of* Σ *is defined as the monic least common denominator of all nonzero entries of* $H(s)$.

The zeros of Σ are the roots of the *zero polynomial* according to this definition.

Definition 19 (zero polynomial [2])**.** *The* zero polynomial $z_H(s)$ *of* Σ *is defined as the monic greatest common divisor of the numerators of all the highest-order nonzero minors of* $H(s)$ *after all their denominators have been set equal to* $p_H(s)$.

Remark that $z_H(s) = c_1 \det \begin{bmatrix} sI-A & -B \\ C & 0 \end{bmatrix}$ and also $z_H(s) = c_2 \det Q(s)$ with suitable scalars $c_1, c_2 \neq 0$.

By assumption, Σ is invertible. Hence, $H^{-1}(s)$ is well defined and

$$H^{-1}(s) = Q^{-1}(s)P(s) = \frac{\operatorname{adj}[Q(s)]P(s)}{\det Q(s)}. \tag{4.11}$$

The polynomial division of every entry of the product $\operatorname{adj}[Q(s)]P(s)$ leads to

$$(\operatorname{adj}[Q(s)]P(s))_{ij} = q_{0,ij}(s)\det Q(s) + p_{0,ij}(s) \tag{4.12}$$

where $q_{0,ij}(s)$ and $p_{0,ij}(s)$ are (unique) polynomials for which $\deg p_{0,ij}(s) < \deg\det Q(s)$. By defying $Q_0(s) := [q_{0,ij}(s)]$, $P_0(s) := [p_{0,ij}(s)]$, and $H_0 := P_0(s)/\det Q(s)$ it follows that

$$H^{-1}(s) = Q_0(s) + H_0(s) \tag{4.13}$$

where $H_0(s)$ is a strictly proper rational matrix that represents the so-called zero dynamics [26]. Remark that matrices $Q_0(s)$, $H_0(s)$ are unique. Furthermore, since $H(s)$ is (strictly) proper, Q_0 is invertible and its inverse is proper (see Lemma 3.12 in [2]).

Lemma 8. *There exists a polynomial matrix* $P_1(s)$ *such that*

$$H_0(s) = Q^{-1}(s)P_1(s). \tag{4.14}$$

Proof: From (4.11) and (4.13) we obtain $Q^{-1}(s)P(s) = Q_0(s) + H_0(s)$. By multiplying this relation by $Q(s)$ from the left $P(s) = Q(s)Q_0(s) + Q(s)H_0(s)$. Here

$Q(s)H_0(s)$ is necessarily a polynomial matrix because it is the difference of two polynomial matrices. Hence, define $P_1(s) := Q(s)H_0(s)$ and multiply this relation by $Q^{-1}(s)$ from the left to obtain (4.14), the left MFD of $H_0(s)$. $\square$

The dual concept of minimal pole polynomial, i.e. the *minimal zero polynomial* is then introduced.

Definition 20 (minimal zero polynomial)**.** *The* minimal zero polynomial $z'_H(s)$ *of* Σ *is defined as the monic least common denominator of all nonzero entries of* $H_0(s)$.

Remark 10. *From the previous definitions and Lemma 8 it follows that the (minimal) zero polynomial of* Σ *is equal to the (minimal) pole polynomial associated to* $H_0(s)$.

Pole and zero modes are crucial notions which are introduced as follows.

Definition 21 (pole and zero modes)**.** *Given a real or complex pole (zero)* $p \in \mathbb{R}$ *or* $p = \sigma \pm j\omega \in \mathbb{C}$ *(*$z \in \mathbb{R}$ *or* $z = \rho \pm j\psi \in \mathbb{C}$*) with multiplicity* μ *(*ν*) as a root of the minimal pole (zero) polynomial* $p'_H(s)$ *(*$z'_H(s)$*), the associated* pole (zero) modes *are the time-functions* $e^{pt}, te^{pt}, \ldots, t^{\mu-1}e^{pt}$ *or* $e^{\sigma t}\cos(\omega t), e^{\sigma t}\sin(\omega t), \ldots, t^{\mu-1}e^{\sigma t}\cos(\omega t)$, $t^{\mu-1}e^{\sigma t}\sin(\omega t)$ *(*$e^{zt}, te^{zt}, \ldots, t^{\nu-1}e^{zt}$ *or* $e^{\rho t}\cos(\psi t), e^{\rho t}\sin(\psi t), \ldots, t^{\nu-1}e^{\rho t}\cos(\psi t)$, $t^{\nu-1}e^{\psi t}\sin(\psi t)$*) respectively. All the pole (zero) modes are denoted by* $m_i^P(t)$ *(*$m_i^Z(t)$*),* $i = 1, \ldots, m_P(m_Z)$ *with* $m_P := \deg p'_H$ *(*$m_Z := \deg z'_H$*).*

Definition 22 (Autonomous behavior)**.** *The autonomous behavior of* Σ *is the set*

$$\mathscr{B}_{\mathtt{aut}} := \{\boldsymbol{y}_{\mathtt{hom}} \in C^{\infty}_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m) : \boldsymbol{y}_{\mathtt{hom}} \text{ is a weak solution of } P(D)\boldsymbol{y}_{\mathtt{hom}}(t) = 0\} \quad (4.15)$$

The following property characterizes the structure of $\mathscr{B}_{\mathtt{aut}}$.

Property 6. *Let* $\boldsymbol{y}_{\mathtt{hom}} \in \mathscr{B}_{\mathtt{aut}}$ *then*

$$\boldsymbol{y}_{\mathtt{hom}}(t) = \sum_{i=1}^{m_P} \boldsymbol{f}_i m_i^P(t), \quad t \in \mathbb{R}, \quad (4.16)$$

with $\boldsymbol{f}_i \in \mathscr{F}_i$ *where* $\mathscr{F}_i$*,* $i = 1, \ldots, m_P$ *are suitable subspaces of* $\mathbb{R}^m$.

Definition 23 (Inverse system autonomous behavior)**.** *The autonomous behavior of the inverse system of* Σ *is the set*

$$\mathscr{B}'_{\mathtt{aut}} := \{\boldsymbol{u}_{\mathtt{hom}} \in C^{\infty}_{\mathrm{p}}(\mathbb{R}, \mathbb{R}^m) : \boldsymbol{u}_{\mathtt{hom}} \text{ is a weak solution of } Q(D)\boldsymbol{u}_{\mathtt{hom}}(t) = 0\} \quad (4.17)$$

Property 7. *Let $\boldsymbol{u}_{\texttt{hom}} \in \mathscr{B}'_{\texttt{aut}}$ then*

$$\boldsymbol{u}_{\texttt{hom}}(t) = \sum_{i=1}^{m_Z} \boldsymbol{g}_i m_i^Z(t) \tag{4.18}$$

with $\boldsymbol{g}_i \in \mathscr{G}_i$ where $\mathscr{G}_i$, $i = 1, \ldots, m_Z$ are suitable subspaces of $\mathbb{R}^m$.

Mathematically Properties 6 and 7 are actually the same result but conjugated with respect to the $P(D)$ and $Q(D)$ matrix operators respectively. For brevity the proof is omitted. An alternative, equivalent formulation of this result is reported in [65, Theorem 3.2.16, p. 77].

Remark 11. *The time-function (4.16) is the free output response or output's pole dynamics. It is the system output when the input is kept to zero: $(0, \boldsymbol{y}_{\texttt{hom}}) \in \mathscr{B}$. Dually, the time-function (4.18) is the input's zero dynamics. It is the system input when the output is kept to zero: $(\boldsymbol{u}_{\texttt{hom}}, 0) \in \mathscr{B}$.*

Define $h(t) \in \mathbb{R}^{m \times m}$ and $h_0(t) \in \mathbb{R}^{m \times m}$ as the analytical extensions over $\mathbb{R}$ of $\mathscr{L}^{-1}[H(s)]$ and $\mathscr{L}^{-1}[H_0(s)]$ respectively ($h(t)1(t)$ and $h_0(t)1(t)$ are the impulse matrix responses of Σ and of the zero dynamics system respectively). Also define q_i, $i = 1, \ldots, m$ as the degree of the i-th column of $Q_0(s)$, i.e. $q_i := \max_{j=1,\ldots,m} \deg q_{0,ji}(s)$, so that $\boldsymbol{q} := [q_1 \cdots q_m]^T$.

Lemma 9. *Given system Σ, then $\boldsymbol{q} \geq \boldsymbol{r}$ (component-wise inequality). If the decoupling matrix Γ is nonsingular, $\boldsymbol{q} = \boldsymbol{r}$.*

Proof. It follows from (4.8) that

$$H(s) = \begin{bmatrix} s^{-r_1} & & \\ & \ddots & \\ & & s^{-r_m} \end{bmatrix} \tilde{H}(s), \tag{4.19}$$

where $\tilde{H}(s) := \Gamma + \Psi(sI - A)^{-1}B$. Clearly, since $H(s)$ is invertible then $\tilde{H}(s)$ is invertible too. Recall that $H^{-1}(s)$ can be decomposed as in (4.13) with $Q_0(s)$ invertible. In the same way, we decompose $\tilde{H}^{-1}(s)$ as:

$$\tilde{H}^{-1}(s) = \tilde{Q}_0(s) + \tilde{H}_0(s) \tag{4.20}$$

where $\tilde{H}_0(s) \in \mathbb{R}^{m\times m}(s)$ is strictly proper and $\tilde{Q}_0(s) \in \mathbb{R}^{m\times m}(s)$ is invertible. Then, in follows from (4.19) that:

$$\tilde{Q}_0(s) - Q_0(s) \begin{bmatrix} s^{-r_1} & & \\ & \ddots & \\ & & s^{-r_m} \end{bmatrix} = -\tilde{H}_0(s) + H_0 \begin{bmatrix} s^{-r_1} & & \\ & \ddots & \\ & & s^{-r_m} \end{bmatrix}.$$

Since the right-hand side of the previous expression is the sum of strictly proper matrices, it follows:

$$\lim_{s\to\infty} \left(\tilde{Q}_0(s) - Q_0(s) \begin{bmatrix} s^{-r_1} & & \\ & \ddots & \\ & & s^{-r_m} \end{bmatrix} \right) = 0,$$

or, stated component-wise:

$$\lim_{s\to\infty} \left(\frac{\tilde{q}_{0,ij}(s)s^{r_j} - q_{0,ij}(s)}{s^{r_j}} \right) = 0. \tag{4.21}$$

Since $\tilde{Q}_0(s)$ is invertible, for any column $j \in \{1,\dots,m\}$ there exists an index i, which we denote as $i(j)$, such that $\tilde{q}_{0,i(j)j}(s) \neq 0$. Hence, it follows from (4.21) that $\deg q_{0,i(j)j} \geq r_j$. Therefore, for each j-th column of $Q_0(s)$ there exists an element with degree at least r_j, i.e. $q_j \geq r_j$ or equivalently $\boldsymbol{q} \geq \boldsymbol{r}$.

It remains to be shown that when Γ is invertible then $\boldsymbol{q} = \boldsymbol{r}$. In order to do so, remark that, since $\tilde{H}(s)$ is proper and $\lim_{s\to\infty} \tilde{H}(s) = \Gamma$, with $\det\Gamma \neq 0$, it follows that $\tilde{H}^{-1}(s)$ is proper (see Corollary 3.13 in [2]). This means that $\tilde{Q}_0 \in \mathbb{R}^{m\times m}$, i.e. $\tilde{Q}_0$ is a real (invertible) matrix. Hence, taking into account (4.21) we deduce $\boldsymbol{q} = \boldsymbol{r}$. □

The following results emphasize relevant particular solutions of the differential equation (4.6) [12].

Proposition 5 (Output's particular solution)**.** *Let $\boldsymbol{u} \in C^\infty_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m)$ and define $\boldsymbol{y}_{\mathtt{par}}(t) := \int_0^t h(t-v)\boldsymbol{u}(t)dv$, $t \in \mathbb{R}$. Then $(\boldsymbol{u},\boldsymbol{y}_{\mathtt{par}}) \in \mathscr{B}$.*

Proposition 6 (Input's particular solution)**.** *Let* $\boldsymbol{y} \in C_{\mathtt{p}}^{\infty}(\mathbb{R},\mathbb{R}^m)$ *and* $y_i \in C^{q_i-1}$, $i = 1,\dots,m$. *Define*

$$\boldsymbol{u}_{\mathtt{par}}(t) := Q_0(D)\boldsymbol{y}(t^+) + \int_0^t h_0(t-v)\boldsymbol{y}(t)dv, \quad t \in \mathbb{R}.$$

Then $(\boldsymbol{u}_{\mathtt{par}},\boldsymbol{y}) \in \mathscr{B}$.

Useful characterizations of the behavior $\mathscr{B}$ are the following.

Theorem 6. *Define*

$$\begin{aligned}\mathscr{B}_{\mathtt{i/o}} := \{(\boldsymbol{u},\boldsymbol{y}) \in C_{\mathtt{p}}^{\infty}(\mathbb{R},\mathbb{R}^m) \times C_{\mathtt{p}}^{\infty}(\mathbb{R},\mathbb{R}^m) :& \\ \boldsymbol{y}(t) = \int_0^t h(t-v)\boldsymbol{u}(v)dv + \boldsymbol{y}_{\mathtt{hom}}(t), \quad t \in \mathbb{R},\ \boldsymbol{y}_{\mathtt{hom}} \in \mathscr{B}_{\mathtt{aut}}\}&.\end{aligned} \tag{4.22}$$

Then $\mathscr{B}_{\mathtt{i/o}} = \mathscr{B}$.

Theorem 7. *Define*

$$\begin{aligned}\mathscr{B}_{\mathtt{o/i}} := \{(\boldsymbol{u},\boldsymbol{y}) \in C_{\mathtt{p}}^{\infty}(\mathbb{R},\mathbb{R}^m) \times C_{\mathtt{p}}^{\infty}(\mathbb{R},\mathbb{R}^m) : y_i \in C^{q_i-1}, i = 1,\dots,m,& \\ \boldsymbol{u}(t) = Q_0(t)\boldsymbol{y}(t^+) + \int_0^t h_0(t-v)\boldsymbol{y}(v)dv + \boldsymbol{u}_{\mathtt{hom}}(t), \quad t \in \mathbb{R},\ \boldsymbol{u}_{\mathtt{hom}} \in \mathscr{B}'_{\mathtt{aut}}\}&.\end{aligned} \tag{4.23}$$

Then $\mathscr{B}_{\mathtt{o/i}} \subset \mathscr{B}$.

Corollary 3. *Assume that the decoupling matrix* Γ *is nonsingular. Then* $\mathscr{B}_{\mathtt{o/i}} = \mathscr{B}$.

Proof. A glimpse of the proofs of the above results is proposed. Theorems 6 and 7 follow from Properties 6 and 7 (on the set of solutions of the homogeneous differential equations $P(D)\boldsymbol{y}_{\mathtt{hom}}(t) = 0$ and $Q(D)\boldsymbol{u}_{\mathtt{hom}}(t) = 0$) and Propositions 5 and 6 (on the output's and input's particular solutions of the differential equation $P(D)\boldsymbol{y}(t) = Q(D)\boldsymbol{u}(t)$) respectively. Indeed, as known [65], the set of (weak) solutions of (4.6) (i.e. the behavior $\mathscr{B}$) is given by a particular solution plus the set of all solutions of the associated homogeneous differential equation. Corollary 3 follows from Theorems 7 and 5. In the scalar case, complete proofs of Theorem 6 and Corollary 3 are reported in [12]. □

4.4 Stable input-output inversion

The stable input-output inversion problem can be addressed as follows.

Problem 2. *Given a desired, bounded, sufficiently smooth output* $\boldsymbol{y}_{\mathtt{d}} \in C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m)$ *find a bounded input* $\boldsymbol{u}_{\mathtt{d}} \in C^{\infty}_{\mathtt{p}}(\mathbb{R},\mathbb{R}^m)$ *such that* $(\boldsymbol{u}_{\mathtt{d}},\boldsymbol{y}_{\mathtt{d}}) \in \mathscr{B}$.

By assumption, the zero dynamics of Σ is hyperbolic, i.e. the real parts of the system zeros are negative or positive. Hence, the zero polynomial (cf. Definition 19) can be factorized as $z_H(s) = z_H^-(s)z_H^+(s)$ where $z_H^-(s)$ and $z_H^+(s)$ are monic polynomials with root's real parts that are all negative and positive respectively. It follows that $\det Q(s) = cz_H^-(s)z_H^+(s)$ with a suitable scalar $c \neq 0$.

The matrix transfer function of the zero dynamics of Σ is $H_0(s) = \frac{P_0(s)}{\det Q(s)} = \left[\frac{p_{0,ij}(s)}{\det Q(s)}\right]$ (cf. (4.12) and (4.13); $i,j = 1,\dots,m$). By partial fraction expansion, the entries of $H_0(s)$ can be rewritten as

$$\frac{p_{0,ij}(s)}{\det Q(s)} = \frac{p^-_{0,ij}(s)}{z_H^-(s)} + \frac{p^+_{0,ij}(s)}{z_H^+(s)} \tag{4.24}$$

where $p^-_{0,ij}(s)$, $p^+_{0,ij}(s)$ are polynomials with $\deg p^-_{0,ij}(s) < \deg z_H^-(s)$, $\deg p^+_{0,ij}(s) < \deg z_H^+(s)$. Define $P_0^-(s) := [p^-_{0,ij}(s)]$, $P_0^+(s) := [p^+_{0,ij}(s)]$ and $H_0^- := \frac{P_0^-(s)}{z_H^-(s)}$, $H_0^+ := \frac{P_0^+(s)}{z_H^+(s)}$. Hence, $H_0(s)$ is split into stable and unstable parts:

$$H_0(s) = H_0^-(s) + H_0^+(s). \tag{4.25}$$

The strictly proper rational matrices $H_0^-(s)$ and $H_0^+(s)$ can be represented by left MFDs according to the following result.

Lemma 10. *There exist polynomial matrices* $P_1^-(s)$ *and* $P_1^+(s)$ *such that*

$$H_0^-(s) = Q^{-1}(s)P_1^-(s), \quad H_0^+(s) = Q^{-1}(s)P_1^+(s). \tag{4.26}$$

Proof. It is omitted for brevity. □

Let $h_0^-(t) \in \mathbb{R}^{m\times m}$ and $h_0^+(t) \in \mathbb{R}^{m\times m}$ be the analytical extensions over $\mathbb{R}$ of $\mathscr{L}^{-1}[H_0^-(s)]$ and $\mathscr{L}^{-1}[H_0^+(s)]$ for which

$$h_0(t) = h_0^-(t) + h_0^+(t),\ t \in \mathbb{R}. \tag{4.27}$$

All the matrix functions $h_0(t)$, $h_0^-(t)$, $h_0^+(t)$ satisfy the homogeneous matrix differential equation associated to the operator $Q(D)$.

Lemma 11. *The following relations hold:* $Q(D)h_0(t) = 0$, $Q(D)h_0^-(t) = 0$ *and* $Q(D)h_0^+(t) = 0$, $t \in \mathbb{R}$.

Proof: It is based on the MFDs provided by Lemma 8 and Lemma 10 and on the concept of *impulse response matrix* [2]. Function $h_0(t)1(t)$ is the impulse response matrix of the zero system of Σ whose matrix transfer function is $H_0(s) = Q^{-1}(s)P_1(s)$ (cf. (4.14)). This system is then described by the differential equation $Q(D)\boldsymbol{\eta}(t) = P_1(D)\boldsymbol{y}(t)$ where $\boldsymbol{y}(t)$ and $\boldsymbol{\eta}(t)$ are the input and the output respectively. Hence, it holds $Q(D)h_0(t) = 0$ for $t > 0$ and by analytical extension for all $t \in \mathbb{R}$. A similar reasoning leads to $Q(D)h_0^-(t) = 0$ and $Q(D)h_0^+(t) = 0$ for all $t \in \mathbb{R}$. □

Denote by $m_i^{Z-}(t)$, $i = 1,\dots m_Z^-$ and $m_i^{Z+}(t)$, $i = 1,\dots m_Z^+$ the stable and unstable zero modes ($m_Z^- + m_Z^+ = m_Z$; cf. Definition 21). Taking into account Lemma 11 and Property 7, there exist matrices $G_i^-, G_i^+ \in \mathbb{R}^{m\times m}$ such that $\operatorname{im} G_i^- \subseteq \mathscr{G}_i^-$, $\operatorname{im} G_i^+ \subseteq \mathscr{G}_i^+$ and

$$h_0^-(t) = \sum_{i=1}^{m_Z^-} G_i^- m_i^{Z-}(t), \;\; h_0^+(t) = \sum_{i=1}^{m_Z^+} G_i^+ m_i^{Z+}(t) \tag{4.28}$$

where $\mathscr{G}_i^-$ and $\mathscr{G}_i^+$ are the (input) subspaces of $\mathbb{R}^m$ associated to the modes $m_i^{Z-}(t)$ and $m_i^{Z+}(t)$.

The next result gives an explicit closed-form expression of the inverse input $\boldsymbol{u}_{\mathtt{d}}$ solving Problem 2.

Theorem 8 (Stable inversion formula)**.** *Let be given a desired output* $\boldsymbol{y}_{\mathtt{d}} \in C_{\mathtt{p}}^\infty(\mathbb{R},\mathbb{R}^m)$ *for which* $y_{\mathtt{d},i} \in C^{q_i-1}$ *and* $y_{\mathtt{d},i}, y_{\mathtt{d},i}^{(1)}, \dots, y_{\mathtt{d},i}^{(q_i)}$ *are all bounded* ($i = 1,\dots,m$)*. Then a solution to Problem 2 can be expressed as follows:*

$$\boldsymbol{u}_{\mathtt{d}}(t) = Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+) + \int_{-\infty}^{t} h_0^-(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv - \int_{t}^{+\infty} h_0^+(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv, \; t \in \mathbb{R}. \tag{4.29}$$

Proof. From (4.28), the right side of (4.29) can be written as

$$Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+) + \sum_{i=1}^{m_Z^-} G_i^- \int_{-\infty}^{t} \boldsymbol{y}_{\mathtt{d}}(v) m_i^{Z-}(t-v)dv \\ - \sum_{i=1}^{m_Z^+} G_i^+ \int_{t}^{+\infty} \boldsymbol{y}_{\mathtt{d}}(v) m_i^{Z+}(t-v)dv. \quad (4.30)$$

All the addends of the above expression are bounded and so is the resulting $\boldsymbol{u}_{\mathtt{d}}(t)$. Indeed, $Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+)$ involves derivatives of $\boldsymbol{y}_{\mathtt{d}}$ that are bounded by assumption and all the the integrals appearing in (4.30) are bounded too. For simplicity, in the following, we consider that the minimal zero polynomial $m^Z(s)$ (cf. Definition 20) has only simple real roots. For example, assume $m_i^{Z+}(t) = e^{z_i^+ t}$ with $z_i^+ > 0$ and define $\boldsymbol{y}_{\mathtt{d,sup}} := \sup_{t\in\mathbb{R}} |\boldsymbol{y}_{\mathtt{d}}(t)| \in \mathbb{R}^m$ (the absolute value and the supremum are applied componentwise). Hence, $|\int_t^{+\infty} \boldsymbol{y}_{\mathtt{d}}(v) m_i^{Z+}(t-v)dv| \le \boldsymbol{y}_{\mathtt{d,sup}} \int_t^{+\infty} e^{z_i^+(t-v)}dv = \frac{1}{z_i^+} \cdot \boldsymbol{y}_{\mathtt{d,sup}}$, $t \in \mathbb{R}$. Similarly, the boundedness of $\int_{-\infty}^{t} \boldsymbol{y}_{\mathtt{d}}(v) m_i^{Z-}(t-v)dv$ can be ascertained.

We will now show that $(\boldsymbol{u}_{\mathtt{d}}, \boldsymbol{y}_{\mathtt{d}}) \in \mathscr{B}_{o/i}$ (cf. Theorem 7). In (4.29), break the integrals into two parts at zero and rearrange them by taking into account (4.27) to obtain:

$$\boldsymbol{u}_{\mathtt{d}}(t) = Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+) + \int_0^t h_0(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv \\ + \int_{-\infty}^{0} h_0^-(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv - \int_0^{+\infty} h_0^+(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv.$$

The last two integrals above are suitable linear combinations of the zero modes of Σ. Indeed, by (4.28) and still considering $m_i^{Z-}(t) = e^{z_i^- t}$, $m_i^{Z+}(t) = e^{z_i^+ t}$ ($z_i^- < 0$, $z_i^+ > 0$) these integrals can be expressed as

$$\sum_{i=1}^{m_Z^-} G_i^- \int_{-\infty}^{0} e^{z_i^-(t-v)} \boldsymbol{y}_{\mathtt{d}}(v)dv - \sum_{i=1}^{m_Z^+} G_i^+ \int_0^{+\infty} e^{z_i^+(t-v)} \boldsymbol{y}_{\mathtt{d}}(v)dv = \sum_{i=1}^{m_Z^-} G_i^- \boldsymbol{l}_i^- e^{z_i^- t} - \sum_{i=1}^{m_Z^+} G_i^+ \boldsymbol{l}_i^+ e^{z_i^+ t} \quad (4.31)$$

where $\boldsymbol{l}_i^- := \int_{-\infty}^0 e^{-z_i^- t} \boldsymbol{y}_{\mathtt{d}}(v)dv \in \mathbb{R}^m$ and $\boldsymbol{l}_i^+ := \int_0^{+\infty} e^{-z_i^+ t} \boldsymbol{y}_{\mathtt{d}}(v)dv \in \mathbb{R}^m$ having taken into account that $\boldsymbol{y}_{\mathtt{d}}$ is bounded over $\mathbb{R}$. From $\operatorname{im} G_i^- \subseteq \mathscr{G}_i^-$ and $\operatorname{im} G_i^+ \subseteq \mathscr{G}_i^+$ we obtain $\operatorname{im} G_i^- \boldsymbol{l}_i^- \in \mathscr{G}_i^-$ and $\operatorname{im} G_i^+ \boldsymbol{l}_i^+ \in \mathscr{G}_i^+$ so that (4.31) is equal to $\sum_{i=1}^{m_Z} \boldsymbol{g}_i m_i^Z(t)$, $\boldsymbol{g}_i \in \mathscr{G}_i$ (cf. (4.28) and Property 7). Hence $\boldsymbol{u}_{\mathtt{d}}(t) = Q_0(D)\boldsymbol{y}_{\mathtt{d}}(t^+) + \int_0^t h_0(t-v)\boldsymbol{y}_{\mathtt{d}}(v)dv + \sum_{i=1}^{m_Z} \boldsymbol{g}_i m_i^Z(t)$ and along with $y_{\mathtt{d},i} \in C^{q_i - 1}$, $i = 1, \ldots, m$ this proves that $(\boldsymbol{u}_{\mathtt{d}}, \boldsymbol{y}_{\mathtt{d}}) \in \mathscr{B}_{\mathtt{o/i}}$. Theorem 7 states that $\mathscr{B}_{\mathtt{o/i}} \subseteq \mathscr{B}$ so that $(\boldsymbol{u}_{\mathtt{d}}, \boldsymbol{y}_{\mathtt{d}}) \in \mathscr{B}$. □

Remark 12. *Formula (4.29) can be applied to both decoupable and nondecoupable systems. In the former case, it appears that the inverse input (4.29) is equal to that obtained by the state-space approaches [15, 25].*

4.5 An example

Consider the system Σ with

$$A = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{4.32}$$

This system is controllable and observable. The zero polynomial and the zero minimal polynomial coincides, i.e. $z_H(s) = z'_H(s) = s-1$. Hence, Σ is nonminimum-phase and its zero dynamics is hyperbolic. The decoupling matrix $\Gamma = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is singular (cf. (4.9)) so that the system is nondecoupable. This precludes the possibility of applying the state-space inversion procedure of [15, 25]. Nevertheless, Σ is invertible and the stable input-output inversion provided by formula (4.29) can be applied.

The vector relative degree is $\boldsymbol{r} = [1\ 2]^T$ and the vector of the column degrees of $Q_0(s)$ is $\boldsymbol{q} = [3\ 4]^T$ (cf. (4.13) and Lemma 9). We desire a set-point transition on the outputs with a smooth planning given by the *transition polynomials* [44]. Specifically, outputs 1 and 2 should move from 0 to $y_{\mathrm{dc},1} := 2$, and $y_{\mathrm{dc},2} := 4$ with interval times $\tau_1 = 1$ s and $\tau_2 = 2$ s respectively. To obtain a continuous inverse input $\boldsymbol{u}_{\mathrm{d}}(t)$ (cf. (4.29)) we choose $y_{\mathrm{d},1}(t)$ and $y_{\mathrm{d},2}(t)$ with smoothness degrees (cf. Definition 16) equal to 3 and 4 respectively. The resulting expressions for $y_{\mathrm{d},1}$, $y_{\mathrm{d},2}$ are:

$$y_{\mathrm{d},1}(t) := \begin{cases} 0, & t < 0 \\ [-20(\frac{t}{\tau_1})^7 + 70(\frac{t}{\tau_1})^6 - 84(\frac{t}{\tau_1})^5 + 35(\frac{t}{\tau_1})^4]2, & 0 \le t \le 1\,, \\ 2 & t > 1 \end{cases} \tag{4.33}$$

$$y_{\mathrm{d},2}(t) := \begin{cases} 0, & t < 0 \\ [70(\frac{t}{\tau_2})^9 - 315(\frac{t}{\tau_2})^8 + 540(\frac{t}{\tau_2})^7 - 420(\frac{t}{\tau_2})^6 + 126(\frac{t}{\tau_2})^5]4, & 0 \le t \le 2\,\cdot \\ 4 & t > 2 \end{cases} \tag{4.34}$$

The inversion procedure (cf. Section 4.4) requires to compute the differential operator $Q_0(D)$:

$$Q_0(D) = \begin{bmatrix} D+1 & 1 \\ D^3 + 6D^2 + 14D + 19 & -D^4 - 6D^3 - 15D^2 - 25D - 32 \end{bmatrix},$$

$h_0^-(t) = \mathbf{0} \in \mathbb{R}^{2\times 2}$, and

$$h_0^+(t) = \begin{bmatrix} 0 & 0 \\ 18\,\mathrm{e}^t & -36\,\mathrm{e}^t \end{bmatrix}, \quad t \in \mathbb{R}.$$

The inverse input $\boldsymbol{u}_\mathrm{d}(t)$, which is determined by (4.29), and output $\boldsymbol{y}_\mathrm{d}(t)$ are then plotted in Figure 4.1. Note that, in this case, $\boldsymbol{u}_\mathrm{d}(t)$ does not have *postaction* (or *postactuation*) [18] (at time $\max\{\tau_1, \tau_2\} = 2$ s the system is at the equilibrium) because there are no zeros with negative real part. On the other hand, input $\boldsymbol{u}_\mathrm{d}$ exhibits *preaction* (or *preactuation*) [36, 15] which is due to the positive real zero 1 (see $u_{\mathrm{d},2}(t)$, $t \in [-1, 0]$, in Figure 4.1).

4.6 Conclusions

Input-output inversion allows *virtual* decoupling by feedforward control [46], i.e. by appropriately designing the desired (vector) output it is possible to decouple all

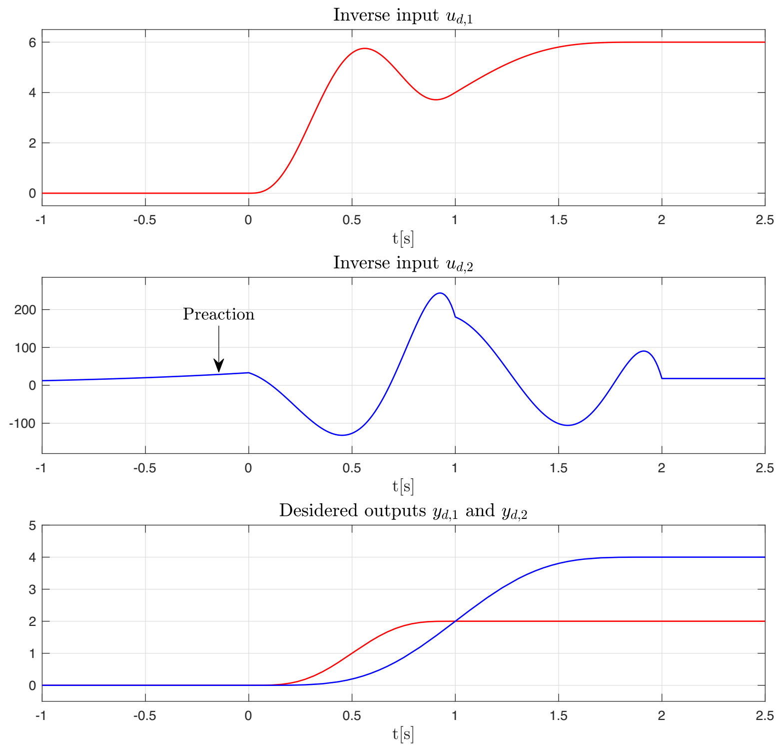

Figure 4.1: Input and output components $u_{\mathrm{d},1}(t)$, $y_{\mathrm{d},1}(t)$ and $u_{\mathrm{d},2}(t)$, $y_{\mathrm{d},2}(t)$ are plotted with red and blue lines respectively.

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the scalar outputs from each other. Hence, the found inversion formula (4.29) that is based on the matrix transfer function inverse $H^{-1}(s)$ (cf. (4.13)) allows input-output decoupling also for nonminimum-phase linear systems that are not decoupable by state feedback. To deal with these nondecoupable systems a possible state-space alternative approach may be based on the dynamic extension algorithm [26]. In such a way the decoupling matrix in the augmented state-space becomes nonsingular so that the stable inversion method of [15, 25] can be applied. However, the resulting overall inversion procedure may be cumbersome. This possible approach should be investigated in a future research.

Chapter 5

On the equivalence of inversion-based control architectures for scalar systems

5.1 Introduction

Feedforward control can improve the performances of feedback control systems. As we described in the previous chapter, when specific signal features at the output of a controlled system are required, inversion-based control may be a very effective feedforward technique to be used.

Given a desired output signal, an inversion-based controller computes the inverse input that, in absence of modeling errors and disturbances, allows to obtain the desired signal at the output of the controlled system. However, inversion-based controllers cannot compensate for tracking errors caused by modeling inaccuracies or disturbances [14]. In practice, a feedback controller is almost always needed either for increasing the control system robustness or because the plant (the controlled system) has to be stabilized. The structure in which the inversion-based and feedback controllers are arranged is referred to as an inversion-based control architecture or briefly inversion architecture. There are two main inversion architectures:

- the *plant inversion architecture* [14], [67] (cf. Fig. 5.1) and
- the *closed-loop inversion architecture* [43], [45] (cf. Fig. 5.2).

In absence of modeling errors and disturbances they both yield the same output, the desired output signal. However, in practice, modeling errors due to the plant's uncertainties, perturbations and disturbances always occur. Hence, in both architectures the actual plant's output differs from the desired one.

As a consequence, from the viewpoint of the control applications, the relevant question is: which one of the two architectures performs better? Comparisons addressing this question have appeared in the control literature for scalar (single-input single-output) discrete-time linear systems [50, 51, 8].

In [50], for the minimum-phase case (the nominal plant and the feedback controller are both minimum-phase) an algebraic analysis shows both architectures to be equivalent in presence of uncertain plant dynamics. For the nonminimum-phase case (more specifically, the plant is nonminimum-phase whereas the controller is minimum-phase) the comparison shows that the two architectures perform differently when approximate stable inverses are used. For settle time applications (i.e. setpoint regulation) the closed-loop inversion architecture appears to achieve superior performances. Further comparisons, still using approximate stable inverses, confirm the better performance of the closed-loop inversion architecture [51, 8]. However these comparisons seem not to be conclusive nor fully general because they depend on specific applications. As a consequence, the plant inversion architecture has still been used as reference architecture in several subsequent studies e.g. [61, 6].

For scalar continuous-time linear systems, we show that the two inversion architectures are fully equivalent when *exact* stable inverse inputs are used as feedforward controllers. In the general nonminimum-phase case — in which both the plant and the feedback controller can be nonminimum-phase — an equivalence result is established. Essentially, it can be stated as follows. *For any desired output (signal) and any disturbance and plant perturbation for which the feedback controller ensures closed-loop stability, the inputs and the outputs of the perturbed plant are the same bounded signals in both architectures* (Theorem 9).

The equivalence result is deduced within the C_p^∞ behavioral framework introduced in Chapter 2. Specific keystones to achieve the result are:

- the concept of *forced response of a system initially at rest at time* $-\infty$ (cf. Section 5.3);
- the *algebraic identity* (5.37) relating the feedback controller C to the unstable zero dynamics of the nominal plant P and closed-loop system H (cf. Proposition 14);
- the *preaction signals* of the closed-loop system H and plant P (related to the stable inverses, cf. Remark 13, (5.13), and (5.25)) being shown to be an input and the corresponding output of the feedback controller C respectively (Proposition 15).

The equivalence still holds in practice when a careful truncation of the preaction control occurs (cf. Remark 13 and Subsection 5.7.1). Nevertheless the ease of implementation and different *preaction* and *postaction times* of the stable inverses may indicate that one architecture is preferable than the other depending on the addressed control application (cf. Subsection 5.7.2). To help in choosing the preferable architecture new more precise rules to set the truncations of the preaction and postaction control are introduced. These are given by the *output-error-based preaction* and *postaction times* (Definitions 27 and 28).

Chapter organization: This chapter is organized as follows. Section 5.2 recaps the necessary preliminaries: a behavioral presentation of a scalar system (Subsection 5.2.1) and the related stable input-output inversion (Subsection 5.2.2). Section 5.3 introduces the concept of forced response from time $-\infty$ and associated properties. The inversion-based control architectures are presented in Section 5.4. Equivalence results between the architectures by using standard inverses are given in Section 5.5. The next Section 5.6 reports key algebraic results in Subsection 5.6.1 and consequently, in Subsection 5.6.2, the full equivalence result when stable inverses are used. An example with simulation comparisons highlights the results in Subsection 5.7.1 and a related discussion is reported in Subsection 5.7.2. Final remarks end the paper in Section 5.8.

The content of this chapter is based on [32].

Notation: Given a real function $f : \mathbb{R} \to \mathbb{R}$, the following shorthand notations are used: $f(-\infty) := \lim_{t\to-\infty} f(t)$, $f(+\infty) := \lim_{t\to+\infty} f(t)$. We say that f is a *causal* signal when $f(t) = 0$, $t < 0$. The symbol $\equiv$ denotes an identity that holds over $\mathbb{R}$: $f \equiv 0$, i.e. $f(t) = 0$, $t \in \mathbb{R}$; $f \equiv g$, i.e. $f(t) = g(t)$, $t \in \mathbb{R}$. The Laplace transform of f is $F(s) := \mathscr{L}[f(t)]$. The analytical extension over $\mathbb{R}$ of the inverse Laplace transform is denoted by $\mathscr{L}^{-1}_{\mathtt{ae}}[\cdot]$ (defined by analytic continuation for negative times).

The polynomial $p(s) \in \mathbb{R}[s]$ is said to be Hurwitz if all its roots have negative real parts. The *left half-plane* (LHP) and *right half-plane* (RHP) denote the sets of complex numbers having negative and positive real parts respectively. A linear, time-invariant system is said to be *minimum-phase* if all its zeros lie on the LHP or there are no (finite) zeros at all. It is said to be *nonminimum-phase* if there exists a system's zero on the RHP. System's zeros that lie on the RHP are said to be *nonminimum-phase* or *unstable*.

5.2 Preliminaries

5.2.1 System's behavioral presentation

As in the previous chapters, we consider the linear time-invariant continuous-time system H defined by its transfer function

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}. \tag{5.1}$$

Polynomials $a(s)$ and $b(s)$ (with $\deg a = n \geq \deg b = m$) are coprime and $b(s)$ has no roots on the imaginary axis (the zero dynamics of H is hyperbolic). The *relative degree* of H is $r := n - m$. The scalar input and output of H, u and y respectively, belong to $C^\infty_{\mathtt{p}}(\mathbb{R})$, the set of piecewise C^∞-functions (see Section 2.2)

The *behavior* of H (see Chapter 2 and in particular Section 2.3), which is here denoted by $\mathscr{B}_H$, is the set of ordered pairs $(u, y) \in C^\infty_{\mathtt{p}}(\mathbb{R})^2$ that are *weak solutions* of the differential equation

$$\sum_{i=0}^{n} a_i D^i y = \sum_{i=0}^{m} b_i D^i u \,. \tag{5.2}$$

Actually, a pair (u,y) is a weak solution of (5.2) if there exists a polynomial $g(t)$ with $\deg g \leq n-1$ such that the integral equation (see Section 2.3)

$$\sum_{i=0}^{n} a_i \int^{n-i} y(t) = \sum_{i=0}^{m} b_i \int^{n-i} u(t) + g(t) \tag{5.3}$$

is satisfied for all $t \in \mathbb{R}$. A relevant property of the behavior $\mathscr{B}_H$ is the following ($C^{-1} := C^{\infty}_{\mathtt{p}}(\mathbb{R})$).

Proposition 7 ([12])**.** *Consider a pair $(u,y) \in \mathscr{B}_H$ and let $p \in \mathbb{Z}$ with $p \geq -1$. Then $u \in C^p$ if and only if $y \in C^{r+p}$.*

The pole and zero modes of H can be introduced as follows [12].

Definition 24 (pole and zero modes)**.** *Given a real (complex) pole of H, $p \in \mathbb{R}$ ($p = \sigma \pm j\omega \in \mathbb{C}$) with multiplicity μ, the associated modes are: e^{pt}, te^{p}, $\ldots$, $t^{\mu-1}e^{pt}$ ($e^{\sigma t}\cos(\omega t)$, $e^{\sigma t}\sin(\omega t)$, $\ldots$, $t^{\mu-1}e^{\sigma t}\cos(\omega t)$, $t^{\mu-1}e^{\sigma t}\sin(\omega t)$). All the pole modes of H are denoted by $m_i^{\mathtt{p}}(t)$, $i = 1,\ldots,n$.*
Given a real (complex) zero of H, $z \in \mathbb{R}$ ($z = \rho \pm j\psi \in \mathbb{C}$) with multiplicity ν, the associated modes are: e^{zt}, te^{z}, $\ldots$, $t^{\nu-1}e^{zt}$ ($e^{\rho t}\cos(\psi t)$, $e^{\rho t}\sin(\psi t)$, $\ldots$, $t^{\nu-1}e^{\sigma t}\cos(\psi t)$, $t^{\mu-1}e^{\sigma t}\sin(\psi t)$). All the zero modes of H are denoted by $m_i^{\mathtt{z}}(t)$, $i = 1,\ldots,m$.
Pole and zero modes are said to be stable (unstable) if they converge to 0 as t goes to $+\infty$ ($-\infty$).

By polynomial division of $b(s)$ by $a(s)$, we rewrite the system's transfer function as

$$H(s) = h_{\mathtt{hg}} + H_{\mathtt{dy}}(s)$$

where $h_{\mathtt{hg}} := \lim_{s\to\infty} H(s)$ is the system's high-frequency gain [37] and $H_{\mathtt{dy}}(s)$ is a strictly proper rational function representing the (purely) dynamic part of H. (Note that if the relative degree $r \geq 1$ then $h_{\mathtt{hg}} = 0$ and $H(s) = H_{\mathtt{dy}}(s)$.) Then define

$$h_{\mathtt{dy}}(t) := \mathscr{L}_{\mathtt{ae}}^{-1}[H_{\mathtt{dy}}(s)] \tag{5.4}$$

and a useful characterization of the behavior set $\mathscr{B}_H$ is the following [12].

Proposition 8 (Input-output representation of $\mathscr{B}_H$)**.**

$$\mathscr{B}_H = \{(u,y) \in C^{\infty}_{\mathtt{p}}(\mathbb{R})^2 : y(t) = h_{\mathtt{hg}} u(t) + \int_0^t h_{\mathtt{dy}}(t-v)u(v)dv + \sum_{i=1}^{n} f_i m_i^{\mathtt{P}}(t), t \in \mathbb{R}; f_i \in \mathbb{R}\}. \tag{5.5}$$

By polynomial division, write $a(s) = q(s)b(s) + c(s)$ with $\deg c \le m-1$ so that the transfer function inverse becomes

$$H^{-1}(s) = q(s) + H_0(s) \tag{5.6}$$

with $H_0(s) := c(s)/b(s)$ representing the zero dynamics of H. Define $h_0(t) := \mathscr{L}^{-1}_{\mathtt{ae}}[H_0(s)]$ and introduce the output-input representation of the behavior $\mathscr{B}_H$ [12].

Proposition 9 (Output-input representation of $\mathscr{B}_H$)**.**

$$\mathscr{B}_H = \{(u,y) \in C^{\infty}_{\mathtt{p}}(\mathbb{R})^2 : y \in C^{r-1},\ u(t) = q(D)y(t^+) + \int_0^t h_0(t-v)y(v)dv + \sum_{i=1}^{m} g_i m_i^{\mathtt{Z}}(t), t \in \mathbb{R}; g_i \in \mathbb{R}\}. \tag{5.7}$$

5.2.2 Stable input-output inversion for scalar systems

Let be given a causal desired output (signal) $y_{\mathtt{d}}$ of H and assume that $y_{\mathtt{d}} \in C^{\infty}_{\mathtt{p}}(\mathbb{R}) \cap C^{r-1}$ and $y_{\mathtt{d}}, y_{\mathtt{d}}^{(1)}, \ldots, y_{\mathtt{d}}^{(r)}$ are all bounded time-functions on $\mathbb{R}$. The *standard inverse* (input) can be expressed as (cf. Proposition 9)

$$u'_{H,\mathtt{d}}(t) = q(D)y_{\mathtt{d}}(t^+) + \int_0^t h_0(t-v)y_{\mathtt{d}}(v)dv,\ t \in \mathbb{R}. \tag{5.8}$$

As known, if H is nonminimum-phase $u'_{H,\mathtt{d}}(t)$ diverges exponentially because the zero dynamics $H_0(s)$ is unstable [15]. In this case $u'_{H,\mathtt{d}}(t)$ cannot be used for control applications. Nevertheless, a bounded noncausal inverse input exists [15, 25]. By partial fraction decomposition, $H_0(s)$ can be split as

$$H_0(s) = H_0^-(s) + H_0^+(s) \tag{5.9}$$

where $H_0^-(s)$ and $H_0^+(s)$ represent the stable and unstable zero dynamics respectively. Hence, define $h_0^-(t) := \mathscr{L}_{\mathtt{ae}}^{-1}[H_0^-(s)]$, $h_0^+(t) := \mathscr{L}_{\mathtt{ae}}^{-1}[H_0^+(s)]$ so that

$$h_0(t) = h_0^-(t) + h_0^+(t),\ t \in \mathbb{R} \tag{5.10}$$

and the bounded noncausal inverse input — called the *stable inverse* — can be expressed as [12]

$$\begin{aligned} u_{H,\mathtt{d}}(t) = q(D)y_{\mathtt{d}}(t^+) + \int_0^t h_0^-(t-v)y_{\mathtt{d}}(v)dv \\ - \int_t^{+\infty} h_0^+(t-v)y_{\mathtt{d}}(v)dv,\ t \in \mathbb{R}. \end{aligned} \tag{5.11}$$

A relationship between $u'_{H,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ can be established (cf. [41]).

Proposition 10. *The stable inverse $u_{H,\mathtt{d}}$ can be expressed as*

$$u_{H,\mathtt{d}}(t) = u'_{H,\mathtt{d}}(t) + u_{H,\mathtt{ps}}(t) \quad t \in \mathbb{R}, \tag{5.12}$$

where

$$u_{H,\mathtt{ps}}(t) := -\int_0^{+\infty} h_0^+(t-v)y_d(v)dv \tag{5.13}$$

is a (linear) combination of the system's unstable zero modes.

Remark 13. *Note that the standard inverse $u'_{H,\mathtt{d}}$ is always a causal signal whereas the stable inverse $u_{H,\mathtt{d}}$ is noncausal if H is nonminimum-phase, i.e. $h_0^+ \not\equiv 0$. In this case, by relation (5.12) $u_{H,\mathtt{d}}(t) = u_{H,\mathtt{ps}}(t)$ for negative times and this input is the so-called* preaction *(or* preactuation*)* control *that corresponds to the output being kept to zero [15, 12]. We denote $u_{H,\mathtt{ps}}$ extended over $\mathbb{R}$ as the* preaction signal.

Remark 14. *Note that both $u'_{H,\mathtt{d}}(t)$ and $u_{H,\mathtt{d}}(t)$ in (5.8) and (5.11) can be interpreted as the outcomes of linear operators applied to $y_{\mathtt{d}}(t)$.*

5.3 The system's forced response from time $-\infty$

The following definitions are introduced.

Definition 25 (System initially at rest)**.** *Let be given a system H with input u and output y, i.e. $(u,y) \in \mathscr{B}_H$ (cf. Subsection 5.2.1). H is said to be initially at rest (at time $-\infty$) if $u(-\infty) = 0$ and $y(-\infty) = 0$.*

Definition 26 (Forced response from time $-\infty$)**.** *Assume that H is asymptotically stable. Let be given a pair $(u,y) \in \mathscr{B}_H$ with $u(-\infty) = 0$ and $y(-\infty) = 0$. Then y is called a forced response from time $-\infty$ or, more precisely, a forced response to input u of H initially at rest (at time $-\infty$).*

A key result is the following.

Proposition 11 (Uniqueness of the forced response)**.** *Assume that H is asymptotically stable and let be given a pair $(u,y) \in \mathscr{B}_H$ with $u(-\infty) = 0$ and $y(-\infty) = 0$. Then, y is the unique forced response to input u of H initially at rest (at time $-\infty$).*

Proof. Suppose there exists another forced response $y_1 \in C^\infty_{\mathrm{p}}(\mathbb{R})$ such that $y_1(-\infty) = 0$ and $(u, y_1) \in \mathscr{B}_H$. Since $\mathscr{B}_H$ is a linear space (cf. [12]) the difference of two pairs in $\mathscr{B}_H$ is still a pair of $\mathscr{B}_H$. Hence, $(0, y_1 - y) \in \mathscr{B}_H$ and $\lim_{t\to-\infty} y_1(t) - y(t) = 0$. By Proposition 8, there exist coefficients $f_i \in \mathbb{R}$, $i = 1, \ldots n$ such that

$$y_1(t) - y(t) = \sum_{i=1}^{n} f_i m_i^{\mathrm{p}}(t),\ t \in \mathbb{R} \tag{5.14}$$

where the $m_i^{\mathrm{p}}(t)$, $i = 1, \ldots n$ are the pole modes of H (cf. Definition 24). The asymptotic stability of H implies the following. If $m_i^{\mathrm{p}}(t)$ is associated to a real pole we have $\lim_{t\to-\infty} m_i^{\mathrm{p}}(t) = +\infty$ whereas if $m_i^{\mathrm{p}}(t)$ is associated to a complex pole then the mode $m_i^{\mathrm{p}}(t)$ oscillates between $-\infty$ and $+\infty$ as t goes to $-\infty$. By taking into account that $m_i^{\mathrm{p}}(t)$, $i = 1, \ldots n$ is a linear independent base in C^∞ it follows that $f_i = 0$, $i = 1, \ldots n$ necessarily, otherwise the limit $\lim_{t\to-\infty} y_1(t) - y(t) = 0$ cannot hold. Hence, from (5.14) we obtain $y_1 \equiv y$ and this proves the sought uniqueness. □

Given an input $u \in C^\infty_{\mathrm{p}}(\mathbb{R})$ with $u(-\infty) = 0$ the computation of the forced response from time $-\infty$ is addressed by the following result.

Proposition 12 (Forced response operator)**.** *Assume that H is asymptotically stable. Let be given an input $u \in C^\infty_{\mathrm{p}}(\mathbb{R})$ with $u(-\infty) = 0$ and introduce the following operator:*

$$H(u)(t) := h_{\mathrm{hg}} u(t) + \int_{-\infty}^{t} h_{\mathrm{dy}}(t-v)u(v)dv,\ t \in \mathbb{R}. \tag{5.15}$$

Then $(u, H(u)) \in \mathscr{B}$ and $H(u)(-\infty) = 0$, i.e. $H(u)(t)$ is the forced response to input u of H from time $-\infty$.

Proof. Assume for simplicity that all the poles of $H(s)$ are real and simple so that $h_{\mathrm{dy}}(t) = \sum_{i=1}^{n} d_i e^{p_i t}$, $t \in \mathbb{R}$ (cf. (5.4)). First we show that $(u, H(u)) \in \mathscr{B}$. Rewrite relation (5.15) as $H(u)(t) = h_{\mathrm{hg}} u(t) + \int_{-\infty}^{0} h_{\mathrm{dy}}(t-v)u(v)dv + \int_{0}^{t} h_{\mathrm{dy}}(t-v)u(v)dv$ and note that $\int_{-\infty}^{0} h_{\mathrm{dy}}(t-v)u(v)dv = \sum_{i=1}^{n} f_i e^{p_i t}$, $t \in \mathbb{R}$ with

$$f_i := d_i \int_{-\infty}^{0} e^{-p_i v} u(v) dv.$$

Coefficient f_i is well-defined because the involved integral is finite. Indeed $p_i < 0$ (H is asymptotically stable) and $u(t)$ is bounded over $(-\infty, 0]$ because $u \in C^\infty_{\mathrm{p}}(\mathbb{R})$ and $u(-\infty) = 0$. Therefore, by Proposition 8 it follows that $(u, H(u)) \in \mathscr{B}$.

In the following we prove that $H(u)(-\infty) = 0$, i.e. $\lim_{t \to -\infty} H(u)(t) = 0$. The hypothesis $u(-\infty) = 0$ means that for any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that $|u(t)| < \varepsilon$ $\forall t < -T_\varepsilon$. From (5.15)

$$|H(u)(t)| \leq |h_{\mathrm{hg}}||u(t)| + \sum_{i=1}^{n} e^{p_i t} |d_i| \left| \int_{-\infty}^{t} e^{-p_i v} u(v) dv \right|,$$

$t \in \mathbb{R}$. If $t < -T_\varepsilon$ then $\left| \int_{-\infty}^{t} e^{-p_i v} u(v) dv \right| \leq \frac{\varepsilon}{-p_i} e^{-p_i t}$. Hence

$$|H(u)(t)| < \left(|h_{\mathrm{hg}}| + \sum_{i=1}^{n} \frac{|d_i|}{-p_i} \right) \cdot \varepsilon \qquad \forall t < -T_\varepsilon\,. \tag{5.16}$$

Since ε can be chosen arbitrarily small, inequality (5.16) proves that $H(u)(-\infty) = 0$. □

Remark 15. *With a slight abuse of notation we denote the forced response operator introduced in (5.15) with the same symbol H used to denote the system itself and its transfer function $H(s)$. The context clarifies what H stands for.*

A related technical result that does not require the system's asymptotic stability follows.

Lemma 12. *Assume that H is initially at rest at time $-\infty$ with $(u,y) \in \mathscr{B}_H$. The input u be identically zero, i.e. $u \equiv 0$, and the output y be bounded over $\mathbb{R}$. Then $y \equiv 0$, i.e. the output is identically zero too.*

Proof. Since $u \equiv 0$, by Proposition 8, $y(t) = \sum_{i=1}^{n} f_i m_i^{\mathrm{p}}(t)$, $t \in \mathbb{R}$. Then, by a contradiction argument we prove $f_i = 0$, $i = 1, \ldots n$ so that $y \equiv 0$. Indeed, suppose that there exists $f_j \neq 0$ for some $j \in \{1, \ldots, n\}$. There are two cases for the corresponding pole mode $m_j^{\mathrm{p}}(t)$: (1) It is unbounded over $\mathbb{R}$ because the associated pole has negative or positive real part or it is a multiple pole on the imaginary axis with multiplicity greater or equal to two. (2) It is bounded over $\mathbb{R}$ because the associated pole is simple and purely imaginary. In this case $m_j^{\mathrm{p}}(t) = 1$ or $\sin(\omega t)$ or $\cos(\omega t)$ (cf. Definition 24).

For the case (1) $f_j m_j^{\mathrm{p}}(t)$ is unbounded over $\mathbb{R}$ and so is $y(t)$ because any linear combination of other pole modes cannot make $y(t)$ bounded over $\mathbb{R}$ (the set $m_i^{\mathrm{p}}(t)$, $i = 1, \ldots, n$ is a linear independent base) and so we have a contradiction. For the case (2) $\lim_{t \to -\infty} f_j m_j^{\mathrm{p}}(t) \neq 0$ and this implies that $\lim_{t \to -\infty} y(t) \neq 0$ because any linear combination of other pole modes cannot make $y(-\infty) = 0$. Since the system is initially at rest, here there is a contradiction. □

5.4 The inversion-based control architectures

In the framework of inversion-based control, consider a plant to be controlled whose nominal transfer function is $P(s)$ and its relative degree is r_P (≥ 0). The actual plant that may differ from the nominal plant due to model inaccuracies and perturbations is denoted by $\tilde{P}$ and its transfer function is $\tilde{P}(s)$. The set of all possible perturbed plants $\tilde{P}$ belongs to $\mathscr{P}$, the *uncertain plant set* (also $P \in \mathscr{P}$).

There are two main inversion-based control architectures (also called schemes in the following): 1) the *plant inversion* architecture [14, 67] and 2) the *closed-loop inversion* architecture [43, 45]. They are depicted by block diagrams in Fig. 5.1 and 5.2 respectively.

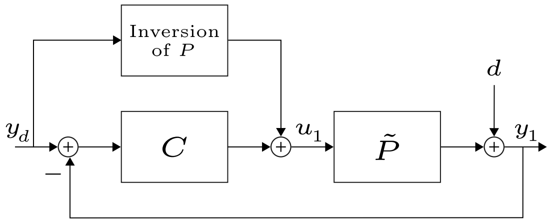

Figure 5.1: Plant inversion architecture or scheme 1.

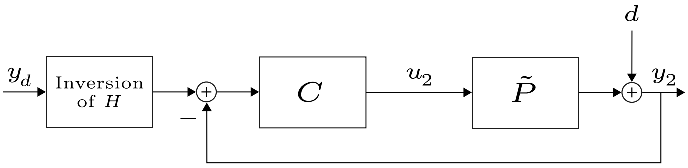

Figure 5.2: Closed-loop inversion architecture or scheme 2 in which $H := (1+CP)^{-1}CP$.

Both architectures use a (feedback) controller C having relative degree $r_C \geq 0$. This controller is designed to ensure the well-posedness [62] and the internal asymptotic stability of the closed-loop system for any $\tilde{P} \in \mathscr{P}$. Thus, the following assumption is considered [60].

Assumption 1. *For any $\tilde{P} \in \mathscr{P}$: (i) $\lim_{|s|\to\infty} 1 + C(s)\tilde{P}(s) \neq 0$; (ii) all the roots of $1 + C(s)\tilde{P}(s) = 0$ have negative real parts; (iii) there are no pole-zero cancellations between $C(s)$ and $\tilde{P}(s)$ on the closed RHP.*

Further assumptions are the following.

Assumption 2. *The zero dynamics of plant P and controller C are both hyperbolic, i.e. there are no zeros of $P(s)$ and $C(s)$ on the imaginary axis of the complex plane.*

Assumption 3. *The disturbance $d \in C_{\mathrm{p}}^{\infty}(\mathbb{R})$ is bounded over $\mathbb{R}$.*

The nominal closed-loop transfer function is $H(s) := [1 + C(s)P(s)]^{-1}C(s)P(s)$

and its relative degree is $r = r_C + r_P$. Let the causal desired output be

$$y_{\mathtt{d}} \in C^{\infty}_{\mathtt{p}}(\mathbb{R}) \cap C^{r-1} \tag{5.17}$$

and assume that $y_{\mathtt{d}}$ and its derivatives $y_{\mathtt{d}}^{(1)}, \ldots, y_{\mathtt{d}}^{(r)}$ are all bounded time-functions (cf. Subsection 5.2.2). The inverse inputs applied to H that cause $y_{\mathtt{d}}$ are the standard inverse $u'_{H,\mathtt{d}}$ and the stable inverse $u_{H,\mathtt{d}}$. These inverses whose closed-form expressions are reported in (5.8) and (5.11) are used in the closed-loop inversion architecture or scheme 2.

The nominal plant and the controller transfer function are written as

$$P(s) = \frac{b_P(s)}{a_P(s)} \text{ and } C(s) = \frac{b_C(s)}{a_C(s)} \tag{5.18}$$

with $a_P(s)$, $b_P(s)$ and $a_C(s)$, $b_C(s)$ being both coprime polynomial pairs. By Assumption 2, polynomials $b_P(s)$ and $b_C(s)$ can be factorized as

$$b_P(s) = b_P^-(s) b_P^+(s) \text{ and } b_C(s) = b_C^-(s) b_C^+(s) \tag{5.19}$$

with $b_P^-(s)$, $b_C^-(s)$, $b_P^+(-s)$, and $b_C^+(-s)$ being all Hurwitz polynomials.

The input-output inversion of the nominal plant follows (cf. Subsection 5.2.2). By polynomial division $a_P(s) = q_P(s) b_P(s) + c_P(s)$ with $\deg c_P < \deg b_P$ so that the plant's transfer function inverse is expressed by

$$P^{-1}(s) = q_P(s) + P_0(s) \tag{5.20}$$

with $P_0(s) = \frac{c_P(s)}{b_P(s)}$ representing the plant's zero dynamics. $P_0(s)$ can be split into stable and unstable parts by partial fraction expansion as

$$P_0(s) = \frac{c_P(s)}{b_P^-(s) b_P^+(s)} = P_0^-(s) + P_0^+(s) \tag{5.21}$$

having set $P_0^-(s) = \frac{c_P^-(s)}{b_P^-(s)}$, $P_0^+(s) = \frac{c_P^+(s)}{b_P^+(s)}$ with c_P^- and c_P^+ being suitable polynomials. Define $p_0(t) := \mathscr{L}_{\mathtt{ae}}^{-1}[P_0(s)]$, $p_0^-(t) := \mathscr{L}_{\mathtt{ae}}^{-1}[P_0^-(s)]$, and $p_0^+(t) := \mathscr{L}_{\mathtt{ae}}^{-1}[P_0^+(s)]$ so that the standard inverse input and the stable inverse one can be expressed respectively as ($t \in \mathbb{R}$)

$$u'_{P,\mathtt{d}}(t) = q_P(D) y_{\mathtt{d}}(t^+) + \int_0^t p_0(t-v) y_{\mathtt{d}}(v) dv, \tag{5.22}$$

$$u_{P,\mathtt{d}}(t) = q_P(D)y_{\mathtt{d}}(t^+) + \int_0^t p_0^-(t-v)y_{\mathtt{d}}(v)dv$$
$$- \int_t^{+\infty} p_0^+(t-v)y_{\mathtt{d}}(v)dv. \tag{5.23}$$

Analogously to (5.12), the stable inverse $u_{P,\mathtt{d}}$ can be written as

$$u_{P,\mathtt{d}}(t) = u'_{P,\mathtt{d}}(t) + u_{P,\mathtt{ps}}(t),\, t \in \mathbb{R} \tag{5.24}$$

in which

$$u_{P,\mathtt{ps}}(t) := -\int_0^{+\infty} p_0^+(t-v)y_d(v)dv \tag{5.25}$$

that is defined on $\mathbb{R}$ is the *plant's preaction signal* (cf. Remark 13).

The inverses (5.22) and (5.23) are used in the plant inversion architecture or scheme 1.

Remark 16. *To spare symbols to be used, H denotes both a generic system in the preliminaries in Section 5.2 and the nominal closed-loop system of scheme 2.*

Remark 17. *Note that the desired output $y_{\mathtt{d}}$ (5.17) has a continuity order sufficiently high to be inverted on the nominal plant P (cf. [12]). Indeed, $y_{\mathtt{d}} \in C^{r-1}$ implies $y_{\mathtt{d}} \in C^{r_P-1}$ because $r = r_C + r_P \geq r_P$.*

5.5 Equivalence results with standard inverses

When the inversion architectures 1 and 2 (cf. Fig. 5.1 and 5.2) are used with standard inverses — which are always causal signals — the following result follows.

Proposition 13. *Assume the inversion architectures 1 and 2 are at rest at time $-\infty$ and apply the desired output $y_{\mathtt{d}}$ and the standard inverses $u'_{P,\mathtt{d}}$ and $u'_{H,\mathtt{d}}$ respectively. Then $u_1 \equiv u_2$ and $y_1 \equiv y_2$ for any perturbed plant $\tilde{P} \in \mathscr{P}$ and any disturbance d.*

Proof. The linearity of both the control schemes 1 and 2 and the inversion operators (5.22) and (5.8) (cf. Remark 14) implies that signals u_1, u_2 and y_1, y_2 can be determined as with $j = 1,2$

$$u_j(t) = u_j(t)|_{y_{\mathtt{d}}\equiv 0,d} + u_j(t)|_{y_{\mathtt{d}},d\equiv 0},\; t \in \mathbb{R}; \tag{5.26}$$

$$y_j(t) = y_j(t)|_{y_{\mathtt{d}}\equiv 0,d} + y_j(t)|_{y_{\mathtt{d}},d\equiv 0},\; t \in \mathbb{R}. \tag{5.27}$$

When the desired output is identically zero over $\mathbb{R}$, evidently $u_1(t)|_{y_d\equiv 0,d} = u_2(t)|_{y_d\equiv 0,d}$ and $y_1(t)|_{y_d\equiv 0,d} = y_2(t)|_{y_d\equiv 0,d}$ for all $t \in \mathbb{R}$. Moreover, since the control schemes are at rest at time $-\infty$ and y_d, $u'_{P,d}$, and $u'_{H,d}$ are all causal, it follows that $u_1(t)|_{y_d,d\equiv 0} = u_2(t)|_{y_d,d\equiv 0} = 0$ and $y_1(t)|_{y_d,d\equiv 0} = y_2(t)|_{y_d,d\equiv 0} = 0$, $t < 0$. To complete these identities for nonnegative times consider the following. Let us define $Y_d(s) := \mathscr{L}[y_d(t)]$ and with $j = 1,2$

$$U_j(s) := \mathscr{L}[u_j(t)|_{y_d,d\equiv 0}],\ Y_j(s) := \mathscr{L}[y_j(t)|_{y_d,d\equiv 0}].$$

The Laplace transforms of the standard inverses are simply expressed by

$$\mathscr{L}[u'_{P,d}(t)] = P^{-1}(s)Y_d(s),\ \ \mathscr{L}[u'_{H,d}(t)] = H^{-1}(s)Y_d(s)$$

and the control schemes dictate the following algebraic relations (for simplicity the argument s is dropped):

$$U_1 = P^{-1}Y_d + C(Y_d - \tilde{P}U_1)\ , \tag{5.28}$$

$$U_2 = C(H^{-1}Y_d - \tilde{P}U_2)\ . \tag{5.29}$$

Since $H^{-1} = C^{-1}P^{-1}(1+CP)$, equality (5.29) becomes after some algebraic manipulations

$$U_2 = P^{-1}Y_d + C(Y_d - \tilde{P}U_2)\ . \tag{5.30}$$

Therefore, from (5.28) and (5.30) it follows that

$$U_1 = U_2 = (1 + C\tilde{P})^{-1}(P^{-1} + C)Y_d\ .$$

Hence, by Laplace anti-transformation $u_1(t)|_{y_d,d\equiv 0} = u_2(t)|_{y_d,d\equiv 0}$, $t \geq 0$. Since $Y_1 = \tilde{P}U_1$ and $Y_2 = \tilde{P}U_2$ we also deduce $Y_1 = Y_2$ and finally $y_1(t)|_{y_d,d\equiv 0} = u_2(t)|_{y_d,d\equiv 0}$, $t \geq 0$. □

Proposition 13 states an equivalence between the two architectures regardless of whether or not the (nominal) plant and the controller are minimum-phase. However, when the plant is nonminimum-phase the standard inverses $u'_{P,d}$ and $u'_{H,d}$ are both exponentially unbounded so that these cannot be used in a real implementation.

On the other hand, when the plant and controller are minimum-phase the standard inverses $u'_{P,\mathrm{d}}$ and $u'_{H,\mathrm{d}}$ are both bounded (over $\mathbb{R}$) because the inverse systems P^{-1} and H^{-1} are asymptotically stable (cf. Assumption 2). Hence, architectures 1 and 2 can use these standard inverses and the resulting signals (in particular u_1, y_1 and u_2, y_2) are all bounded due to the standing assumptions (cf. Assumptions 1 and 3). Therefore, Proposition 13 directly leads to the following result.

Corollary 4 (Minimum-phase case equivalence)**.** *Let P and C be minimum-phase systems. Assume the inversion architectures 1 and 2 are at rest at time $-\infty$ and apply the desired output y_d and the standard inverses $u'_{P,\mathrm{d}}$ and $u'_{H,\mathrm{d}}$ respectively. Then $u_1 \equiv u_2$ and $y_1 \equiv y_2$ for any perturbed plant $\tilde{P} \in \mathscr{P}$ and any disturbance d. Moreover, all these signals are bounded.*

Corollary 4 means that when P and C are minimum-phase, the plant and the closed-loop inversion architectures deliver the same performance regardless of uncertainties, perturbations, and disturbances on the plant. Essentially, this result was originally stated in [50] for discrete-time systems.

Remark 18. *The statement of Corollary 4 also holds by using the stable inverses $u_{P,\mathrm{d}}$ and $u_{H,\mathrm{d}}$. Indeed, when P and C are minimum-phase $u_{P,\mathrm{d}}$ and $u_{H,\mathrm{d}}$ coincide with the standard inverses $u'_{P,\mathrm{d}}$ and $u'_{H,\mathrm{d}}$ respectively, i.e. $u_{P,\mathrm{d}} \equiv u'_{P,\mathrm{d}}$, $u_{H,\mathrm{d}} \equiv u'_{H,\mathrm{d}}$ (cf. Proposition 10 and Remark 13).*

5.6 Equivalence of the inversion architectures with stable inverses

For the general nonminimum-phase case, this section addresses the equivalence of the inversion-based control architectures when stable inverses are used. Before introducing the main result (Theorem 9 in Subsection 5.6.2) some crucial algebraic relations are first presented.

5.6.1 Algebraic results

The inverse transfer function of the closed-loop system H can be expressed as (cf. (5.6), (5.9) and (5.18), (5.19))

$$H^{-1}(s) = q(s) + H_0^-(s) + H_0^+(s) \tag{5.31}$$

with

$$H_0^-(s) = \frac{c_H^-(s)}{b_C^-(s)b_P^-(s)}, \quad H_0^+(s) = \frac{c_H^+(s)}{b_C^+(s)b_P^+(s)} \tag{5.32}$$

where c_H^-, c_H^+ are suitable polynomials. Factorize the polynomial $b_C^+(s)$ as

$$b_C^+(s) = b_{CC}^+(s)b_{CP}^+(s) \tag{5.33}$$

such that b_{CC}^+ and b_{CP}^+ are coprime and b_{CP}^+ has all the common roots of b_C^+ and b_P^+ (i.e. if z^+ satisfies $b_C^+(z^+) = 0$, $b_P^+(z^+) = 0$ then $b_{CP}^+(z^+) = 0$). Hence, the closed-loop's unstable zero dynamics $H_0^+(s)$ can be split up as follows

$$H_0^+(s) = H_{0,CC}^+(s) + H_{0,P}^+(s) \tag{5.34}$$

with

$$H_{0,CC}^+(s) = \frac{d_{CC}^+(s)}{b_{CC}^+(s)}, \;\; H_{0,P}^+(s) = \frac{d_P^+(s)}{b_{CP}^+(s)b_P^+(s)} \tag{5.35}$$

where d_{CC}^+, and d_P^+ are suitable polynomials. $H_{0,CC}^+$ and $H_{0,P}^+$ represent the closed-loop's unstable zero dynamics due to different sets of nonminimum-phase zeros of H. $H_{0,CC}^+$ exhibits the dynamics due to the nonminimum-phase zeros of C that do not coincide with any zero of P. $H_{0,P}^+$ shows the dynamics due to the nonminimum-phase zeros of P that may have increased multiplicity due to the possible presence of shared zeros with C.

Define

$$h_{0,CC}^+(t) := \mathscr{L}_{\mathtt{ae}}^{-1}[H_{0,CC}^+(s)], \;\; h_{0,P}^+(t) := \mathscr{L}_{\mathtt{ae}}^{-1}[H_{0,P}^+(s)]$$

and the algebraic relation (5.34) implies the following useful identity:

$$h_0^+(t) = h_{0,CC}^+(t) + h_{0,P}^+(t), \; t \in \mathbb{R}. \tag{5.36}$$

The fundamental algebraic result leading to Theorem 9 is the following.

Proposition 14. *There exists a polynomial $\lambda(s)$ such that*

$$a_C(s)P_0^+(s) - b_C(s)H_{0,P}^+(s) = \lambda(s)\ . \tag{5.37}$$

Proof. In the following the complex variable s has been omitted for simplicity. The inverse closed-loop transfer function can be expressed as (cf. (5.31), (5.32), (5.34), and (5.35))

$$H^{-1} = q + \frac{c_H^-}{b_C^- b_P^-} + \frac{d_{CC}^+}{b_{CC}^+} + H_{0,P}^+ .$$

Multiplying H^{-1} by b_C, we obtain (cf. (5.19) and (5.33))

$$b_C H^{-1} = b_C q + \frac{b_C^+ c_H^-}{b_P^-} + b_C^- b_{CP}^+ d_{CC}^+ + b_C H_{0,P}^+ . \tag{5.38}$$

By polynomial division of $b_C^+ c_H^-$ by b_P^-, there exist polynomials α, β for which $b_C^+ c_H^- / b_P^- = \alpha + (\beta / b_P^-)$ so that (5.38) becomes

$$b_C H^{-1} = b_C q + \alpha + \frac{\beta}{b_P^-} + b_C^- b_{CP}^+ d_{CC}^+ + b_C H_{0,P}^+ . \tag{5.39}$$

On the other hand, by definition of H it follows that $H^{-1} = 1 + C^{-1}P^{-1}$ and by (5.20) and (5.21) we have

$$b_C H^{-1} = b_C + a_C q_P + \frac{a_C c_P^-}{b_P^-} + a_C P_0^+$$

The polynomial division of $a_C c_P^-$ by b_P^- permits to write $a_C c_P^- / b_P^- = \gamma + (\delta / b_P^-)$ with suitable γ, δ polynomials. Hence

$$b_C H^{-1} = b_C + a_C q_P + \gamma + \frac{\delta}{b_P^-} + a_C P_0^+ \tag{5.40}$$

The right sides of (5.39) and (5.40) represent the same rational function $b_C H^{-1}$ in different ways. Now consider the partial fraction decomposition of $b_C H^{-1}$. The partial fractions associated to the stable poles of $b_C H^{-1}$ can only be found in the partial fraction decompositions of β / b_P^- and δ / b_P^- because the other poles of $b_C H^{-1}$ are to be found in $b_C H_{0,P}^+$ and $a_C P_0^+$ and they are all unstable. The uniqueness of the partial fraction decomposition along with β/b_P^- and δ/b_P^- being strictly proper rational functions dictates that $\beta/b_P^- = \delta/b_P^-$ so that $\beta = \delta$. Finally, the difference of the right side of (5.40) and that of (5.39) leads to (5.37) having set $\lambda := b_C q + \alpha + b_C^- b_{CP}^+ d_{CC}^+ - b_C - a_C q_P - \gamma$. □

In the following we assume for simplicity that the nonminimum-phase zeros of C and P are simple and real. Accordingly, we set

$$b_C^+(s) = \prod_{i=1}^{m_{CC}^+}(s - z_{CC,i}^+)\prod_{i=1}^{m_{CP}^+}(s - z_{CP,i}^+)$$

$$b_P^+(s) = \prod_{i=1}^{m_{PP}^+}(s - z_{PP,i}^+)\prod_{i=1}^{m_{CP}^+}(s - z_{CP,i}^+)$$

where $z_{CP,i}^+$, $i = 1,\dots,m_{CP}^+$ are the common unstable zeros of C and P whereas $z_{CC,i}^+$, $i = 1,\dots,m_{CC}^+$ and $z_{PP,i}^+$, $i = 1,\dots,m_{PP}^+$ are the unshared unstable zeros of C and P respectively. Hence, by partial fraction decomposition P_0^+, $H_{0,P}^+$, and $H_{0,CC}^+$ can be expressed as (cf. (5.21) and (5.35))

$$P_0^+(s) = \sum_{i=1}^{m_{PP}^+}\frac{\alpha_i}{s - z_{PP,i}^+} + \sum_{i=1}^{m_{CP}^+}\frac{\beta_i}{s - z_{CP,i}^+}, \tag{5.41}$$

$$H_{0,P}^+(s) = \sum_{i=1}^{m_{PP}^+}\frac{\gamma_i}{s - z_{PP,i}^+} + \sum_{i=1}^{m_{CP}^+}\left(\frac{\delta_i}{s - z_{CP,i}^+} + \frac{\varepsilon_i}{(s - z_{CP,i}^+)^2}\right), \tag{5.42}$$

$$H_{0,CC}^+(s) = \sum_{i=1}^{m_{CC}^+}\frac{\eta_i}{s - z_{CC,i}^+}. \tag{5.43}$$

Then, the algebraic relation (5.37) leads to the following lemma.

Lemma 13. *Coefficients of the decompositions of P_0^+ and $H_{0,P}^+$ in (5.41) and (5.42) satisfy the following identities*

$$\alpha_i a_C(z_{PP,i}^+) = \gamma_i b_C(z_{PP,i}^+), \quad i = 1,\dots,m_{PP}^+, \tag{5.44}$$

$$\beta_i a_C(z_{CP,i}^+) = \varepsilon_i b_{C,i}(z_{CP,i}^+), \quad i = 1,\dots,m_{CP}^+, \tag{5.45}$$

where $b_{C,i}(s)$, $i = 1, \ldots, m_{CP}^+$ *are polynomials defined by* $b_{C,i}(s) := b_C(s)/(s - z_{CP,i}^+)$.

Proof. Let us insert the partial fraction decompositions of $P_0^+(s)$ and $H_{0,P}^+(s)$ (cf. (5.41), (5.42)) in the identity (5.37). The resulting equality is then multiplied by $s - z_{PP,i}^+$, $i = 1, \ldots, m_{PP}^+$ to obtain

$$a_C(s)\left[\alpha_i + \sum_{j\neq i}\frac{\alpha_j(s-z_{PP,i}^+)}{s-z_{PP,j}^+} + \sum_{j=1}^{m_{CP}^+}\frac{\beta_j(s-z_{PP,i}^+)}{s-z_{CP,j}^+}\right] - b_C(s)\left\{\gamma_i + \sum_{j\neq i}\frac{\gamma_j(s-z_{PP,i}^+)}{s-z_{PP,j}^+} + \sum_{j=1}^{m_{CP}^+}\left[\frac{\delta_j(s-z_{PP,i}^+)}{s-z_{CP,j}^+} + \frac{\varepsilon_j(s-z_{PP,i}^+)}{(s-z_{CP,j}^+)^2}\right]\right\} = \lambda(s)(s-z_{PP,i}^+).$$

By setting $s = z_{PP,i}^+$ the identity (5.44) follows.

In a similar way, insert decompositions (5.41) and (5.42) in the identity (5.37) having expressed $b_C(s) = b_{C,i}(s)(s - z_{CP,i}^+)$, $i = 1, \ldots, m_{CP}^+$. The resulting equality is multiplied by $s - z_{CP,i}^+$ to obtain

$$a_C(s)\left[\sum_{j=1}^{m_{PP}^+}\frac{\alpha_j(s-z_{CP,i}^+)}{s-z_{PP,j}^+} + \beta_i + \sum_{j\neq i}\frac{\beta_j(s-z_{CP,i}^+)}{s-z_{CP,j}^+}\right] - b_{C,i}(s)\left[\sum_{j=1}^{m_{PP}^+}\frac{\gamma_j(s-z_{CP,i}^+)^2}{s-z_{PP,j}^+} + \delta_i(s-z_{CP,i}^+) + \sum_{j\neq i}\frac{\delta_j(s-z_{CP,i}^+)^2}{s-z_{CP,j}^+} + \varepsilon_i + \sum_{j\neq i}\frac{\varepsilon_j(s-z_{CP,i}^+)^2}{(s-z_{CP,j}^+)^2}\right] = \lambda(s)(s-z_{CP,i}^+).$$

Finally, set $s = z_{CP,i}^+$ to obtain the identity (5.45). □

5.6.2 Comparison with stable inverses

The stable inverses $u_{P,\mathrm{d}}$ and $u_{H,\mathrm{d}}$ to be applied to the control architectures 1 and 2 are obtained by adding to the standard inverses the *preaction signals* $u_{P,\mathrm{ps}}$ and $u_{H,\mathrm{ps}}$ respectively (cf. (5.25), (5.13) and Remark 13). Closed-form expressions of these signals can be determined as follows. First, recall that $p_0^+(t) = \mathscr{L}_{\mathrm{ae}}^{-1}[P_0^+(s)]$,

$h_0^+(t) = \mathscr{L}_{\mathtt{ae}}^{-1}[H_0^+(s)]$ and from (5.41), (5.42), (5.43) and (5.34) obtain ($t \in \mathbb{R}$)

$$p_0^+(t) = \sum_{i=1}^{m_{PP}^+} \alpha_i e^{z_{PP,i}^+ t} + \sum_{i=1}^{m_{CP}^+} \beta_i e^{z_{CP,i}^+ t}\,, \tag{5.46}$$

$$h_0^+(t) = \sum_{i=1}^{m_{CC}^+} \eta_i e^{z_{CC,i}^+ t} + \sum_{i=1}^{m_{PP}^+} \gamma_i e^{z_{PP,i}^+ t} + \sum_{i=1}^{m_{CP}^+} \left(\delta_i e^{z_{CP,i}^+ t} + \varepsilon_i t e^{z_{CP,i}^+ t}\right). \tag{5.47}$$

Then, by the direct computation of the integrals (5.25) and (5.13) we have ($t \in \mathbb{R}$)

$$u_{P,\mathtt{ps}}(t) = -\sum_{i=1}^{m_{PP}^+} \alpha_i f_i \cdot e^{z_{PP,i}^+ t} - \sum_{i=1}^{m_{CP}^+} \beta_i g_i \cdot e^{z_{CP,i}^+ t}\,, \tag{5.48}$$

$$u_{H,\mathtt{ps}}(t) = -\sum_{i=1}^{m_{CC}^+} \eta_i x_i \cdot e^{z_{CC,i}^+ t} - \sum_{i=1}^{m_{PP}^+} \gamma_i f_i \cdot e^{z_{PP,i}^+ t} - \sum_{i=1}^{m_{CP}^+} (\delta_i g_i - \varepsilon_i w_i + \varepsilon_i g_i \cdot t) \cdot e^{z_{CP,i}^+ t} \tag{5.49}$$

with $f_i := \int_0^{+\infty} e^{-z_{PP,i}^+ v} y_{\mathtt{d}}(v) dv$, $g_i := \int_0^{+\infty} e^{-z_{CP,i}^+ v} \cdot y_{\mathtt{d}}(v) dv$, $x_i := \int_0^{+\infty} e^{-z_{CC,i}^+ v} y_{\mathtt{d}}(v) dv$, $w_i := \int_0^{+\infty} e^{-z_{CP,i}^+ v} \cdot v \cdot y_{\mathtt{d}}(v) dv$.

Remarkably, the preaction signals $u_{P,\mathtt{ps}}$ and $u_{H,\mathtt{ps}}$ are tightly related one to the other by the controller dynamics as herein shown.

Proposition 15. *The preaction signals $u_{P,\mathtt{ps}}$ and $u_{H,\mathtt{ps}}$ satisfy the differential equation associated to the controller C:*

$$a_C(D) u_{P,\mathtt{ps}}(t) = b_C(D) u_{H,\mathtt{ps}}(t),\ t \in \mathbb{R}. \tag{5.50}$$

Proof. Given a polynomial $w(s)$ then $w(D)e^{zt} = w(z)e^{zt}$, $t \in \mathbb{R}$ so that if z is a root of $w(s)$ then $w(D)e^{zt} \equiv 0$. Hence, from (5.48) we obtain (in the following $t \in \mathbb{R}$)

$$a_C(D) u_{P,\mathtt{ps}}(t) = -\sum_{i=1}^{m_{PP}^+} \alpha_i f_i a_C(z_{PP,i}^+) e^{z_{PP,i}^+ t} - \sum_{i=1}^{m_{CP}^+} \beta_i g_i a_C(z_{CP,i}^+) e^{z_{CP,i}^+ t}\,. \tag{5.51}$$

On the other hand, from (5.49) we have

$$\begin{aligned} b_C(D)u_{H,\mathrm{ps}}(t) = &-\sum_{i=1}^{m_{PP}^+} \gamma_i f_i b_C(z_{PP,i}^+) e^{z_{PP,i}^+ t} \\ &-\sum_{i=1}^{m_{CP}^+} \varepsilon_i g_i b_C(D)(t e^{z_{CP,i}^+ t}). \end{aligned} \tag{5.52}$$

Taking into account that (cf. Lemma 13) $b_C(D) = b_{C,i}(D)(D - z_{CP,i}^+)$, $i = 1, \dots, m_{CP}^+$ and $(D - z_{CP,i}^+)(t e^{z_{CP,i}^+ t}) = e^{z_{CP,i}^+ t}$ it follows that $b_C(D)(t e^{z_{CP,i}^+ t}) = b_{C,i}(z_{CP,i}^+) e^{z_{CP,i}^+ t}$. Hence, by recalling identities (5.44) and (5.45) we conclude that expressions (5.51) and (5.52) coincide. □

Remark 19. *The differential identity (5.50) has a noteworthy system interpretation. When the closed-loop's preaction signal $u_{H,\mathrm{ps}}(t)$, $t \in \mathbb{R}$ is injected at the controller input, the corresponding output is the plant's preaction signal $u_{P,\mathrm{ps}}(t)$, $t \in \mathbb{R}$, i.e. $(u_{H,\mathrm{ps}}, u_{P,\mathrm{ps}}) \in \mathscr{B}_C$ (cf. Fig. 5.3).*

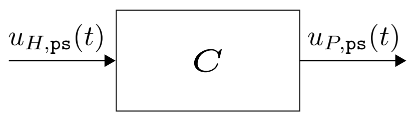

Figure 5.3: A special input-output pair of the controller C: the preaction signals $u_{H,\mathrm{ps}}(t)$ and $u_{P,\mathrm{ps}}(t)$.

The equivalence of the inversion-based control architectures asserted by Corollary 4 in the restricted minimum-phase case is extended to the general nonminimum-phase case when using stable inverses. This is the main finding herein stated.

Theorem 9. *Assume the inversion architectures 1 and 2 are at rest at time $-\infty$ and apply the desired output y_d and the stable inverses $u_{P,\mathrm{d}}$ and $u_{H,\mathrm{d}}$ respectively. Then $u_1 \equiv u_2$ and $y_1 \equiv y_2$ for any perturbed plant $\tilde{P} \in \mathscr{P}$ and any disturbance d. Moreover, all these signals are bounded.*

Proof. From (5.24) and (5.12) the stable inverses to be applied to the control architectures 1 and 2 can be written as $u_{P,\mathtt{d}} = u'_{P,\mathtt{d}} + u_{P,\mathtt{ps}}$ and $u_{H,\mathtt{d}} = u'_{H,\mathtt{d}} + u_{H,\mathtt{ps}}$ respectively. Hence, by the linearity of these architectures the control inputs u_1 and u_2 can be determined as

$$u_1(t) = u_1(t)\Big|_{\substack{y_\mathtt{d}, u'_{P,\mathtt{d}}, d \\ u_{P,\mathtt{ps}} \equiv 0}} + u_1(t)\Big|_{\substack{y_\mathtt{d} \equiv 0, u'_{P,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{P,\mathtt{ps}}}} , \tag{5.53}$$

$$u_2(t) = u_2(t)\Big|_{\substack{y_\mathtt{d}, u'_{H,\mathtt{d}}, d \\ u_{H,\mathtt{ps}} \equiv 0}} + u_2(t)\Big|_{\substack{y_\mathtt{d} \equiv 0, u'_{H,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{H,\mathtt{ps}}}} . \tag{5.54}$$

The equivalence provided by Proposition 13 for standard inverses showed that

$$u_1\Big|_{\substack{y_\mathtt{d}, u'_{P,\mathtt{d}}, d \\ u_{P,\mathtt{ps}} \equiv 0}} \equiv u_2\Big|_{\substack{y_\mathtt{d}, u'_{H,\mathtt{d}}, d \\ u_{H,\mathtt{ps}} \equiv 0}} .$$

Thus, to prove that $u_1 \equiv u_2$ we need to ascertain

$$u_1(t)\Big|_{\substack{y_\mathtt{d} \equiv 0, u'_{P,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{P,\mathtt{ps}}}} = u_2(t)\Big|_{\substack{y_\mathtt{d} \equiv 0, u'_{H,\mathtt{d}} \equiv 0, d \equiv 0 \\ u_{H,\mathtt{ps}}}} , t \in \mathbb{R} \tag{5.55}$$

and in the following the signals in (5.55) are simply denoted by u_1 and u_2.

Under the assumption of $y_\mathtt{d} \equiv 0$, $u'_{P,\mathtt{d}} \equiv 0$, $u'_{H,\mathtt{d}} \equiv 0$, and $d \equiv 0$ the only external inputs in schemes 1 and 2 are the preaction signals $u_{P,\mathtt{ps}}$ and $u_{H,\mathtt{ps}}$ respectively. In this scenario, the transfer functions from $u_{P,\mathtt{ps}}$ to u_1 and from $u_{H,\mathtt{ps}}$ to u_2 are

$$T_1(s) := \frac{1}{1 + C(s)\tilde{P}(s)} , \; T_2(s) := \frac{C(s)}{1 + C(s)\tilde{P}(s)} \tag{5.56}$$

respectively. Set $\tilde{P}(s) = b_{\tilde{P}}(s)/a_{\tilde{P}}(s)$ with $a_{\tilde{P}}$, $b_{\tilde{P}}$ being suitable polynomials. Hence

$$T_1(s) = \frac{a_{\tilde{P}}(s) a_C(s)}{a_\mathtt{cl}(s)} , \; T_2(s) = \frac{a_{\tilde{P}}(s) b_C(s)}{a_\mathtt{cl}(s)} \tag{5.57}$$

with $a_\mathtt{cl}(s) := a_C(s) a_{\tilde{P}}(s) + b_C(s) b_{\tilde{P}}(s)$ being an Hurwitz polynomial by Assumption 1. The inversion architectures are initially at rest, i.e. both C and $\tilde{P}$ are initially at rest, and $u_{P,\mathtt{ps}}(-\infty) = 0$, $u_{H,\mathtt{ps}}(-\infty) = 0$ (cf. (5.48), (5.49)). Therefore $u_1(-\infty) = 0$, $u_2(-\infty) = 0$ and u_1, u_2 are the forced responses to inputs $u_{P,\mathtt{ps}}$, $u_{H,\mathtt{ps}}$ of T_1, T_2

from time $-\infty$ respectively, i.e. $u_1(t) = T_1(u_{P,\mathtt{ps}})(t)$, $u_2(t) = T_2(u_{H,\mathtt{ps}})(t)$, $t \in \mathbb{R}$ (cf. Proposition 12).

The pairs $(u_{P,\mathtt{ps}}, u_1)$ and $(u_{H,\mathtt{ps}}, u_2)$ are weak solutions of the differential equations associated to T_1 and T_2 respectively, but since $u_{P,\mathtt{ps}}$, $u_{H,\mathtt{ps}} \in C^\infty$ and by virtue of Proposition 7 u_1, $u_2 \in C^\infty$, they are also *strong solutions* (cf. [12]), i.e.

$$a_{\mathtt{cl}}(D)u_1(t) = a_{\tilde{P}}(D)a_C(D)u_{P,\mathtt{ps}}(t),\ t \in \mathbb{R}, \tag{5.58}$$

$$a_{\mathtt{cl}}(D)u_2(t) = a_{\tilde{P}}(D)b_C(D)u_{H,\mathtt{ps}}(t),\ t \in \mathbb{R}. \tag{5.59}$$

Since $a_C(D)u_{P,\mathtt{ps}}(-\infty) = 0$, $b_C(D)u_{H,\mathtt{ps}}(-\infty) = 0$ from (5.58) and (5.59) it follows that u_1 and u_2 are the forced responses to input $a_C(D)u_{P,\mathtt{ps}}$ and $b_C(D)u_{H,\mathtt{ps}}$ respectively of the system associated to the transfer function $a_{\tilde{P}}(s)/a_{\mathtt{cl}}(s)$. But signals $a_C(D)u_{P,\mathtt{ps}}$ and $b_C(D)u_{H,\mathtt{ps}}$ coincide as shown in (5.50) (cf. Proposition 15). Hence, by Proposition 11 the forced responses u_1 and u_2 must be the same unique signal, i.e. identity (5.55) holds. It is then proved that $u_1 \equiv u_2$ under the application of $y_{\mathtt{d}}$, $u_{P,\mathtt{d}}$ (in scheme 1) and $u_{H,\mathtt{d}}$ (in scheme 2) for any perturbed plant $\tilde{P}$ and disturbance d.

Moreover, u_1, u_2 as well as y_1, y_2 are bounded over $\mathbb{R}$. Indeed, the external inputs of schemes 1 and 2, i.e. $y_{\mathtt{d}}$, $u_{P,\mathtt{d}}$, $u_{H,\mathtt{d}}$ and d, are all bounded over $\mathbb{R}$ (cf. Subsection 5.2.2 and Assumption 3) and the closed-loop system in schemes 1 and 2 is internally asymptotically stable (cf. Assumption 1). Since schemes 1 and 2 are initially at rest, all the signals are zero at time $-\infty$. Hence, the formal outputs of $\tilde{P}$ in these schemes, i.e. $y_1 - d$ and $y_2 - d$ respectively (the plant's outputs unaffected by the disturbance d), are both zero at time $-\infty$ and bounded over $\mathbb{R}$. Then $(u_1, y_1 - d)$, $(u_2, y_2 - d) \in \mathscr{B}_{\tilde{P}}$. From $u_1 \equiv u_2$ the difference between these pairs is $(0, y_2 - y_1)$ and it still belongs to $\mathscr{B}_{\tilde{P}}$ with $y_2 - y_1$ being bounded over $\mathbb{R}$. Hence, by Lemma 12 $y_2 - y_1 \equiv 0$, i.e. $y_1 \equiv y_2$. This concludes the proof. □

5.7 Simulation comparison and discussion

5.7.1 Simulation example

A set-point regulation problem for a nonminimum-phase plant affected by perturbations is addressed by inversion-based control. The nominal plant is [47]

$$P(s) = -4\frac{(s-1)(s+1)}{(s+2)(s^2+s+2)}$$

with relative degree $r_P = 1$. For the simulative implementation we compare the plant inversion architecture (cf. Fig. 5.1) with the closed-loop inversion architecture (cf. Fig. 5.2). Both schemes use the same controller

$$C(s) = -7\frac{(s+2)(s^2+s+2)(s-2)}{s(s+1)(s+10)(s+20)}$$

(its relative degree is $r_C = 0$) for which the closed-loop internal asymptotic stability is ensured for any perturbed plant belonging to an uncertain set $\mathscr{P}$ (Assumption 1). In the simulations, the following perturbed plant is considered

$$\tilde{P}(s) = -3.8\frac{(s-1.05)(s+0.97)}{(s+2.1)(s^2+1.05s+1.9)(0.01s+1)}$$

in which both parametric and high-frequency perturbations are present. Both the plant P and controller C have an hyperbolic zero dynamics (Assumption 2) and for simplicity, the disturbance d is set to be identically zero, i.e. $d \equiv 0$ (cf. Assumption 3).

The nominal closed-loop system is $H = [1+CP]^{-1}CP$ with relative degree $r = r_C + r_P = 1$. The desired output $y_{\mathtt{d}}$ is a monotonically increasing function defined by a *transition polynomial* [44],[38] whose *smoothness degree* [12] is chosen equal to 1 ($y_{\mathtt{d}} \in C^\infty_{\mathrm{p}}(\mathbb{R}) \cap C^1$; cf. Proposition 7 and Remark 17):

$$y_{\mathtt{d}}(t) := \begin{cases} 0 & t < 0 \\ 3\left(\frac{t}{\tau}\right)^2 - 2\left(\frac{t}{\tau}\right)^3 & t \in [0,\tau] \\ 1 & t > \tau \end{cases} .$$

The chosen transition time is $\tau = 3$ s (cf. Fig. 5.4). The stable inverse inputs $u_{P,\mathtt{d}}$

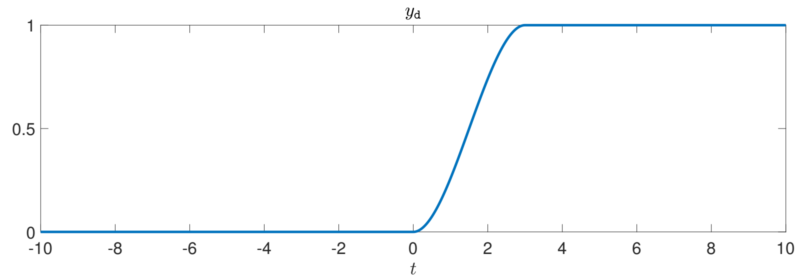

Figure 5.4: Plot of the desired output.

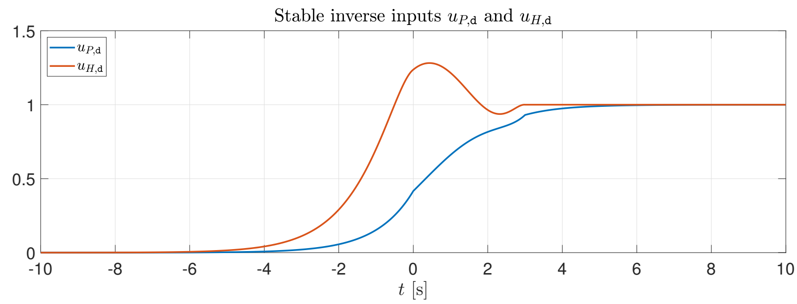

Figure 5.5: Plots of the stable inverses $u_{P,\mathrm{d}}$ (in blue) and $u_{H,\mathrm{d}}$ (in red) for the plant and closed-loop inversion architectures respectively.

and $u_{H,\mathrm{d}}$ to be applied on schemes 1 and 2 are determined by formulas (5.23) and (5.11) respectively (cf. Fig. 5.5). Stable inverses differ from standard inverses by the preaction signals (cf. Remark 13 and (5.25), (5.13)). In this example

$$u_{P,\mathrm{ps}}(t) = \frac{1+5e^{-3}}{3}e^{t}, \tag{5.60}$$

$$u_{H,\mathrm{ps}}(t) = \frac{11+55e^{-3}}{6}e^{t} - \frac{22+44e^{-6}}{21}e^{2t}. \tag{5.61}$$

Significantly, the controller's unstable zero mode e^{2t} only appears in the closed-loop's preaction signal whereas e^{t} (the plant's unstable zero mode) appears in both the plant's and closed-loop's preaction signals.

To perform the simulations it is necessary to truncate the exact infinite preaction control starting from time $-\infty$. Hence, a *preaction time* $t_{\mathtt{pre}}$ is set for which the preaction control is almost identically zero on $(-\infty, -t_{\mathtt{pre}})$. This means that the preaction control can be applied at time $-t_{\mathtt{pre}}$ with a negligible error on the resulting system response. Following a rule of thumb proposed in [38] (also cf. [42])

$$t_{\mathtt{pre}} = f_{\mathtt{pre}}/d_{\mathtt{rhp}} \tag{5.62}$$

where $f_{\mathtt{pre}}$ is a selectable factor to be chosen in the interval $[5, 10]$ and $d_{\mathtt{rhp}}$ is the minimum distance of the unstable zeros from the imaginary axis. We choose $f_{\mathtt{pre}} = 10$ and in this example $t_{\mathtt{pre}} = 10$ s for both signals (5.60) and (5.61). Consequently, for both schemes the preaction control starts at time -10 s.

The simulation results are reported in Fig. 5.6 and 5.7. These highlight that the plots of the plant's input-output pairs (u_1, y_1) and (u_2, y_2) in schemes 1 and 2 almost coincide. Indeed, with a negligible error due to the preaction truncations ($t_{\mathtt{pre}} = 10$ s) we almost have $u_1 \equiv u_2$ and $y_1 \equiv y_2$: $\max_{t\in\mathbb{R}} |u_1(t) - u_2(t)| = 7.47 \times 10^{-4}$ and $\max_{t\in\mathbb{R}} |y_1(t) - y_2(t)| = 4.40 \times 10^{-5}$. This is the expected result because the equivalence theorem (Theorem 1) predicts that the two inversion-based control architectures deliver the same performances.

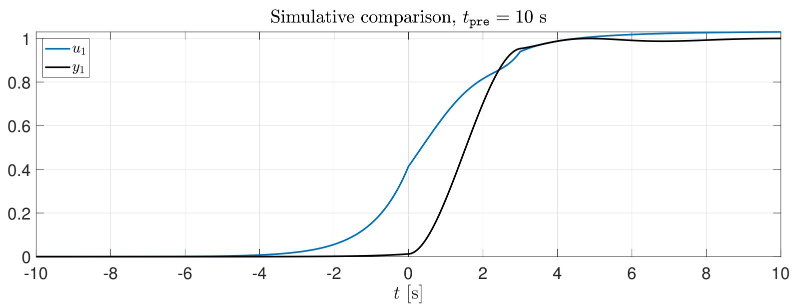

Figure 5.6: Plant's input (u_1) and output (y_1) in the plant inversion architecture (scheme 1) with preaction time $t_{\mathtt{pre}} = 10$ s.

On the other hand, it may be interesting to see what a rough truncation of the preaction controls determines. To this aim set e.g. $t_{\mathtt{pre}} = 2$ s (and so not following

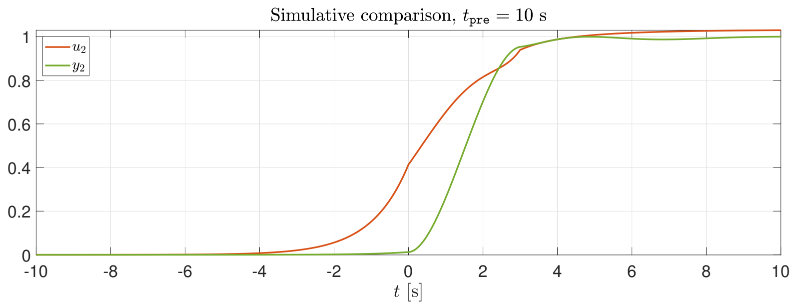

Figure 5.7: Plant's input (u_2) and output (y_2) in the closed-loop inversion architecture (scheme 2) with preaction time $t_{\mathtt{pre}} = 10$ s.

the rule (5.62)) in both schemes. Simulation results in Fig. 5.8 and 5.9 show what happens. The pairs (u_1, y_1) and (u_2, y_2) are sharply different, especially the inputs:

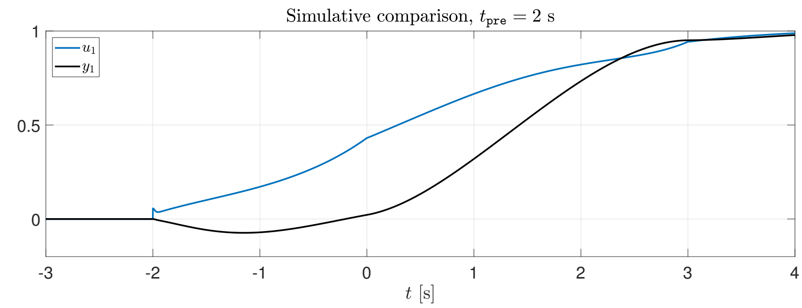

Figure 5.8: Plant's input (u_1) and output (y_1) in the plant inversion architecture (scheme 1) with preaction time $t_{\mathtt{pre}} = 2$ s.

$\max_{t\in\mathbb{R}} |u_1(t) - u_2(t)| = 2.09$ and $\max_{t\in\mathbb{R}} |y_1(t) - y_2(t)| = 0.123$. Hence, when (substantially) inexact inversion is implemented the two inversion-based control architectures are no longer equivalent.

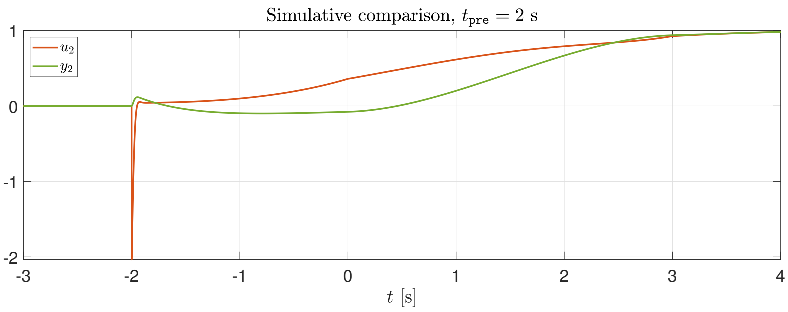

Figure 5.9: Plant's input (u_2) and output (y_2) in the closed-loop inversion architecture (scheme 2) with preaction time $t_{\texttt{pre}} = 2$ s.

5.7.2 Discussion

When measuring the performances on the perturbed plant's input and output the two inversion-based control architectures are equivalent provided that a careful truncation of the preaction control is made (cf. the example in Subsection 5.7.1). However, still holding this equivalence a control engineer may choose one or the other architecture on the grounds of implementation issues. These may regard 1) the ease of implementation and 2) the span times of the stable inverses.

(1) The inversion required by the plant inversion does not need knowledge of the controller. This may be a possible advantage over the closed-loop inversion architecture in which the inversion also depends on the controller. This may be relevant in application where supervisory control dictates a change of the controller depending on operational conditions.

On the other hand, in the minimum-phase case (both the plant and the controller do not have unstable zeros) with a high-gain controller the closed-loop inversion architecture may be advantageous. Indeed, in this case we may use an approximate inverse input simply given by the desired output itself [20].

(2) The span times of a stable inverse are the *preaction time*, the *transition time* and the *postaction time*. These times measure three different consecutive periods in which the stable inverse differ from the beginning and final steady-state inputs (sim-

ply 0 and 1 in the previous example; a more general context is described in [38]).

The preaction time $t_{\mathtt{pre}}$ that sets the starting of the preaction control may be different in the two architectures. Following rule (5.62) the preaction time is different in the two architectures only if

$$d_{\mathtt{rhp}}(C) < d_{\mathtt{rhp}}(P) \tag{5.63}$$

where $d_{\mathtt{rhp}}(C)$ and $d_{\mathtt{rhp}}(P)$ are the minimum distances of the unstable zeros from the imaginary axis for the controller and the nominal plant respectively (having chosen the same factor $f_{\mathtt{rhp}}$ for both schemes). If (5.63) is satisfied $t_{\mathtt{pre}}$ is smaller for the plant inversion architecture and so this architecture may be preferable in an actual implementation. However, nonminimum-phase controllers are somewhat infrequent in feedback systems [19] and satisfaction of condition(5.63) may be even more rare.

Hence, in the majority of cases, i.e. when the controller is minimum-phase or $d_{\mathtt{rhp}}(C) \geq d_{\mathtt{rhp}}(P)$, the preaction time set by rule (5.62) is the same in both architectures. However, this rule appears a rough formula that cannot be tuned to choose the acceptable mismatch between the plant's output and the desired output. Hence, the following more precise new rule is introduced to set preaction times.

Definition 27 (Output-error-based preaction time)**.** *Let be given a threshold $\varepsilon > 0$. Then set*

$$\begin{aligned} t_{P,\mathtt{pre}} = \min \Big\{ & \eta \geq 0 : \max_{t\in\mathbb{R}} |y_1(t) - y_{\mathtt{d}}(t)| \leq \varepsilon\,, \\ & y_1(t) = (1+CP)^{-1} P(\check{u}_{P,\mathtt{d}})(t) + H(y_{\mathtt{d}})(t),\, t \in \mathbb{R} \Big\}, \\ t_{H,\mathtt{pre}} = \min \Big\{ & \eta \geq 0 : \max_{t\in\mathbb{R}} |y_2(t) - y_{\mathtt{d}}(t)| \leq \varepsilon\,, \\ & y_2(t) = H(\check{u}_{H,\mathtt{d}})(t), t \in \mathbb{R} \Big\} \end{aligned}$$

with $\check{u}_{P,\mathtt{d}}(t) := 0$, $\check{u}_{H,\mathtt{d}}(t) := 0$ if $t < -\eta$ and $\check{u}_{P,\mathtt{d}}(t) := u_{P,\mathtt{d}}(t)$, $\check{u}_{H,\mathtt{d}}(t) := u_{H,\mathtt{d}}(t)$ otherwise (cf. the subsequent Remark 20).

In our example with the new rule by choosing $\varepsilon = 10^{-4}$ we obtain the different preaction times

$$t_{P,\mathtt{pre}} = 8.75 \text{ s} \ \text{ and } \ t_{H,\mathtt{pre}} = 9.13 \text{ s} \tag{5.64}$$

for schemes 1 and 2 respectively. Significantly, these truncated preactions still mean an almost exact inversion of $y_{\mathtt{d}}$ in both schemes. As a confirmation, the simulations with the perturbed plant $\tilde{P}$ show that u_1 and u_2 almost coincide ($\max_{t\in\mathbb{R}}|u_1(t)-u_2(t)| = 1.74\cdot 10^{-3}$) as well as y_1 and y_2 do ($\max_{t\in\mathbb{R}}|y_1(t)-y_2(t)| = 1.12\cdot 10^{-4}$). Therefore, figures in (5.64) show that scheme 1 (the plant inversion architecture) has a preaction time that is 4.2% shorter than that of scheme 2 and this may be a possible advantage of scheme 1 over scheme 2.

Typically, a desired output $y_{\mathtt{d}}$ is designed to join two distinct steady-state output segments by a desired transition output signal [42, 38]. Denoting by τ the corresponding transition time, the stable inverse $u_{\mathtt{d}}$ of a given system can be decomposed as

$$u_{\mathtt{d}}(t) = u_{\mathtt{ss}}(t) + u_{\mathtt{post}}(t),\, t \geq \tau$$

in which $u_{\mathtt{ss}}$ is the steady-state input and $u_{\mathtt{post}}$ is the so-called *postaction* (or *postactuation*) *control* [42, 12]. This $u_{\mathtt{post}}$ is a linear combination of the system's stable zero modes. Hence, $u_{\mathtt{d}}(t) \cong u_{\mathtt{ss}}(t)$ for $t \geq \tau + t_{\mathtt{post}}$ where $t_{\mathtt{post}}$ is the *postaction time* of $u_{\mathtt{d}}$ [42, 38]. In analogy to Definition 27, the postaction times can be set by the following new rule.

Definition 28 (Output-error-based postaction time)**.** *Let be given a threshold $\varepsilon > 0$. Then set*

$$\begin{aligned} t_{P,\mathtt{post}} = \min\Big\{ & \theta \geq 0 : \max_{t\in\mathbb{R}} |y_1(t) - y_{\mathtt{d}}(t)| \leq \varepsilon, \\ & y_1(t) = (1+CP)^{-1}P(\hat{u}_{P,\mathtt{d}})(t) + H(y_{\mathtt{d}})(t),\, t\in\mathbb{R}\Big\}, \\ t_{H,\mathtt{post}} = \min\Big\{ & \theta \geq 0 : \max_{t\in\mathbb{R}} |y_2(t) - y_{\mathtt{d}}(t)| \leq \varepsilon, \\ & y_2(t) = H(\hat{u}_{H,\mathtt{d}})(t),\, t\in\mathbb{R}\Big\} \end{aligned}$$

with $\hat{u}_{P,\mathtt{d}}(t) := u_{P,\mathtt{d}}(t)$, $\hat{u}_{H,\mathtt{d}}(t) := u_{H,\mathtt{d}}(t)$ if $t < \tau + \theta$ and $\hat{u}_{P,\mathtt{d}}(t) := u_{P,\mathtt{ss}}(t)$, $\hat{u}_{H,\mathtt{d}}(t) := u_{H,\mathtt{ss}}(t)$ otherwise.

Remark 20. *In Definition 27 (Definition 28), signals y_1 and y_2 are the outputs of the nominal plant P in schemes 1 and 2 when the preaction controls (postaction controls) of $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ are truncated at time $-\eta$ ($\tau+\theta$).*

In our example, still choosing $\varepsilon = 10^{-4}$ the new rule sets the following postaction times (cf. Fig. 5.5):

$$t_{P,\mathtt{post}} = 6.96 \text{ s} \;\text{ and }\; t_{H,\mathtt{post}} = 0 \text{ s}. \tag{5.65}$$

Actually, $t_{H,\mathtt{post}}$ is zero for any $\varepsilon > 0$ because there is no postaction control due to the absence of stable zeros in the nominal closed-loop system ($H(s) = [28(s-2)(s-1)]/(s^3+58s^2+116s+56)$). The truncation of the postaction control of $u_{P,\mathtt{d}}$ with $t_{P,\mathtt{post}} = 6.96$ s gives a negligible error in simulation. Indeed, along with no truncations of the preaction controls of $u_{P,\mathtt{d}}$ and $u_{H,\mathtt{d}}$ the comparison between the schemes with the perturbed plant shows $u_1 \cong u_2$ and $y_1 \cong y_2$ ($\max_{t\in\mathbb{R}} |u_1(t)-u_2(t)| = 1.01\cdot 10^{-4}$, $\max_{t\in\mathbb{R}} |y_1(t)-y_2(t)| = 8.89\cdot 10^{-5}$). The zero postaction time associated to scheme 2 shows that scheme 2 may be better than scheme 1.

The *total span time* of a stable inverse defined as the sum of the preaction, transition and postaction times may be a comprehensive figure to help to assess the implementation of the schemes. In the presented example $t_{P,\mathtt{span}} = t_{P,\mathtt{pre}} + \tau + t_{P,\mathtt{post}} = 18.71$ s and $t_{H,\mathtt{span}} = t_{H,\mathtt{pre}} + \tau + t_{H,\mathtt{post}} = 12.13$ s (the transition time τ is the same in both schemes since it is a feature of $y_{\mathtt{d}}$ only). The reason why a shorter interval $[-t_{\mathtt{pre}}, \tau + t_{\mathtt{post}}]$ may be useful mainly lies in the fact that a real-time update of the desired output is of concern in applications for which successive output transitions have to be implemented [38, 3, 7].

Broadly speaking, preaction times are similar in the schemes when the controller is minimum-phase. When it is nonminimum-phase $t_{H,\mathtt{pre}}$ tends to be a bit longer than $t_{P,\mathtt{pre}}$ because in this case the closed-loop system H has more unstable zeros than the plant P. When the controller determines stable pole-zero cancellations with the nominal plant, the postaction time $t_{H,\mathtt{post}}$ is usually shorter than $t_{P,\mathtt{post}}$ in many instances. Even $t_{H,\mathtt{post}} = 0$ when all the stable poles of P are canceled by the controller and no other stable zeros are added by the controller (as it happens in our example). In this case, seeking a shorter total span time the closed-loop inversion architecture is preferable.

5.8 Conclusions

The two main inversion-based control architectures, i.e. the plant and closed-loop inversion architectures, have been shown to be equivalent for the control of nonminimum-phase plants. Remarkably, this equivalence still holds when the architectures' feedback controller is nonminimum-phase too. By using a behavioral approach, the equivalence has been proved by showing that in both architectures the inputs and outputs of a perturbed plant are the same signals.

Beyond the equivalence, output-error-based rules to set the truncations of the preaction and postaction control of a stable inverse have been introduced. Still holding the architectures' equivalence when careful truncations are adopted, the preaction and postaction times are different in the architectures. This and other considerations may imply preferring one architecture with respect to the other in the practical implementation.

When inexact or approximate inversion is used (such as e.g. when the preaction control is abruptly truncated) the architectures' equivalence is lost. Hence, this case which is relevant in many instances deserves further investigations and comparisons between the architectures (cf. [50, 51, 8, 13]). On the other hand, with the aim to achieve high performances in control applications, future research on inversion architectures may focus on the (feedback) controller design methodology. Indeed, apparently very few contributions have appeared in the control literature on this topic (cf. [43, 48]).

Appendix A

Useful results about polynomial matrices and matrix fraction descriptions

The content of this chapter is mainly taken from [66], [28] and [55].

A.1 Background on polynomial matrices

Let $U(s) \in \mathbb{R}^{p\times p}[s]$ be a polynomial matrix, if $\det U(s)$ is a nonzero constant then we say $U(s)$ is a *unimodular matrix*. It is straightforward to show that $U(s)$ is unimodular if and only if $U^{-1}(s)$ is a polynomial matrix. Some examples of unimodular matrices are: the identity matrix, invertible matrices on $\mathbb{R}$ and lower/upper triangular square polynomial matrices with nonzero constants on the main diagonal. Further, observe that the product of unimodular matrices is unimodular.

We say that two polynomial matrices $Q_1(s), Q_2(s) \in \mathbb{R}^{p\times m}$ are *row equivalent* (*column equivalent*) if there exists a unimodular matrix $U(s)$ such that $Q_1(s) = U(s)Q_2(s)$ ($Q_1(s) = Q_2(s)U(s)$).

Given a polynomial matrix $Q(s) \in \mathbb{R}^{p\times m}[s]$, the *degree of the i-th row of* $Q(s)$ is the degree of the highest degree polynomial in $\boldsymbol{q}^i(s)$ (i.e. in the i-th row of $Q(s)$).

We denote the degree of the i-th row of $Q(s)$ by n_Q^i. The degree of $Q(s)$ is defined as $n_Q := \max\{n_Q^1, \ldots, n_Q^p\}$.

Let $P(s) \in \mathbb{R}^{p\times p}$. The *row degree coefficient matrix* (leading coefficient matrix) $P_{\mathtt{hr}} \in \mathbb{R}^{p\times p}$ of $P(s)$, is the real matrix whose elements are the coefficients of the highest degree terms in each row of $P(s)$, i.e. the (i,j) entry of $P_{\mathtt{hr}}$ is the coefficient of the monomial of degree n_P^i in $\big(P(s)\big)_{ij}$. Observe that, if $P(s)$ has row degrees $n_P^i \geq 0$, $i = 1, \ldots, m$, then

$$P(s) = \begin{bmatrix} s^{n_P^1} & & & \\ & s^{n_P^2} & & \\ & & \ddots & \\ & & & s^{n_P^p} \end{bmatrix} P_{\mathtt{hr}} + P_1(s), \tag{A.1}$$

where $P_1(s)$ is a suitable polynomial matrix whose i-th row degree is less then n_P^i, $i = 1, \ldots, p$. Remark that we required $n_P^i \geq 0$ in order to make sure that all elements in (A.1) are polynomials. For instance, if $P(s)$ is invertible then it is always possible to express it as in (A.1).

Property 8 ([66])**.** *If $P(s)$ is such that $n_P^i \geq 0$, $i = 1, \ldots, p$, then:*

$$\det P(s) = \det P_{\mathtt{hr}}(s) \cdot s^{n_P^1 + n_P^2 + \cdots + n_P^p} + g(s), \quad \deg g(s) < s^{n_P^1 + n_P^2 + \cdots + n_P^p}.$$

We deduce from Property 8 that, for an invertible matrix, $\deg\det P(s)$ can not be higher than $\sum_{i=1}^p n_P^i$, however it can be lower. This means that two invertible matrices can have determinants of same degree but different row degrees as these can be artificially high (e.g. $P_1(s) = U(s)P_2(s)$ with $U(s)$ unimodular).

Definition 29. *A square nonsingular matrix is* row reduced *(or* row proper*) if* $\det P_{\mathtt{hr}} \neq 0$.

It follows that a nonsingular matrix $P(s)$ is row reduced if and only if $\deg\det P(s) = \sum_{i=0}^p n_P^i$.

Theorem 10 ([66])**.** *A nonsingular polynomial matrix $P(s)$ is row equivalent to a row proper matrix, i.e. one can always find a unimodular matrix $U(s)$ sch that $U(s)P(s)$ is row reduced.*

Theorem 11 ([66])**.** *Any polynomial matrix $Z(s) \in \mathbb{R}^{p\times m}[s]$, with $p \leq m$, is column equivalent to the lower triangular matrix:*

$$\hat{Z}(s) = \left[\begin{array}{cccc|c} \hat{z}_{1,1}(s) & 0 & 0 & \dots & 0_{1\times(m-p)} \\ \hat{z}_{2,1}(s) & \hat{z}_{2,2}(s) & 0 & \dots & 0_{1\times(m-p)} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \hat{z}_{p,1}(s) & \hat{z}_{p,2}(s) & \dots & \hat{z}_{p,p}(s) & 0_{1\times(m-p)} \end{array}\right], \tag{A.2}$$

where, if $\deg \hat{z}_{i,i}(s) \geq 0$ *then* $\hat{z}_{i,1}(s), \hat{z}_{i,2}(s), \dots, \hat{z}_{i,i-1}(s)$ *have lower degree than* $\hat{z}_{i,i}(s)$*, $i = 2, \dots, p$.*

If $Z(s)$ in Theorem 11 is of rank p, then the polynomials $\hat{z}_{i,i}(s)$, $i = 1, \dots, p$ are nonzero and, without loss of generality, they can be assumed to be monic (if they are not, simply multiply $\hat{Z}(s)$ by a suitable diagonal matrix to make them monic). In this case, the structure of $\hat{Z}(s)$ described in Theorem 11 is known as the *row Hermite form* of $Z(s)$ (see Theorem 16.7 in [55]). Furthermore, observe that if $Z(s) = P(s)$ is invertible then $\hat{Z}(s) = \hat{P}(s)$ is row reduced. It follows from Theorem 10 and Theorem 11 that any nonsingular polynomial matrix $P(s)$ can be reduced to row proper form by either row or column operations.

Let $L(s) \in \mathbb{R}^{p\times p}[s]$ be such that:

$$P(s) = L(s)Q(s),$$

then we say that:

- $L(s)$ is a *left divisor* of $P(s)$;
- $P(s)$ is a *right multiple* of $L(s)$.

Observe that if $P(s)$ is a non singular matrix, then any divisor of $P(s)$ is nonsingular.

Definition 30. *A* greatest common left divisor *of $P(s)$, $Q(s)$ is a left divisor of $P(s)$ and $Q(s)$ (i.e. a* common *left divisor) which is a right multiple of every common left divisor of $P(s)$, $Q(s)$.*

A useful way to find a greatest common left divisor is the following.

Theorem 12 ([66])**.** *If* $Z(s) := [P(s)\ Q(s)] \in \mathbb{R}^{p\times(p+m)}[s]$ *is reduced to the lower triangular form* $\hat{Z}(s) = [L(s)\ 0_{p\times m}]$ *as described in Theorem 11, then* $L(s)$ *is a greatest common divisor of* $P(s)$*,* $Q(s)$*.*

The previous theorem describes how to find only *one* greatest common left divisor. The next result characterizes how greatest common left divisors are related to one other.

Property 9. *[66] If* $L(s)$ *is a greatest common left divisor of* $P(s)$ *and* $Q(s)$*, with* $P(s)$ *nonsingular, then any other greatest common left divisor* $G(s)$ *of* $P(s)$ *and* $Q(s)$ *is column equivalent to* $L(s)$*, i.e.* $G(s) = L(s)U(s)$ *with* $U(s)$ *unimodular.*

Definition 31. *The polynomial matrices* $P(s), Q(s)$*, are said* relatively left prime *(or simply* left coprime*) if all greatest common left divisors are unimodular matrices.*

A.2 Polynomial matrix fraction descriptions and their role in realization theory

Let $H(s) \in \mathbb{R}^{p\times m}(s)$ be a rational matrix and consider two polynomial matrices $P(s) \in \mathbb{R}^{p\times p}[s]$, $Q(s) \in \mathbb{R}^{p\times m}[s]$ where $P(s)$ is assumed to be invertible. We say that the product $P^{-1}(s)Q(s)$ is a *left matrix fraction description* (abbreviated *lMFD* or simply *MFD*) of $H(s)$ if $H(s) = P^{-1}(s)Q(s)$. We define the degree of the left matrix fraction description to be $\deg\det P(s)$.

Definition 32. *The left matrix fraction description* $P^{-1}(s)Q(s)$ *is said* left coprime *if* $P(s)$*,* $Q(s)$ *are left coprime polynomial matrices.*

Observe that since $P(s)$ is assumed to be invertible, the pair $P(s)$, $Q(s)$ has only nonsingular common left divisors. Next, remark that extracting common left divisors entails reducing the degree of the matrix fraction description. To see this, suppose $L(s)$ is a common left divisor of $P(s)$ and $Q(s)$, i.e. $P(s) = L(s)P_2(s)$, $Q(s) = L(s)Q_2(s)$, where $P_2(s), Q_2(s)$ are suitable polynomial matrices. Clearly, $\deg\det P(s) = \deg\det L(s) + \deg\det P_2(s)$ which, if $L(s)$ is not unimodular, implies $\deg\det P(s) >$

$\deg\det P_2(s)$. Hence, $P_2^{-1}(s)Q_2(s)$ is a matrix fraction description for $H(s)$ of lower degree than $P(s)^{-1}Q(s)$. It follows that if $P(s)$ and $Q(s)$ are left coprime then extracting common left divisors does not result in any reduction in the degree of the matrix fraction description. Otherwise, if they are not left coprime, the maximum reduction occurs by extracting a greatest common left divisor. This discussion should be enough to justify that extracting common left divisors from $P(s)$ and $Q(s)$ is the multivariable counterpart of canceling common factors in scalar transfer functions.

Property 10. *Let $H(s)$ be a strictly proper matrix transfer function described by the left matrix fraction description $P^{-1}(s)Q(s)$. Then there exists a completely observable realization of $H(s)$ with order equal to* $\deg\det P(s)$.

Proof. See for instance the *Observability form realization*, Section 6.4.4 in [28] (or the Observer form realization, Section 6.4.3, in [28]). □

Theorem 13 ([55])**.** *Let $H(s) = P^{-1}(s)Q(s)$ be a strictly proper matrix transfer function and (A,B,C) be a realization of order* $\deg\det P(s)$ *of $H(s)$. Then, (A,B,C) is a* minimal realization *of $H(s)$ if and only if the matrix fraction description $P^{-1}(s)Q(s)$ is left coprime.*

It follows from the previous theorem that all minimal realizations of $H(s)$ have order $\deg\det P(s)$, where $P^{-1}(s)Q(s)$ is any left coprime matrix fraction description of $H(s)$.

Theorem 14 ([55])**.** *Let $H(s) = P^{-1}(s)Q(s) = P_1^{-1}(s)Q_1(s)$ with $H(s)$ strictly proper and $(P(s),Q(s))$, $(P_1(s),Q_1(s))$ left coprime. Then, there exists a unimodular polynomial matrix $U(s) \in \mathbb{R}^{p\times p}[s]$ such that*

$$P_1(s) = U(s)P(s), \quad Q_1(s) = U(s)Q(s). \tag{A.3}$$

Appendix B

The observability canonical form and its link to the *Beghelli-Guidorzi input-output form*

The content of this chapter is taken from [22], [5] and [23].

B.1 The observability canonical form for completely observable systems

Consider the state-space model

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x}\end{aligned}, \tag{B.1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ denotes the state of the system, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the input and $\boldsymbol{y}(t) \in \mathbb{R}^p$ is the output. We assume $\operatorname{rank} C = p$ and that the system is observable, i.e. the matrix

$$O = \begin{bmatrix} C^\top & A^\top C^\top & \dots & A^{\top(n-1)} C^\top \end{bmatrix}^\top$$

is full column rank. The observability assumption implies that we can select n linearly independent row vectors form O. Clearly, such a choice can be made in many different ways but it is useful for us to follow a specific fixed order in selecting these independent vectors. To do so, we start by selecting the first row vector of O and we evaluate in ascending order each of the subsequent row vectors in the observability matrix. Evaluating these vectors means to check if these are linearly independent from all the previous rows of O. Once a certain row vector is found to be linearly independent from the previous ones, it is selected and added to a sequence that contains all previously selected vectors which are associated to the same output component, i.e. let $\boldsymbol{c}^i A^j$ be the selected vector, then it is added at the end of the sequence of form:

$$\boldsymbol{c}^i,\ \boldsymbol{c}^i A,\ \boldsymbol{c}^i A^2,\ \ldots, \boldsymbol{c}^i A^{j-1}.$$

Following this setting, we end up with m sequences (remark that by assumption all rows of C are linearly independent), one for each output component:

$$\boldsymbol{c}^1,\ \boldsymbol{c}^1 A,\ \boldsymbol{c}^1 A^2,\ \ldots, \boldsymbol{c}^1 A^{\nu_1 - 1}, \tag{B.2}$$
$$\boldsymbol{c}^2,\ \boldsymbol{c}^2 A,\ \boldsymbol{c}^2 A^2,\ \ldots, \boldsymbol{c}^2 A^{\nu_2 - 1}, \tag{B.3}$$
$$\vdots$$
$$\boldsymbol{c}^p,\ \boldsymbol{c}^p A,\ \boldsymbol{c}^p A^2,\ \ldots, \boldsymbol{c}^p A^{\nu_p - 1}, \tag{B.4}$$

where $\nu_1, \nu_2, \ldots, \nu_p$ are the number of vectors that have been added to each of the sequences (B.2)-(B.4). Observe that these sequences do not have 'holes', i.e. $\boldsymbol{c}^i A^{\nu_i}$ is the *first* row vector that does not enter the i-th sequence in (B.2)-(B.4). Indeed, if we denote by $\boldsymbol{c}^i A^{\nu_i}$ the first vector of the i-th sequence in (B.2)-(B.4) that is linearly dependent from the previous rows of O, then we can write:

$$\boldsymbol{c}^i A^{\nu_i} = \sum_{j=1}^{p} \sum_{k=1}^{\nu_{ij}} a_{ij,k} \boldsymbol{c}^j A^{k-1}, \tag{B.5}$$

where $a_{ij,k} \in \mathbb{R}$ are suitable coefficients and, because of the order followed in selecting independent vectors, the parameters v_{ij} are given by:

$$v_{ij} := \begin{cases} v_i & \text{if} \quad i = j; \\ \min\{v_i + 1,\ v_j\} & \text{if} \quad i > j; \\ \min\{v_i,\ v_j\} & \text{if} \quad i < j. \end{cases} \tag{B.6}$$

It follows from (B.5) that all subsequent vectors $\boldsymbol{c}^i A^{v_i + j}$, $j = 1, \ldots$ will not be selected, i.e. they will not enter the i-th sequence. In other words, for each of the sequences (B.2)-(B.4), once the first dependent vector $\boldsymbol{c}^i A^{v_i}$ is found, then the sequence stops increasing its size (cf. Property 3.3.4 in [4]).

Remark that, because of the observability assumption, $v_1 + v_2 + \cdots + v_p = n$, i.e. the vectors that form the sequences (B.2)-(B.4) are a base of $\mathbb{R}^n$. The tuple $(v_1, v_2, \ldots, v_p)$ is called the *ordered set of Kronecker invariants* of (C,A). The adjective ordered is here used to emphasize the fact that the i-th element of the tuple, v_i, is associated to the i-th output component. The integers v_1, v_2, $\ldots v_p$ are called *Kronecker invariants* of the pair (C,A). Note that, since $\operatorname{rank} C = p$ then $v_i \geq 1$. The scalars $a_{ij,k}$ that appear in (B.5) are called *characteristic parameters* of (C,A). The vectors that have made their way into sequences (B.2)-(B.4) are said *regular vectors*.

Note that the pair $(CT, T^{-1}AT)$, for any invertible T, has same Kronecker invariants and characteristics parameters as the pair (C,A). This means that the function which associates to the pair (C,A) its corresponding Kronecker invariants and characteristic parameters is invariant with respect to the equivalence relation resulting by a change of coordinates (for a detailed discussion, see [23]).

Set

$$T^{-1} := \begin{bmatrix} \boldsymbol{c}^{1\top} & (\boldsymbol{c}^1 A)^\top & \ldots & (\boldsymbol{c}^1 A^{v_1 - 1})^\top & \ldots & \boldsymbol{c}^{p\top} & (\boldsymbol{c}^p A)^\top & \ldots & (\boldsymbol{c}^p A^{v_p - 1})^\top \end{bmatrix}^\top, \tag{B.7}$$

then, performing the change of coordinates $\boldsymbol{x} = T\boldsymbol{w}$ yields:

$$\begin{aligned} \dot{\boldsymbol{w}} &= A_o \boldsymbol{w} + B_o \boldsymbol{u} \\ \boldsymbol{y} &= C_o \boldsymbol{w} \end{aligned}, \tag{B.8}$$

with

$$A_o = T^{-1}AT = [A_{o,ij}], \quad i,j = 1,\dots,p, \tag{B.9}$$

where matrices $A_{o,ii} \in \mathbb{R}^{\nu_i \times \nu_i}$, which form the block diagonal part of A_o, are in companion form, i.e.

$$A_{o,ii} = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \dots & & 1 \\ \hline a_{ii,1} & a_{ii,2} & \dots & & a_{ii,\nu_i} \end{array}\right], \quad i = 1,\dots,p, \tag{B.10}$$

and the other block matrices $A_{o,ij} \in \mathbb{R}^{\nu_i \times \nu_j}$ are of form:

$$A_{o,ij} = \left[\begin{array}{ccccccc} & & & 0_{(\nu_i-1)\times\nu_j} & & & \\ \hline a_{ij,1} & a_{ij,2} & \dots & a_{ij,\nu_{ij}} & 0 & \dots & 0 \end{array}\right], \quad i \neq j, \ i,j = 1,\dots,p. \tag{B.11}$$

The input matrix $B_o = T^{-1}B$ does not have a specific structure (in general, it is a full matrix), while the output matrix $C_o = CT \in \mathbb{R}^{p\times n}$ is:

$$C_o = \left[\begin{array}{c|c|c|c|c|c|c} 1 & & 0 & & & 0 & \\ 0 & & 1 & & & 0 & \\ \vdots & 0_{p\times(\nu_1-1)} & \vdots & 0_{p\times(\nu_2-1)} & \dots & \vdots & 0_{p\times(\nu_p-1)} \\ 0 & & 0 & & & 1 & \end{array}\right]. \tag{B.12}$$

We call the resulting model (A_o, B_o, C_o) the *observability form* of system (A,B,C). The name 'observability form' stems from the fact that we can extract the canonical base of $\mathbb{R}^n$ from the observability matrix of (A_o, C_o). Indeed, taking into account the simple structure of the pair (A_o, C_o), it is easy to verify that:

$$I_{n\times n} = \left[\boldsymbol{c}_{\boldsymbol{o}}^{1\top} \quad (\boldsymbol{c}_{\boldsymbol{o}}^1 A_o)^\top \quad \dots \quad (\boldsymbol{c}_{\boldsymbol{o}}^1 A_o^{\nu_1-1})^\top \quad \dots \quad \boldsymbol{c}_{\boldsymbol{o}}^{p\top} \quad (\boldsymbol{c}_{\boldsymbol{o}}^p A_o)^\top \quad \dots \quad (\boldsymbol{c}_{\boldsymbol{o}}^p A_o^{\nu_p-1})^\top\right]^\top. \tag{B.13}$$

B.2 The Beghelli-Guidorzi input-output form

Let $H(s) = C(sI - A)^{-1}B$ be the strictly proper matrix transfer function associated to the state space model (B.1) and $P^{-1}(s)Q(s)$ be any matrix fraction description of $H(s)$ with $\deg P(s) = n$. We are now going to see that it is possible to manipulate $P(s), Q(s)$ so as to obtain a new matrix fraction description of $H(s)$ which exhibits a strong link with the observability form of (A, B, C).

It was shown in [5] that there always exists a unimodular matrix $U(s) \in \mathbb{R}^{p\times p}[s]$ such that the pair

$$\big(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s)\big), \tag{B.14}$$

where $P_{\mathrm{g}}(s) := U(s)P(s)$, $Q_{\mathrm{g}}(s) := U(s)Q(s)$, satisfies all the following conditions:

1. The polynomials on the main diagonal of $P_g(s)$ are monic;
2. The degrees of the polynomial entries of $P_g(s)$ satisfy the following constraints:

$$\deg P_{\mathrm{g},ii}(s) \geq \deg P_{\mathrm{g},ij}(s) \quad \text{if } i > j; \tag{B.15}$$

$$\deg P_{\mathrm{g},ii}(s) > \deg P_{\mathrm{g},ij}(s) \quad \text{if } i < j; \tag{B.16}$$

$$\deg P_{\mathrm{g},ii}(s) > \deg P_{\mathrm{g},ji}(s) \quad \text{if } i \neq j. \tag{B.17}$$

3. The degrees of the entries of $P_{\mathrm{g}}(s)$ and $Q_{\mathrm{g}}(s)$ are such that:

$$\deg P_{\mathrm{g},ii}(s) > \det Q_{\mathrm{g},ij}(s). \tag{B.18}$$

It should be noted that, because of conditions (B.15), (B.16), the i-th row degree of $P_{\mathrm{g}}(s)$ is $n^i_{P_{\mathrm{g}}} = \deg P_{\mathrm{g},ii} \geq 0$. Furthermore, since the elements on the main diagonal $P_{\mathrm{g},ii}(s)$ are monic (recall condition 1.), it follows from (B.15), (B.16), that $P_{\mathrm{g}}(s)$ is row reduced. Indeed the leading coefficient matrix of $P_{\mathrm{g}}(s)$, $P_{\mathrm{g_{hr}}}$, is a lower triangular matrix of form:

$$P_{\mathrm{g_{hr}}} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & 1 & 0 & \vdots \\ \vdots & * & \ddots & 0 \\ * & \dots & * & 1 \end{bmatrix}.$$

Since $P_{\mathrm{g}}(s)$ is row reduced, this means that condition (B.18) is equivalent to requiring that $H(s)$ is strictly proper (see for instance Lemma 6.3-11 in [28]).

We call $(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s))$ the *Beghelli-Guidorzi form* of $(P(s), Q(s))$. It can be shown that there is one and only one pair $(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s))$ row equivalent to $(P(s), Q(s))$. Actually, more can be shown. Namely, $(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s))$ is a canonical form of $(P(s), Q(s))$, see [23] for more details. An algorithm to obtain the unimodular matrix $U(s)$ which transforms the pair $(P(s), Q(s))$ into $(P_{\mathrm{g}}(s), Q_{\mathrm{g}}(s))$ is described in [5].

Recall that we defined $P(s)$, $Q(s)$ such that $H(s) = P^{-1}(s)Q(s)$, $\deg\det P(s) = n$ where $H(s)$ is the matrix transfer function of model (B.1). It turns out that the Beghelli-Guidorzi form of $(P(s), Q(s))$ has some straightforward algebraic links with the observability form of (B.1). In other words, the knowledge of matrices (A_o, B_o, C_o), which form the observability form of (B.1), is sufficient for constructing the Beghelli-Guidorzi form of $(P(s), Q(s))$. This is summarized in what follows. Let

$$P_{\mathrm{g}} = [p_{\mathrm{g},ij}(s)], \quad i, j = 1, \ldots, p, \tag{B.19}$$

then

$$p_{\mathrm{g},ii}(s) = s^{v_i} - a_{ii,v_i} s^{v_i - 1} - \cdots - a_{ii,2} s - a_{ii,1}, \quad i = 1, \ldots p, \tag{B.20}$$

$$p_{\mathrm{g},ij}(s) = -a_{ij,v_{ij}} s^{v_{ij} - 1} - \cdots - a_{ij,2} s - a_{ij,1}, \quad i, j = 1, \ldots p, \tag{B.21}$$

where v_i denote the observability indexes of (C, A), v_{ij} are defined in (B.6) and $a_{ij,k}$, $k = 1, \ldots, v_{ij}$, are the characteristic parameters of (C, A). This means that the observability indexes and the characteristic parameters of (C, A) are everything that is needed to construct $P_{\mathrm{g}}(s)$. Remark that the row degrees of $P_{\mathrm{g}}(s)$ coincide with the *ordered* set of observability indexes of (C, A).

As for $Q_{\mathrm{g}}(s)$, let us denote the entries of $Q_{\mathrm{g}}(s)$ by

$$Q_{\mathrm{g}}(s) = [q_{\mathrm{g},ij}(s)], \quad i = 1, \ldots, p, j = 1, \ldots, m, \tag{B.22}$$

then these are expressed as

$$q_{\mathrm{g},ij}(s) = \beta_{ij,v_i} s^{v_i - 1} + \cdots + \beta_{ij,2} s + \beta_{ij,1}, \tag{B.23}$$

where the coefficients $\beta_{ij,k}$, $k = 1,\dots,v_i$ are determined by the entries of the matrix

$$B_g = MB_o = \begin{bmatrix} B_{g,1} \\ B_{g,2} \\ \vdots \\ B_{g,p} \end{bmatrix}, \quad B_{g,i} = \begin{bmatrix} \beta_{i1,1} & \beta_{i2,1} & \dots & \beta_{im,1} \\ \beta_{i1,2} & \beta_{i2,2} & \dots & \beta_{im,2} \\ \vdots & & & \\ \beta_{i1,v_i} & \beta_{i2,v_i} & \dots & \beta_{im,v_i} \end{bmatrix} \tag{B.24}$$

where

$$M = [M_{ij}], \quad i,j = 1,\dots,p \tag{B.25}$$

is a structurally nonsingular matrix whose constituting blocks have form:

$$M_{ii} = \begin{bmatrix} -a_{ii,2} & -a_{ii,3} & \dots & -a_{ii,v_i} & 1 \\ -a_{ii,3} & -a_{ii,4} & \dots & 1 & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 & 0 \\ -a_{ii,v_i} & \cdot^{\cdot^{\cdot}} & 0 & \dots & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{bmatrix}, \tag{B.26}$$

$$M_{ij} = \begin{bmatrix} -a_{ij,2} & -a_{ij,3} & \dots & -a_{ij,v_{ij}} & 0_{1\times v_j - v_{ij}} \\ -a_{ij,3} & -a_{ij,4} & \dots & 0 & 0 \\ \vdots & & \cdot^{\cdot^{\cdot}} & 0 & 0 \\ -a_{ij,v_{ij}} & \cdot^{\cdot^{\cdot}} & 0 & \dots & \vdots \\ 0_{v_i - v_{ij}\times 1} & 0 & \dots & \dots & 0 \end{bmatrix}. \tag{B.27}$$

Bibliography

[1] S. Ahuja and R. K. Arya. Laplace transform treatment for chemical engineering systems: whether to use 0^- or 0^+? *Chemical Engineering Technology*, 41(4):875–882, 2018.

[2] P. Antsaklis and A. N. Michel. *Linear systems*. McGraw-Hill, New York, 1997.

[3] K. Åström. Event-based control. In A. Astolfi and L. Marconi, editors, *Analysis and Design of Nonlinear Control Systems*, pages 127–147. Springer, Berlin, 2008.

[4] G. Basile and G. Marro. *Controlled and conditioned invariants in linear system theory*. Prentice Hall Englewood Cliffs, NJ, 1992.

[5] S. Beghelli and R. Guidorzi. A new input-output canonical form for multivariable systems. *IEEE Transactions on Automatic Control*, 21(5):692–696, 1976.

[6] L. Blanken and T. Oomen. Kernel-based identification of non-causal systems with application to inverse model control. *Automatica*, 114:108830, 2020.

[7] A. Boekfah and S. Devasia. Output-boundary regulation using event-based feedforward for nonminimum-phase systems. *IEEE Transactions on Control Systems Technology*, 24(1):265–275, Jan 2016.

[8] J. Butterworth, L. Pao, and D. Abramovitch. A comparison of control architectures for atomic force microscopes. *Asian Journal of Control*, 11(2):175–181, March 2009.

[9] C.-T. Chen. *Analog and digital control system design: transfer-function, state-space, and algebraic methods*. Saunders College Publishing, 1993.

[10] C.-T. Chen. *Linear System Theory and Design*. Oxford University Press, Inc., New York, NY, USA, 3rd edition, 1998.

[11] D. Chen. An iterative solution to stable inversion of nonminimum phase systems. In *Amer. Control Conf.*, pages 2960–2964, June 1993.

[12] A. Costalunga and A. Piazzi. A behavioral approach to inversion-based control. *Automatica*, 95:433–445, 2018.

[13] L. Dai, X. Li, Y. Zhu, and M. Zhang. Quantitative analysis on tracking error under different control architectures and feedforward methods. In *2019 American Control Conference (ACC)*, pages 5680–5686, 2019.

[14] S. Devasia. Should model-based inverse inputs be used as feedforward under plant uncertainty? *IEEE Transaction on Automatic Control*, 47(11):1865–1871, November 2002.

[15] S. Devasia, D. Chen, and B. Paden. Nonlinear inversion-based output tracking. *IEEE Trans. Autom. Control*, AC-41(7):930–942, July 1996.

[16] R. C. Dorf and R. H. Bishop. *Modern Control Systems*. Pearson Prentice Hall, eleventh edition, 2008.

[17] P. Falb and W. Wolovich. Decoupling in the design and synthesis of multivariable control systems. *IEEE Trans. Autom. Control*, 12(6):651–659, 1967.

[18] M. Farid and S. Lukasiewicz. On trajectory control of multi-link robots with flexible links and joints. In *Proc. of 1996 Canadian Conf. on Electrical and Computer Engineering*, pages 513–516, May 1996.

[19] A. Ferrante, W. Krajewski, A. Lepschy, and U. Viaro. Analytic stability margin design for unstable and nonminimum-phase plants. *IEEE Transactions on Automatic Control*, 47(12):2117–2121, 2002.

[20] G. Goodwin, S. Graebe, and M. Salgado. *Control System Design*. Prentice-Hall, Upper Saddle River, NJ, 2001.

[21] J. W. Grizzle. Linear Time-Invariant Systems by Martin Schetzen. *IEEE Control Systems Magazine*, 24(3):87–89, June 2004. Book review.

[22] R. Guidorzi. Canonical structures in the identification of multivariable systems. *Automatica*, 11(4):361–374, 1975.

[23] R. P. Guidorzi. Invariants and canonical forms for systems structural and parametric identification. *Automatica (Journal of IFAC)*, 17(1):117–133, 1981.

[24] R. M. Hirschorn. Invertibility of nonlinear control systems. *SIAM Journal on Control and Optimization*, 17(2):289–297, 1979.

[25] L. Hunt, G. Meyer, and R. Su. Noncausal inverses for linear systems. *IEEE Transaction on Automatic Control*, AC-41:608–611, 1996.

[26] A. Isidori. *Nonlinear Control Systems*. Springer, London, third edition edition, 1995.

[27] L. Jetto, V. Orsini, and R. Romagnoli. Almost perfect tracking through mixed numerical-analytical stable pseudo-inversion of non minimum phase plants. In *52nd IEEE Conference on Decision and Control*, pages 1453–1460, 2013.

[28] T. Kailath. *Linear systems*, volume 156. Prentice-Hall Englewood Cliffs, NJ, 1980.

[29] D. Kalman. The generalized Vandermonde matrix. *Mathematics Magazine*, 57(1):15–21, 1984.

[30] J. Kavaja, A. Minari, and A. Piazzi. Stable input-output inversion for nondecoupable nonminimum-phase linear systems. In *2018 European Control Conference (ECC)*, pages 2855–2860. IEEE, 2018.

[31] J. Kavaja and A. Piazzi. Input-output jumps of scalar linear systems. *IFAC-PapersOnLine*, 52(17):13–18, 2019.

[32] J. Kavaja and A. Piazzi. On the equivalence of model inversion architectures for control applications. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 5173–5179. IEEE, 2020.

[33] W. S. Levine, editor. *The Control Handbook: Control System Fundamentals*. CRC Press, second edition, 2010.

[34] K. H. Lundberg, H. R. Miller, and D. L. Trumper. Initial conditions, generalized functions, and the Laplace transform. *IEEE Control Systems Magazine*, 27(1):22–35, 2007.

[35] P. M. Mäkilä. A note on the Laplace transform method for intial value problems. *International Journal of Control*, 79(1):36–41, 2006.

[36] G. Marro and A. Piazzi. A geometric approach to multivariable perfect tracking. In *Proceedings of the 13th IFAC World Congress*, volume C, pages 241–246, San Francisco, USA, July 1996.

[37] R. Middleton, S. Graebe, A. Alén, and J. Shamma. Design methods. In W. Levine, editor, *The Control Handbook: Control Systems Fundamentals*, pages **19**–1–**19**–35. CRC Press, second edition, 2011.

[38] A. Minari, A. Piazzi, and A. Costalunga. Polynomial interpolation for inversion-based control. *European Journal of Control*, 56:62–72, November 2020.

[39] M. Mueller. Normal form for linear systems with respect to its vector relative degree. *Linear Algebra and its Appl.*, 430(4):1292–1312, 2009.

[40] N. S. Nise. *Control Systems Engineering*. Wiley, 4th edition, 2004.

[41] D. Pallastrelli and A. Piazzi. Stable dynamic inversion of nonminimum-phase scalar linear systems. In *Proc. of the 16th IFAC World Congress*, pages 107–112, Prague, Czech Republic, July 2005.

[42] H. Perez and S. Devasia. Optimal output-transitions for linear systems. *Automatica*, 39(2):181–192, February 2003.

[43] A. Piazzi and A. Visioli. Optimal inversion-based control for the set-point regulation of nonminimum-phase uncertain scalar systems. *IEEE Transaction on Automatic Control*, 46(10):1654–1659, October 2001.

[44] A. Piazzi and A. Visioli. Optimal noncausal set-point regulation of scalar systems. *Automatica*, 37(1):121–127, January 2001.

[45] A. Piazzi and A. Visioli. Robust set-point constrained regulation via dynamic inversion. *International Journal of Robust and Nonlinear Control*, 11(1):1–22, January 2001.

[46] A. Piazzi and A. Visioli. Pareto optimal feedforward constrained regulation of mimo linear systems. In *Proceedings of the 16th IFAC World Congress*, pages 419–424, Prague, Czech Republic, July 2005.

[47] A. Piazzi and A. Visioli. Using stable input-output inversion for minimum-time feedforward constrained regulation of scalar systems. *Automatica*, 41(2):305–313, 2005.

[48] A. Piazzi and A. Visioli. Combining H_∞ control and dynamic inversion for robust constrained set-point regulation. *International Journal of Tomography & Statistics*, 6(S07):63–68, 2007.

[49] K. S. Ramani, M. Duan, C. E. Okwudire, and A. Galip Ulsoy. Tracking control of linear time-invariant nonminimum phase systems using filtered basis functions. *Journal of Dynamic Systems, Measurement, and Control*, 139(1), 2017.

[50] B. P. Rigney, L. Pao, and D. Lawrence. Model inversion architectures for settle time applications with uncertainty. In *45th IEEE Conference on Decision and Control*, pages 6518–6524, 2006.

[51] B. P. Rigney, L. Y. Pao, and D. A. Lawrence. Nonminimum phase dynamic inversion for settle time applications. *IEEE Transactions on Control Systems Technology*, 17(5):989–1005, 2009.

[52] R. Romagnoli and E. Garone. A general framework for approximated model stable inversion. *Automatica*, 101:182–189, 2019.

[53] H. Rosenbrock. *State Space and Multivariable Theory*, volume 26. Wiley, 1970.

[54] W. Rudin et al. *Principles of mathematical analysis*, volume 3. McGraw-hill New York, 1964.

[55] W. J. Rugh. *Linear system theory*. Prentice-Hall, Inc., 1996.

[56] J. M. Schumacher. Transformations of linear systems under external equivalence. *Linear Algebra and its Applications*, 102:1–33, 1988.

[57] W. M. Siebert. *Circuit, Signals, and Systems*. The MIT Press, 1986.

[58] L. M. Silverman. Inversion of multivariable linear systems. *IEEE Trans. on Automatic Control*, 14:270–276, 1969.

[59] T. Singh. *Optimal Reference Shaping for Dynamical Systems: theory and applications*. CRC Press, 2010.

[60] S. Skogestad and I. Postlethwaite. *Multivariable Feedback Control: analysis and design*. Wiley-Interscience, second edition, 2005.

[61] J. van Zundert and T. Oomen. On inversion-based approches for feedforward and ILC. *Mechatronics*, 50:282–291, 2018.

[62] J. Willems. *The Analysis of Feedback Systems*. The M.I.T. Press, 1971.

[63] J. C. Willems. Input-output and state-space representations of finite-dimensional linear time-invariant systems. *Linear Algebra and its Applications*, 50:581–608, 1983.

[64] J. C. Willems. From time series to linear system-part i. finite dimensional linear time invariant systems. *Automatica*, 22(5):561–580, 1986.

[65] J. C. Willems and J. W. Polderman. *Introduction to mathematical systems theory: a behavioral approach*, volume 26. Springer Science & Business Media, 1997.

[66] W. A. Wolovich. *Linear multivariable systems*, volume 11. Springer Science & Business Media, 1974.

[67] Y. Wu and Q. Zou. Robust inversion-based 2-DOF control design for output tracking: piezoelectric-actuator example. *IEEE Transactions on Control Systems Technology*, 17(5):1069–1082, September 2009.