PURE SPINOR SUPERSTRING IN AdS$_4$ × $\mathbb{CP}^3$

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Introduction

Pure Spinor superstring is the first successful attempt to write a space-time supersymmetric formalism of string that admits quantization without breaking manifest Poincaré covariance.

The Ramond and Neveu-Schwarz formalism indeed presents supersymmetry only on the world-sheet that the string describes during its movement, on the contrary space-time supersymmetry is not manifest but it is recovered once bosonic and fermionic degrees of freedom have been selected *a posteriori* by the Gliozzi-Scherk-Olive mechanism. The presence of worldsheet fermions and ghosts for local supersymmetry makes quantisation very complicated, particularly at an arbitrary genus, due to the complicated nature of the (super)moduli space. In particular higher-loop amplitudes computation needs ghost insertions and sum over spin structures. Furthermore the fact that states are represented by an infinite tower of vertex operators related by picture changing makes things even more difficult. Moreover this formalism can not be generalised to curved backgrounds with non-zero Ramond-Ramond fields, since the vertex operators for these states involve spin fields which are non-local in terms of the world-sheet fields.

On the contrary, in Green-Schwarz formalism supersymmetry is manifest in ten-dimensional space-time where strings live. However, one encounters serious difficulties in the quantization procedure already in a flat space due to the presence of first and second class constraints. This problem is solved by requiring the presence of a fermionic local symmetry, named κ-symmetry, and fixing the light-cone gauge. Al-

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though light-cone quantization deals with only physical degrees of freedom, it is not completely satisfactory since one would like to preserve Poincaré covariance. On the other hand, an important feature of the Green-Schwarz formulation is that it naturally extends to any curved background which obeys the supergravity equations of motion, namely the corresponding string action is invariant under the κ-symmetry, which in turn is responsible for the space-time supersymmetry of the physical spectrum. In practice, the explicit construction of the Green-Schwarz action for an arbitrary supergravity solution is a technically complicated problem which has not been solved so far. These difficulties can be bypassed if the background has special symmetry properties by observing that the Green-Schwarz action is equivalent to a Wess-Zumino non-linear sigma model on some coset superspace.

The Pure Spinor formalism, proposed by Berkovits in 2000 has both manifest space-time supersymmetry and ten-dimensional Lorentz covariance. This approach, based on previous idea of Siegel (1986), uses a standard fermionic action and it does not need κ-symmetry and light-cone gauge. In addition to supercoordinates, ghost fields - specifically bosonic spinors - and their conjugate momenta are introduced. The non-physical degrees of freedom introduced in the theory are removed through a BRST-like operator Q. In particular the ghosts are constrained by the canonical request of BRST nilpotency and they are are defined *pure* because of the kind of constraint. For the string in a flat background it has been shown that the BRST cohomology determines the physical spectrum which is equivalent to that of the Ramond-Neveu-Schwarz formalism and that of the Green-Schwarz formalism in the light-cone gauge.

The possibility of avoiding light-cone gauge in this formalism allows the string to have manifest Poincaré covariance together with supersymmetry in space-time even at the quantum level. However, there are some hidden sources of possible violation of Lorentz covariance when one solves the pure spinor constraint in terms of independent fields. Finally, the pure spinor formalism is suitable to describe

strings in curved background and we will discuss in particular the case of anti-de Sitter geometries.

From the latest years of 20th century, the study of holographic theories has become a very interesting research field in height energy physics. The possibility of studying a n-dimensional physical system on the boundary of a $(n+1)$-dimensional system has exponentially increased because of the *Maldacena conjecture*, that supposes the equivalence of the type IIB superstring in $\text{AdS}_5 \times \text{S}^5$ space with supersymmetric Yang-Mills $\mathcal{N} = 4$ theory in Minkovski 4-dimensional space in the 't Hooft limit $N \to \infty$. The fundamental aspect of this correspondence between a string theory and a conformal field theory - known as AdS/CFT correspondence - is that the strong coupling regime of a system corresponds to the weak coupling of its holographic dual and vice-versa: in this way it is possible a perturbative approach otherwise unapplicable.

Recently Aharony, Bergman, Jafferis and Maldacena proposed that the $\mathcal{N} = 6$ supersymmetric Chern-Simons theory in three dimensions had a 't Hooft limit whose holographic dual is described by type IIA superstring in $\text{AdS}_4 \times \mathbb{CP}^3$ background. The ABJM conjecture has stimulated the study of the superstring in the space above: likewise the study of superstring in $\text{AdS}_5 \times \text{S}^5$, the model is based on the possibility to describe an homogeneous space - like AdS - in an algebraic way by means of a coset of its symmetry group, $\text{AdS}_4 \cong \text{SO}(3,2)/\text{SO}(3,1)$. Because of internal space has the same property $\mathbb{CP}^3 \cong \text{SU}(4)/\text{U}(3)$, it is possible to give a supersymmetric form to direct product of the spaces:

$$\text{AdS}_4 \times \mathbb{CP}^3 \cong \frac{\text{SO}(3,2) \times \text{SU}(4)}{\text{SO}(3,1) \times \text{U}(3)} \xrightarrow{SUSY} \frac{\text{OSP}(4|6)}{\text{SO}(3,1) \times \text{U}(3)} \quad .$$

In this way we can build a non-linear sigma model action by means of the supercoset Maurer-Cartan forms

$$J = g^{-1}dg \quad , \qquad g \in \frac{\text{OSP}(4|6)}{\text{SO}(3,1) \times \text{U}(3)} \quad .$$

However, differently from $\text{AdS}_5 \times \text{S}^5$, the so-built Green-Schwarz superstring

presents some issues: in fact the supercoset $OSP(4|6)/SO(3,1) \times U(3)$ contains 24 fermions instead of 32 like in the canonical type IIA superstring. A possible solution is to interpret the sigma model as a Green-Schwarz superstring with half-fixed κ symmetry, in which 8 fermions are gauged away. In confirmation of this interpretation, the coset model presents a local fermionic symmetry that takes away 8 fermionic degrees of freedom and gives 16 fermions, as in canonical total-gauged Green-Schwarz superstring. Nevertheless in some peculiar string configurations the rank of this κ symmetry is bigger that 8 and the argument above is not allowed.

The peculiarity of $OSP(4|6)/SO(3,1) \times U(3)$ coset is the existence of a $\mathbb{Z}_4$-grading, i.e. the decomposition of the Lie algebra in four eigenspaces $\mathcal{H}_i$ bosonic type $(i = 0, 2)$ and fermionic type $(i = 1, 3)$. Thus $J = \sum_{i=0}^{3} J_i$: in particular form J_0 represents gauge-field of the transformation $SO(3,1) \times U(3)$, so it can appear only in interaction terms. To complete the matter part of the action we have to add to the sigma model a Wess-Zumino term, typical of superstring, and the final result is

$$S_{matter} = \frac{R^2}{2\pi} \int d^2z \, \mathrm{STr}\left[\frac{1}{2}J_2\overline{J}_2 + \frac{3}{4}J_3\overline{J}_1 + \frac{1}{4}J_1\overline{J}_3\right] \quad . \tag{1}$$

This is very different from Green-Schwarz model that has fermionic currents only in the Wess-Zumino term, to ensure the κ-symmetry and then the light-cone quantization. Introducing a covariant derivative, the ghost term of the action can be written

$$S_{ghost} = -\frac{R^2}{2\pi} \int d^2z \, \mathrm{STr}\left(w_3\overline{\nabla}\lambda_1 + w_1\nabla\lambda_3\right) \quad ,$$

noting that the ghosts λ and their conjugate momenta w are in the fermionic sectors $\mathcal{H}_{1,3}$ of the superalgebra. The action, provided with current-current interation term, is invariant under the BRST transformation generated by the charge

$$Q = \int dz \, \mathrm{STr}(\lambda_3\overline{J}_1) + \int d\overline{z} \, \mathrm{STr}(\lambda_1 J_3) \quad .$$

Physical states are defined in BRST cohomology of Q, i.e. a physical state Ψ satisfies $Q\Psi = 0$ with $\Psi \neq Q\Psi'$, while nilpotency of the charge $(Q^2 = 0)$ gives the

fundamental constraint that defines the ghosts and corresponds to the pure spinor constraint in flat space. It is important to note that in pure spinor formalism the κ-symmetry is absent, so one can suppose that this approach solves the issues of the coset Green-Schwarz superstring in $\text{AdS}_4 \times \mathbb{CP}^3$.

The aim of present work is to provide an alternative formulation of pure spinor action in $\text{AdS}_4 \times \mathbb{CP}^3$ in which ghost fields are free, that is the constraint is already solved. Defining BRST transformation on the generic coset element $Q(g) = g(\lambda_1 + \lambda_3)$, we obtain the expression of the transformation on Maurer-Cartan form J and imposing $Q^2(J) = 0$ modulo a gauge transformation, we can write the ghost constraints on the super-coset manifold:

$$\{\lambda_1, \lambda_1\} = 0 \qquad \{\lambda_3, \lambda_3\} = 0 \quad .$$

Then we choose a basis for $\text{OSP}(4|6)$ generators that makes explicit the anticommutators and so we can solve the constraints. The solution amounts to decomposing the ghosts λ_1, λ_3 and their conjugate momenta w_3, w_1 into the direct product of bosonic $\text{SO}(3,1)$-spinorial variables (namely new ghost) and orthonormal $\text{U}(3)$-vectorial variables (u, v). Because of the orthonormality and two residual phase invariances, the (u, v) variables lie in the $\text{SU}(3)/\text{U}(1) \times \text{U}(1)$ coset. Finally we can substitute the solutions into the ghost action and add a sigma model on the coset $\text{SU}(3)/\text{U}(1) \times \text{U}(1)$ to take account of (u, v) kinematic.

The advantage of the action we build is to have free ghost fields: it allows to compute their propagators. Since $\text{SO}(3,1)$ gauge fields couple only to the ghosts, the $\text{SO}(3,1)$ currents contain only ghost fields and we can compute explicitly their operator product expansion. We use the background field method to treat perturbatively matter and $\text{U}(3)$-variables: in this way we can study the central charge and show that it vanishes up to one-loop.

To outline the present work, in the first chapter we give an outlook of superstring in flat space: starting from the Casalbuoni-Brink-Schwarz superparticle, we define the Green-Schwarz superstring and introduce the light-cone gauge to quantize,

pointing out the difficulties of a non quadratic action. Then we construct the Pure Spinor superstring from the Siegel action and solve explicitly the ghost constraint in order to make some fundamental computations.

In the second chapter we study the superstring in curved background. First we give the general action for the Green-Schwarz superstring, then we study the important case of superspaces that admit supercoset formulation, in particular $AdS_5 \times S^5$. Finally we derive the Pure Spinor superstring in the same background.

In the third chapter we examine the $AdS_4 \times \mathbb{CP}^3$ superspace as the $OSP(4|6)/SO(3,1) \times U(3)$ coset and the characteristics of Pure Spinor superstring in this background. In particular we give an explicit form for the pure spinor constraint and solve it using an original realization of the superalgebra of $OSP(4|6)$. Then we revise the action to write it in terms of unconstrainted fields.

Finally in the fourth chapter we derive the background field expansion and perform some perturbative calculation.

More details on the supercoset and the structure constants of the superalgebra of $OSP(4|6)$ are given in the Appendices.

Chapter 1

Superstring in flat space

1.1 The Casalbuoni-Brink-Schwarz superparticle

The best way to understand the space-time supersimmetric string is through the *Casalbuoni-Brink-Schwarz superparticle*, describing the world-line of a particle in 10 dimensions instead of the world-sheet of the string. The action is [1][2]

$$S = \int d\tau (\Pi^m P_m + e P^m P_m) \tag{1.1}$$

where $\Pi^m = \dot{X}^m - \frac{i}{2}\dot{\theta}^\alpha(\gamma^m)_{\alpha\beta}\theta^\beta$, $m = 0, \ldots, 9$ and $\alpha = 1, \ldots, 16$. As said, its fundamental aspect is the invariance under space-time supersymmetry transformation

$$\delta\theta^\alpha = \epsilon^\alpha \qquad \delta X^m = \frac{i}{2}\theta\gamma^m\epsilon \qquad \delta P^m = \delta e = 0 \tag{1.2}$$

with ϵ^α constant fermionic parameter. The conjugate momenta of the bosonic and fermionic coordinates are

$$\frac{\delta L}{\delta \dot{X}^m} = P_m \qquad \frac{\delta L}{\delta \dot{\theta}^\alpha} = -\frac{i}{2}P_m(\gamma^m\theta)_\alpha \equiv p_\alpha \quad . \tag{1.3}$$

The momenta p_α depend on the θ^α coordinates by the constraint

$$d_\alpha \equiv p_\alpha + \frac{i}{2}P_m(\gamma^m\theta)_\alpha = 0 \quad ; \tag{1.4}$$

if we define the canonical Poisson brackets

$$\{p_\alpha, \theta^\beta\}_P = -i\delta_\alpha{}^\beta \qquad \{p_\alpha, p_\beta\}_P = \{\theta^\alpha, \theta^\beta\}_P = 0 \tag{1.5}$$

we can construct the constraint matrix

$$\mathcal{C}_{\alpha\beta} \equiv \{d_\alpha, d_\beta\}_P = P_m(\gamma^m)_{\alpha\beta} \quad . \tag{1.6}$$

In general we have a set of constraint h_A with $\mathcal{C}_{AB} = \{h_A, h_B\}_P$: h_m are *first class* constraints if $\mathcal{C}$ is zero o *weakly* zero (i.e. $\mathcal{C}$ is a linear combination of the constraints), otherwise h_α are *second class* constraints.

The Dirac quantization for first class constraints imposes the substitution $\{\ ,\ \}_P = -i[\ ,\ ]$, so the constraint operators commute between themselves and the physical states can be consistently defined by

$$h_m|phys\rangle = 0 \quad . \tag{1.7}$$

This position is non consistent for second class constraint[1], so we have to define the Dirac brackets [3]

$$\{A, B\}_D = \{A, B\}_P - \{A, h_\alpha\}_P (\mathcal{C}^{-1})^{\alpha\beta}\{h_\beta, B\}_P \tag{1.8}$$

and to quantize posing $\{\ ,\ \}_D = -i[\ ,\ ]$. In this way the second class constraints can be considered always zero.

Because the equation of motion $P^2 = 0$ we can choose $P^m = (P, 0, \ldots, 0, P)$, so that $\mathcal{C}_{\alpha\beta} \sim (\gamma^0 - \gamma^9)_{\alpha\beta} \sim (\gamma^-)_{\alpha\beta}$. The rank of this matrix is 8, i.e. $\mathcal{C}_{\alpha\beta}$ has only 8 different from zero eigenvalues . It means that an half of the constraints d_α are of first class and an half are of second class. We need to divide the two kinds of constraints in order to quantize, so we define

$$D_\alpha = P_m(\gamma^m d)_\alpha \tag{1.9}$$

and note that $\{D_\alpha, D_\beta\} = 0$ using $P^2 = 0$. The first class constraints D_α generate a fundamental gauge symmetry of the action (1.1), named *kappa symmetry* [4][5]:

$$\delta\theta^\alpha = P_m(\gamma^m\kappa)^\alpha \qquad \delta X^m = -\frac{i}{2}\theta\gamma^m\delta\theta \qquad \delta P^m = 0 \qquad \delta e = i\dot{\theta}\kappa \tag{1.10}$$

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[1]In fact $0 = h_\alpha h_\beta|phys\rangle - h_\beta h_\alpha|phys\rangle = [h_\alpha, h_\beta]|phys\rangle \neq 0$.

where κ is a fermionic local parameter. If we define the *light-cone coordinates*

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^9) \qquad P^\pm = \frac{1}{\sqrt{2}}(P^0 \pm P^9) \qquad \gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^9) \qquad (1.11)$$

by means of the k symmetry it is possible to choose θ so that

$$\gamma^+ \theta = 0 \quad : \qquad (1.12)$$

since the rank of γ^+ is 8, half of the components of θ is totally decoupled from the theory. The action can be written

$$S_{l.c.} = \int d\tau \left(\dot{X}^m P_m + \frac{i}{2} P^+ (\dot{\theta}\gamma^-\theta) + eP^m P_m \right) \quad . \qquad (1.13)$$

A possible choice for the γ^- matrix is $(\gamma^-)_{\alpha\beta} = -\sqrt{2} \begin{pmatrix} \mathbb{1}_8 & 0 \\ 0 & 0 \end{pmatrix}$, then posing $S^a \equiv$ $2^{1/4}\sqrt{P^+}\theta^\alpha\big|_{\alpha=1,\ldots,8}$ the action (1.1) becomes

$$S_{l.c.} = \int d\tau \left(\dot{X}^m P_m - \frac{i}{2} \dot{S}^a S^a + eP^m P_m \right) \quad . \qquad (1.14)$$

This action in *light-cone gauge* is simple to quantize: the conjugate momentum of S^a is

$$p_a = \frac{\delta L}{\delta \dot{S}^a} = -\frac{i}{2} S_a \qquad (1.15)$$

and imposing the canonical Poisson brackets $\{p^a, S^b\}_P = -i\delta^{ab}$, the constraints $d^a = p^a + \frac{i}{2}S_a$ are just the 8 second class constraints, having non-zero matrix

$$\{d^a, d^b\}_P = \delta^{ab} \qquad (1.16)$$

and the Dirac brackets for S are

$$\{S^a, S^b\}_D = \delta^{ab} \quad . \qquad (1.17)$$

It is important to note that the quantization with second class constraints was possible only in the light-cone gauge, i.e. in a non-covariant gauge: the same problem affects the Green-Schwarz superstring.

Notice that the standard description of the massless relativistic superparticle can be obtained from the action (1.1) by using the equation of motion for P_m, that is

$$\Pi^m + 2eP^m = 0 \qquad \Rightarrow \qquad P^m = -\frac{1}{2e}\Pi^m \quad , \tag{1.18}$$

and has the final expression

$$S = -\frac{1}{4}\int d\tau e^{-1}\Pi^m\Pi_m \quad . \tag{1.19}$$

1.2   The Green-Schwarz superstring

The *Green-Schwarz superstring* represents the generalization of the superparticle on a world-sheet with coordinates $(\tau, \sigma) = (\sigma^0, \sigma^1)$. In 10 dimensional space-time, the action is[2] [6]

$$S_{GS} = -\frac{1}{8\pi}\int d^2\sigma\sqrt{h}h^{ij}\Pi_i^m\Pi_j^n\eta_{mn} + \tag{1.20}$$
$$+ \frac{1}{4\pi}\int d^2\sigma\varepsilon^{ij}[-i\partial_i X_m(\theta_L\gamma^m\partial_j\theta_L - \theta_R\gamma^m\partial_j\theta_R) + (\theta_L\gamma^m\partial_i\theta_L)(\theta_R\gamma^m\partial_j\theta_R)]$$

with $i, j = 0, 1$, h_{ij} world-sheet metric, ε^{ij} antisymmetric tensor, η_{mn} flat space-time metric and Π_i^m natural generalization of Π^m in (1.1):

$$\Pi_i^m = \partial_i X^m - i\theta_L\gamma^m\partial_i\theta_L - i\theta_R\gamma^m\partial_i\theta_R \tag{1.21}$$

where θ_L, θ_R are Majorana-Weyl spinors of $SO(9, 1)$, said respectively *left-* and *right-moving*.

Note that the first term of (1.20) is the 2-dimensional analogue to (1.19) and represents the kinetic term of the string, while the second one is a new contribute known as *Wess-Zumino term* and it is necessary to have the κ symmetry. One could show that this procedure works at most for two supersymmetries and this is the reason of two different spinors θ [7]. Further the WZ term is supersymmetric in 10 dimensions only if θ spinors are Majorana-Weyl.

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[2]Compared with usual convention we divide the action by 4, in agreement with next section.

Because of the reparametrization invariance it is possible to choose flat metric on the world-sheet $h_{ij} = (-1, +1)$. As usual, the the world-sheet metric can be written in euclidean signature by the *Wick rotation*[3] $\tau \to -i\sigma^2$, so that $h_{ij} = (+1, +1)$ for the new coordinates (σ^1, σ^2). Introducing complex coordinates

$$z = \sigma^1 + i\sigma^2 \qquad \bar{z} = \sigma^1 - i\sigma^2 \tag{1.23}$$

and the corresponding derivatives

$$\partial = \frac{1}{2}(\partial_1 - i\partial_2) \qquad \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \tag{1.24}$$

the action (1.20) acquires the form

$$S_{GS} = \frac{1}{4\pi} \int d^2z \ \big[\partial X^m \bar{\partial} X_m - 2i\partial X_m (\theta_L \gamma^m \bar{\partial}\theta_L) - 2i\bar{\partial} X_m (\theta_R \gamma^m \partial\theta_R) +$$

$$- (\theta_L \gamma_m \bar{\partial}\theta_L)(\theta_L \gamma^m \partial\theta_L + \theta_R \gamma^m \partial\theta_R) +$$

$$-(\theta_R \gamma_m \partial\theta_R)(\theta_L \gamma^m \bar{\partial}\theta_L + \theta_R \gamma^m \bar{\partial}\theta_R) \big] \quad . \tag{1.25}$$

We can also define the momentum (1.21) in complex coordinates

$$\Pi^m = \partial X^m - i\theta_L \gamma^m \partial\theta_L - i\theta_R \gamma^m \partial\theta_R \tag{1.26}$$

and analogous for $\bar{\Pi}$.

Superstrings take different names depending on the chirality of θ's: in particular, closed string with θ_L, θ_R of opposite or equal chirality are respectively *Type IIA* or *Type IIB*, while open string with θ_L, θ_R of equal chirality are *Type I*. Open strings with spinors of opposite chirality do not preserve supersymmetry, so we do not care about them.

The equations of motion for the metric give the *Virasoro constraints*

$$\Pi^m \Pi_m = 0 \qquad \bar{\Pi}^m \bar{\Pi}_m = 0 \quad , \tag{1.27}$$

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[3]Note that by definition $L_{euclid} \equiv -L_{\tau \to -i\sigma^2}$, in order that

$$S = \int d\sigma^0 d\sigma^1 L \to i \int d\sigma^1 d\sigma^2 L_{euclid} \equiv iS_{euclid} \quad . \tag{1.22}$$

while the equation for X and θ are highly non linear.

The GS superstring is invariant under supersymmetry transformation

$$\delta\theta^\alpha_{L,R} = \epsilon^\alpha_{L,R} \qquad \delta X^m = i\epsilon_L\gamma^m\theta_L + i\epsilon_R\gamma^m\theta_R \tag{1.28}$$

and κ symmetry

$$\delta\theta^\alpha_L = \Pi_m(\gamma^m\kappa_L)^\alpha \qquad \delta\theta^\alpha_R = \bar\Pi_m(\gamma^m\bar\kappa_R)^\alpha \qquad \delta X^m = i\theta_L\gamma^m\delta\theta_L + i\theta_R\gamma^m\delta\theta_R \tag{1.29}$$

To quantize the GS superstring we define the conjugate momentum of θ_L

$$p^L_\alpha \equiv \pi \frac{\delta S_{GS}}{\delta\partial_0\theta^\alpha_L} = \frac{i}{2}(\theta_L\gamma_m)_\alpha \left[\Pi^m + \frac{i}{2}(\theta_L\gamma^m\partial_1\theta_L)\right] \tag{1.30}$$

and the constraint

$$d^L_\alpha \equiv p^L_\alpha - \frac{i}{2}(\theta_L\gamma_m)_\alpha\Pi^m + \frac{1}{4}(\theta_L\gamma_m)_\alpha(\theta_L\gamma^m\partial_1\theta_L) \quad . \tag{1.31}$$

If we impose the canonical brackets, we obtain immediately the fermionic constraint matrix $\{d^L_\alpha, d^L_\beta\}_P = \Pi_m(\gamma^m)_{\alpha\beta}$ and because of the equations of motion we can see that 8 constraints are first class and the remanent 8 are second class, as for the superparticle. In the same way, we have to separate the two classes, in order to invert the constraint matrix and to define the Dirac brackets.

Using the κ symmetry we can assume the light-cone gauge

$$\gamma^+\theta_L = 0 \qquad \gamma^+\theta_R = 0 \tag{1.32}$$

and this position allows to put zero half of the components of both θ; in addition we can use the residual conformal invariance to choose

$$X^+(\tau,\sigma) = x^+ + p^+\tau = x^+ + \frac{1}{2}p^+(\bar z - z) \quad . \tag{1.33}$$

Using the properties

$$\theta\gamma^+\partial\theta = \theta\gamma^+\bar\partial\theta = 0 \quad , \quad \theta\gamma^i\partial\theta = \theta\gamma^i\bar\partial\theta = 0 \qquad i = 1,\ldots,8 \tag{1.34}$$

and

$$\bar\partial X^+ = -\partial X^+ = \frac{1}{2}p^+ \tag{1.35}$$

the action (1.25) takes the form

$$S_{l.c.} = \frac{1}{4\pi} \int d^2z \left[-\partial X^k \bar\partial X^k - \frac{1}{2} p^+ \bar\partial X^- + \frac{1}{2} p^+ \partial X^- + ip^+ (\theta_L \gamma^- \bar\partial \theta_L) - ip^+ (\theta_R \gamma^- \partial \theta_R) \right]$$
(1.36)

where the second and third term are total derivatives and can be cancelled. There exists the representations of γ^-

$$\gamma^- = -\sqrt{2} \begin{pmatrix} \mathbb{1}_8 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{for Weyl spinor} \tag{1.37}$$

$$\gamma^- = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_8 \end{pmatrix} \qquad \text{for anti-Weyl spinor} \quad , \tag{1.38}$$

so that (1.36) becomes

$$S_{l.c.} = \frac{1}{4\pi} \int d^2z \left(-\partial X^k \bar\partial X^k + S_L^a \bar\partial S_L^a + S_R^{\dot a} \partial S_R^{\dot a} \right) \quad \text{IIA} \tag{1.39}$$

$$S_{l.c.} = \frac{1}{4\pi} \int d^2z \left(-\partial X^k \bar\partial X^k + S_L^a \bar\partial S_L^a + S_R^a \partial S_R^a \right) \quad \text{IIB} \tag{1.40}$$

where $k = 1, \dots, 8$, $a/\dot a = 1, \dots, 8$ and

$$S_L^a = 2^{1/4} i \sqrt{ip^+} \theta_L^\alpha \big|_{\alpha=1,\dots,8} \qquad \begin{array}{l} S_R^a = 2^{1/4} \sqrt{ip^+} \theta_R^\alpha \big|_{\alpha=1,\dots,8} \\ S_R^{\dot a} = 2^{1/4} i \sqrt{ip^+} \theta_R^\alpha \big|_{\alpha=9,\dots,16} \end{array} \qquad . \tag{1.41}$$

The eight surviving components of X form the vectorial representation $\mathbf{8_v}$ of SO(8), while the eight surviving components of each θ - labelled S - form a spinorial representation of the same group. In particular, because of the definitions, they are either in Weyl representation $(S_L^a, S_R^a) \in (\mathbf{8_s}, \mathbf{8_s})$ for IIB string and they are in Weyl and anti-Weyl representation $(S_L^a, S_R^{\dot a}) \in (\mathbf{8_s}, \mathbf{8_c})$ for IIA string[4].

From the action (1.39)-(1.40) we obtain the equations of motion in light-cone gauge:

$$\partial \bar\partial X^i(z, \bar z) = 0 \qquad \bar\partial S_L^a(z) = 0 \qquad \begin{array}{l} \partial S_R^{\dot a}(\bar z) = 0 \quad \text{IIA} \\ \partial S_R^a(\bar z) = 0 \quad \text{IIB} \end{array} \quad , \tag{1.42}$$

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[4]Clearly it is a convention and everything is consistent changing $\mathbf{8_s}$ with $\mathbf{8_c}$.

or, in (σ, τ) coordinates,

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2}\right) X^i = 0 \qquad \left(\frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau}\right) S_L^a = 0 \qquad \begin{array}{ll} \left(\dfrac{\partial}{\partial \sigma} - \dfrac{\partial}{\partial \tau}\right) S_R^{\dot{a}} = 0 & \text{IIA} \\[2ex] \left(\dfrac{\partial}{\partial \sigma} - \dfrac{\partial}{\partial \tau}\right) S_R^a = 0 & \text{IIB} \end{array} \quad ,$$

noting the last two equations justify the left/right-moving definitions for S and hence for θ.

Along with the equations of motion we have to specify the boundary conditions. For closed string we have to impose the periodicity of the σ coordinate:

$$X^i(\sigma + 2\pi, \tau) = X^i(\sigma, \tau) \tag{1.43}$$

$$S_L^a(\sigma + 2\pi, \tau) = S_L^a(\sigma, \tau) \qquad \begin{array}{ll} S_R^{\dot{a}}(\sigma + 2\pi, \tau) = S_R^{\dot{a}}(\sigma, \tau) & \text{IIA} \\[1ex] S_R^a(\sigma + 2\pi, \tau) = S_R^a(\sigma, \tau) & \text{IIB} \end{array} \quad . \tag{1.44}$$

For open string we have to relate the bosonic and the fermionic variables at the ends $\sigma = 0$ and $\sigma = \pi$:

$$\frac{\partial}{\partial \sigma} X^i(0, \tau) = \frac{\partial}{\partial \sigma} X^i(\pi, \tau) = 0 \tag{1.45}$$

$$S_L^a(0, \tau) = S_R^a(0, \tau) \qquad S_L^a(\pi, \tau) = S_R^a(\pi, \tau) \quad . \tag{1.46}$$

It is possible to demonstrate that supersymmetry is only possible for this choice; moreover, for the first transformation of (1.28), boundary conditions for S require $\epsilon_L \equiv \epsilon_R$, i.e. supersymmetry decreases for type I superstring from $\mathcal{N} = 2$ to $\mathcal{N} = 1$.

The light-cone actions (1.39) and (1.40) identify immediately the second class constraints and allow to quantize the superstring in the same way of the superparticle.

1.3   The Siegel superstring

From the previous considerations it is evident that the difficulty of a covariant quantization for GS superstring is strictly related to the mixing between first and second class constraints, that originates from the non-quadratic form of the action.

So an attempt to solve the problem must start from a quadratic formulation of the superstring.

In order to illustrate the relation between the GS formulation and the Siegel proposal, we reconsider the action (1.25) and, to simplify the notation, we take only one type of spinor - like in the heterotic string. Posing $\theta_L \equiv \theta$ and $\theta_R = 0$ in (1.25) we get:

$$S_{GS} = \frac{1}{4\pi} \int d^2z \left[\partial X^m \bar{\partial} X_m - 2i\partial X_m(\theta\gamma^m\bar{\partial}\theta) - (\theta\gamma_m\bar{\partial}\theta)(\theta\gamma^m\partial\theta) \right] \quad . \quad (1.47)$$

Defining the conjugate momentum

$$p_\alpha \equiv 2\pi \frac{\delta S_{GS}}{\delta \bar{\partial}\theta^\alpha} = \frac{1}{2} \left(-2i\partial X_m - \theta\gamma_m\partial\theta \right)(\theta\gamma^m)_\alpha \quad (1.48)$$

and the constraint

$$d_\alpha \equiv p_\alpha - \frac{1}{2} \left(-2i\partial X_m - \theta\gamma_m\partial\theta \right)(\theta\gamma^m)_\alpha \quad , \quad (1.49)$$

the GS action becomes

$$S_{GS} = \frac{1}{4\pi} \int d^2z \left[\partial X^m \bar{\partial} X_m - 2(d_\alpha - p_\alpha)\bar{\partial}\theta^\alpha) \right]$$
$$= \frac{1}{2\pi} \int d^2z \left(\frac{1}{2}\partial X^m \bar{\partial} X_m + p_\alpha\bar{\partial}\theta^\alpha \right) - \frac{1}{2\pi} \int d^2z \, d_\alpha\bar{\partial}\theta^\alpha \quad . \quad (1.50)$$

Therefore it is natural the definition [8]

$$S_S \equiv \frac{1}{2\pi} \int d^2z \left(\frac{1}{2}\partial X^m \bar{\partial} X_m + p_\alpha\bar{\partial}\theta^\alpha \right) \quad (1.51)$$

which is related to GS string by

$$S_S = S_{GS} + \frac{1}{2\pi} \int d^2z \, d_\alpha\bar{\partial}\theta^\alpha \quad . \quad (1.52)$$

If p_α is constrained by $d_\alpha = 0$, the actions (1.47) and (1.51) are completely equivalent. Otherwise, if we relax the constraint and consider p_α as an independent variable, we obtain an alternative action known as *Siegel superstring*.

However, this formulation is not yet the solution of a covariant quantization, because the action (1.51) presents at least two serious issues: it is not anomaly

free and the operator product expansion of the Lorentz currents does not reproduce the Ramond and Neveu-Schwarz result. To show them, we recall the definition of *stress-energy tensor* in general metric h_{ab}

$$T_{ab} = -\frac{2\pi}{\sqrt{h}}\frac{\delta S}{\delta h^{ab}} \quad . \tag{1.53}$$

In a conformal field theory - as superstring - this tensor in complex coordinates satisfies [9]:

$$T_{z\bar{z}} = T_{\bar{z}z} = 0 \qquad \partial T_{zz} = \bar{\partial}T_{\bar{z}\bar{z}} = 0$$

so that

$$T_{zz} \equiv T(z) \qquad T_{\bar{z}\bar{z}} \equiv \overline{T}(\bar{z}) \quad . \tag{1.54}$$

The *operator product expansion* (OPE) of T with himself gives

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{1.55}$$

and analogous for $\overline{T}(\bar{z})$. c is a constant depending on the theory, known as *central charge*: the request for an anomaly free theory gives $c = 0$.

The action S_S (1.51) is quadratic, so it is straightforward to compute the stress-energy tensor

$$T(z) = -\frac{1}{2}\partial X^m \partial X_m - p_\alpha \partial \theta^\alpha \quad , \tag{1.56}$$

and the OPE between the fundamental fields [10]

$$X^m(z)X^n(w) = -\eta^{mn}\ln|z - w|^2 \tag{1.57}$$

$$p_\alpha(z)\theta^\beta(w) = \frac{\delta_\alpha{}^\beta}{z - w} \quad . \tag{1.58}$$

Because bosonic and fermionic fields do not mix, we can compute the central charge separately in the two sector: $c_X = 10$ and $c_{p\theta} = -32$. Usually one understands these results associating a contribute $+1$ at every bosonic degree of freedom (i.e. at every space-time dimension) and -2 at every fermionic one. The total central charge is

$$c = c_X + c_{p\theta} = 10 - 32 = -22 \neq 0 \tag{1.59}$$

so the action presents unwanted anomaly.

Let us compute now the Lorentz current of (1.51): an infinitesimal Lorentz transformation gives

$$\delta X^m = \xi^m{}_n X^n \qquad \text{for vectorial fields}$$
$$\delta\theta^\alpha = \tfrac{1}{4}\xi_{mn}(\gamma^{mn})^\alpha{}_\beta \theta^\beta \qquad \text{for Weyl spinorial fields} \qquad (1.60)$$
$$\delta p_\alpha = \tfrac{1}{4}\xi_{mn}(\gamma^{mn})_\alpha{}^\beta p_\beta \qquad \text{for anti-Weyl spinorial fields}$$

where $\xi_{mn} = -\xi_{nm}$ is the infinitesimal parameter of the transformation, i.e. $\Lambda^m{}_n = \delta^m{}_n + \xi^m{}_n$. According to Noether's theorem, the procedure for computing conserved currents is to let ξ become a function of z: integrating by parts when needed, we have

$$\delta S_S = \frac{1}{2\pi}\int d^2z \left(\partial X_m \bar{\partial}\delta X^m + \delta p_\alpha \bar{\partial}\theta^\alpha + p_\alpha \bar{\partial}\delta\theta^a\right)$$

$$= \frac{1}{2\pi}\int d^2z \left(\xi_{mn}\partial X^m \bar{\partial}X^n + \frac{1}{4}\xi_{mn}(\gamma^{mn})_\alpha{}^\beta p_\beta \bar{\partial}\theta^\alpha + \frac{1}{4}\xi_{mn}p_\alpha(\gamma^{mn})^\alpha{}_\beta \bar{\partial}\theta^\beta +\right.$$

$$\left. + \bar{\partial}\xi_{mn}\partial X^m X^n + \frac{1}{4}\bar{\partial}\xi_{mn}p_\alpha(\gamma^{mn})^\alpha{}_\beta \bar{\partial}\theta^\beta\right)$$

$$= \frac{1}{2\pi}\int d^2z \left(\bar{\partial}\xi_{mn}\partial X^m X^n + \frac{1}{4}\bar{\partial}\xi_{mn}p_\alpha(\gamma^{mn})^\alpha{}_\beta \bar{\partial}\theta^\beta\right) \qquad (1.61)$$

where we used the antisymmetry of ξ_{mn} and $(\gamma^{mn})^\alpha{}_\beta = -(\gamma^{mn})_\beta{}^\alpha$ to cancel the term not $\bar{\partial}\xi$-depending. By definition[5]

$$\delta S = \frac{1}{2\pi}\int d^2z \left(\bar{\partial}\xi_{mn}K^{mn} + \frac{1}{2}\bar{\partial}\xi_{mn}L^{mn}\right) \qquad (1.62)$$

so

$$K^{mn} = \frac{1}{2}(\partial X^m X^n - \partial X^n X^m) \qquad (1.63)$$

$$L^{mn} = \frac{1}{2}p_\alpha(\gamma^{mn})^\alpha{}_\beta \bar{\partial}\theta^\beta \quad . \qquad (1.64)$$

We are interested in computing the OPE of the spin contribute to Lorentz current L^{mn} with himself: using (1.58) we have

$$L^{kl}(z)L^{mn}(w) = \frac{\eta^{m[l}L^{k]n} - \eta^{n[l}L^{k]m}}{z-w} + 4\frac{\eta^{k[n}\eta^{m]l}}{(z-w)^2} \quad . \qquad (1.65)$$

---

[5]The coefficient $\frac{1}{2}$ in front of the fermionic current coupling is due to the *a priori* antisymmetry of L^{mn}

This OPE does not coincide with the analogue in the Ramond and Neveu-Schwarz approach, indeed in this case the spinorial Lorentz current $L_{RNS}^{mn} = \psi^m \psi^n$ gives [10]

$$L_{RNS}^{kl}(z)L_{RNS}^{mn}(w) = \frac{\eta^{m[l}L_{RNS}^{k]n} - \eta^{n[l}L_{RNS}^{k]m}}{z - w} + \frac{\eta^{k[n}\eta^{m]l}}{(z - w)^2} \quad . \tag{1.66}$$

To have an acceptable superstring action we must add to the Siegel action S_S a term that cancels the central charge and adjusts the coefficient of the double pole in the OPE (1.65).

1.4   The Pure Spinor superstring

Both the above requests can be satisfied if we add to S_S a ghost term [11]

$$S_\lambda = \frac{1}{2\pi} \int d^2 z \, w_\alpha \bar{\partial} \lambda^\alpha \tag{1.67}$$

where the ghost fields λ are constrained by

$$\lambda^\alpha (\gamma^m)_{\alpha\beta} \lambda^\beta = 0 \tag{1.68}$$

and take the name of *pure spinors* [12]. This constraint appears because we want to provide the action with BRST quantization, as usual in theories with ghosts; to do it, we have to construct a nilpotent BRST charge Q so that physical states Ψ lie in the *cohomology* of Q:

$$Q\Psi = 0 \quad \text{but} \quad \Psi \neq Q\Phi \quad . \tag{1.69}$$

We know that Q raises the ghost number, hence it has to contain λ (ghost number $+1$); in addition we remember that canonical quantization was $d_\alpha \Psi = 0$, therefore it is natural to define the BRST charge

$$Q = \int dz \lambda^\alpha d_\alpha \quad . \tag{1.70}$$

Using (1.58) we can compute the simple pole in the OPE of d_α with himself:

$$d_\alpha(z)d_\beta(w) \to (2i\partial X_m + \theta\gamma_m\partial\theta)(\gamma^m)_{\alpha\beta}\frac{1}{z - w} \equiv \frac{-\Pi_m(\gamma^m)_{\alpha\beta}}{z - w} \tag{1.71}$$

and so

$$Q^2 = \frac{1}{2}\{Q, Q\} = -\frac{1}{2}\int dz \, \Pi_m \lambda^\alpha (\gamma^m)_{\alpha\beta} \lambda^\beta = 0 \quad ; \tag{1.72}$$

it means that *pure spinor constraint* (1.68) *assures the nilpotency of* BRST *charge.*

The ghost action (1.67) contributes to the central charge (1.59) and to the spinorial currents (1.64). Because of the constraint (1.68) we cannot use the naive OPE between w and λ, indeed using

$$w_a(z)\lambda^\beta(w) = \frac{\delta_\alpha{}^\beta}{z - w} \tag{1.73}$$

we get a contradiction with the pure spinor constraint:

$$w(z)(\lambda\gamma^{mn}\lambda)(w) = \frac{2\gamma_{\alpha\beta}^{mn}\lambda^\beta}{z - w} \neq 0 \quad . \tag{1.74}$$

Then we have to solve the constraint and find a new formulation of S_λ in terms of unconstrainted fields.

1.4.1 Decomposition of SO(10) in U(5)

To solve the pure spinor constraint we need a convenient group representation: we want to write vectors, tensors and spinors of the euclidean Lorentz group SO(10) - i.e. the Wick-rotated form of SO(9, 1) - in term of the group U(5) [11][13].

A vector V^m ($m = 1, \ldots, 10$) in the fundamental representation **10** of SO(10) can be decomposed in vector v^i plus a vector v_i ($i = 1, \ldots, 5$) respectively in the fundamental **5** and anti-fundamental **5*** representation of U(5):

$$V^m \;\to\; (v^i, v_i) \qquad (\mathbf{10} = \mathbf{5} \oplus \mathbf{5}^*) \tag{1.75}$$

with

$$v^i = \frac{1}{\sqrt{2}}\left(V^i + iV^{i+5}\right) \tag{1.76}$$

$$v_i = \frac{1}{\sqrt{2}}\left(V^i - iV^{i+5}\right) \quad . \tag{1.77}$$

and the scalar product can be written

$$V^i W_i = v^i w_i + v_i w^i \quad . \tag{1.78}$$

By the position above it is possible to deduce the decomposition of an antisymmetric 2-rank tensor

$$N^{mn} \;\to\; (n^{ij}, n^i_j, n_{ij}, n) \qquad (\mathbf{45} = \mathbf{10} \oplus \mathbf{24} \oplus \mathbf{10}^* \oplus \mathbf{1}) \tag{1.79}$$

with

$$n^{ij} = \frac{1}{2} \left(N^{ij} + iN^{i(j+5)} + iN^{(i+5)j} - N^{(i+5)(j+5)} \right) \tag{1.80}$$

$$n_{ij} = \frac{1}{2} \left(N^{ij} - iN^{i(j+5)} - iN^{(i+5)j} - N^{(i+5)(j+5)} \right) \tag{1.81}$$

$$n^i_j = \frac{1}{2} \left(N^{ij} - iN^{i(j+5)} + iN^{(i+5)j} + N^{(i+5)(j+5)} \right) - \frac{i}{5} \delta^i_j \sum_{i=1}^{5} N^{(i+5)i} \tag{1.82}$$

$$n = \frac{i}{\sqrt{5}} \sum_{i=1}^{5} N^{(i+5)i} \quad . \tag{1.83}$$

To decompose a spinor it is necessary to decompose before the 10-dimensional γ matrices in the way:

$$a^i = \frac{1}{2} \left(\gamma^i + i\gamma^{i+5} \right) \tag{1.84}$$

$$a_i = \frac{1}{2} \left(\gamma^i - i\gamma^{i+5} \right) \quad . \tag{1.85}$$

In 10 dimensions γ^m can be hermitian, so $a^i = a^\dagger_i$; furthermore using the Clifford algebra $\{\gamma^m, \gamma^n\} = 2\delta^{mn}$ we have

$$\{a_i, a^j\} = \delta^j_i \qquad \{a_i, a_j\} = \{a^i, a^j\} = 0 \quad , \tag{1.86}$$

so we can understand a^i and a_i as creation/annihilation operators. If we define the vacuum state $|0\rangle$ by $a_i|0\rangle = 0$ we can construct the generical state applying a^i

$$|A\rangle = [A_0 + \sum_i A_i a^i + \sum_{i<j} A_{ij} a^i a^j + \sum_{i<j<k} A_{ijk} a^i a^j a^k$$
$$+ \sum_{i<j<k<l} A_{ijkl} a^i a^j a^k a^l + A_5 a^1 a^2 a^3 a^4 a^5]|0\rangle \quad . \tag{1.87}$$

Note that the number of components is $\sum_{k=0}^{5} \binom{5}{k} = 1 + 5 + 10 + 10 + 5 + 1 = 32$, as right for a generic spinor in 10 dimensions.

The chirality operator $\gamma^{11} = i \prod_{m=i}^{10} \gamma^m$ can be written

$$\gamma^{11} = - \prod_{i=1}^{5} (a_i + a^i)(a_i - a^i) = - \prod_{i=1}^{5} (2a^i a_i - 1) \tag{1.88}$$

and we have

$$\{\gamma^{11}, a_i\} = \{\gamma^{11}, a^i\} = 0 \quad . \tag{1.89}$$

Trivially $\gamma^{11}|0\rangle = |0\rangle$, so a positive chirality state contains only terms with 0, 2 or 4 a^i, while a negative chirality state contains only terms with 1, 3 or 5 a^i. By (1.67) we can see that λ and w must have opposite chirality to preserve Lorentz invariance, thus we have for Weyl spinor[6]

$$|\lambda\rangle_+ = \lambda^+ |0\rangle + \frac{1}{2} \lambda_{ij} a^j a^i |0\rangle + \frac{1}{4!} \lambda^i \epsilon_{ijklm} a^m a^l a^k a^j |0\rangle \tag{1.90}$$

where the components are

$$\lambda^+ = \langle 0|\lambda\rangle \qquad \lambda_{ij} = \langle 0|a_i a_j|\lambda\rangle \qquad \lambda^i = \frac{1}{4!} \epsilon^{ijklm} \langle 0|a_j a_k a_l a_m|\lambda\rangle \quad . \tag{1.91}$$

For anti-Weyl spinor

$$|w\rangle_- = w_i a^i |0\rangle + \frac{1}{2 \cdot 3!} w^{ij} \epsilon_{ijklm} a^k a^l a^m |0\rangle + w_+ a^1 a^2 a^3 a^4 a^5 |0\rangle \tag{1.92}$$

where the components are

$$w_+ = \langle 0|a_5 a_4 a_3 a_2 a_1|w\rangle \qquad w^{ij} = \frac{1}{3!} \epsilon^{ijklm} \langle 0|a_k a_l a_m|w\rangle \qquad w_i = \langle 0|a_i|w\rangle \quad . \tag{1.93}$$

In this way we obtained the U(5) decomposition for Weyl spinor

$$\lambda^\alpha \rightarrow (\lambda^+, \lambda_{ij}, \lambda^i) \qquad (\mathbf{16} = \mathbf{1} \oplus \mathbf{10^*} \oplus \mathbf{5}) \tag{1.94}$$

and for anti-Weyl spinor

$$w_\alpha \rightarrow (w_i, w^{ij}, w_+) \qquad (\mathbf{16^*} = \mathbf{5^*} \oplus \mathbf{10} \oplus \mathbf{1^*}) \quad . \tag{1.95}$$

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[6]This choice is in agreement with θ and p in (1.51).

1.4.2   Solution of the constraint

Charge conjugation matrix is defined by [14]

$$C\gamma^m C^{-1} = -(\gamma^m)^T \tag{1.96}$$

and in 10-dimensional case a possible choice is

$$C = -i\gamma^6\gamma^7\gamma^8\gamma^9\gamma^{10} = \prod_{i=1}^{5}(a_i - a^i) \quad . \tag{1.97}$$

Remembering that $(\gamma^m)_{\alpha\beta} = (\gamma^m)_\alpha{}^\gamma C_{\gamma\beta}$, the constraint (1.68) means $\lambda\gamma^m C\lambda = 0$ in SO(10) terms: in U(5) it becomes

$$\langle\lambda|\gamma^i C|\lambda\rangle = \langle\lambda|(a_i + a^i)C|\lambda\rangle = 0$$

$$\langle\lambda|\gamma^{i+5}C|\lambda\rangle = i\langle\lambda|(a_i - a^i)C|\lambda\rangle = 0$$

that is

$$\langle\lambda|a_i C|\lambda\rangle = 0 \tag{1.98}$$

$$\langle\lambda|a^i C|\lambda\rangle = 0 \quad . \tag{1.99}$$

Let us consider first (1.98): by means of the decomposition (1.90) we have

$$\langle\lambda|a_p C|\lambda\rangle = \lambda^+\langle\lambda|a_p C|0\rangle + \frac{1}{2}\lambda_{ij}\langle\lambda|a_p C a^j a^i|0\rangle + \frac{1}{4!}\lambda^i\epsilon_{ijklm}\langle\lambda|a_p C a^m a^l a^k a^j|0\rangle$$

$$= \lambda^+\langle\lambda|a_p C|0\rangle + \frac{1}{2}\lambda_{ij}\langle\lambda|a_p a_j a_i C|0\rangle + \frac{1}{4!}\lambda^i\epsilon_{ijklm}\langle\lambda|a_p a_m a_l a_k a_j C|0\rangle$$

$$= \frac{1}{4!}\lambda^+\lambda^{i*}\epsilon_{ijklm}\langle0|a_j a_k a_l a_m a_p C|0\rangle + \frac{1}{2}\cdot\frac{1}{2}\lambda_{ij}\lambda^*_{kl}\langle0||a_k a_l a_p a_j a_i C|0\rangle +$$

$$+ \frac{1}{4!}\lambda^i\lambda^{+*}\epsilon_{ijklm}\langle0|a_p a_m a_l a_k a_j C|0\rangle$$

$$= -\frac{1}{4!}\lambda^+\lambda^{i*}\epsilon_{ijklm}\epsilon_{jklmp} - \frac{1}{4}\lambda_{ij}\lambda^*_{kl}\epsilon_{klpji} - \frac{1}{4!}\lambda^i\lambda^{+*}\epsilon_{ijklm}\epsilon_{pmlkj}$$

$$= -\lambda^+\lambda^{p*} - \frac{1}{4}\epsilon^{pijkl}\lambda_{ij}\lambda^*_{kl} - \lambda^p\lambda^{+*} \tag{1.100}$$

using $Ca^i = -a_i C$, $\langle0|Ca^i a^j a^k a^l a^m|0\rangle = \epsilon^{ijklm}$ and canonical properties of tensor ϵ. So (1.98) becomes

$$\lambda^+\lambda^{p*} + \frac{1}{4}\epsilon^{pijkl}\lambda_{ij}\lambda^*_{kl} + \lambda^p\lambda^{+*} = 0 \quad ; \tag{1.101}$$

noting that $\left(\epsilon^{pijkl}\lambda_{ij}\lambda^*_{kl}\right)^* = \epsilon^{pijkl}\lambda_{ij}\lambda^*_{kl}$ it is simple ti verify that

$$\lambda^p\lambda^{+*} = -\frac{1}{8}\epsilon^{pijkl}\lambda_{ij}\lambda^*_{kl} \qquad (1.102)$$

satisfies the constraint. Further we could show it solves automatically also the constraint (1.99). Therefore a pure spinor in 10 dimensions is given by the decomposition (1.94) in which

$$\lambda^i = -\frac{1}{8}\cdot\frac{\epsilon^{pijkl}\lambda_{ij}\lambda^*_{kl}}{\lambda^{+*}} \quad , \qquad (1.103)$$

for $\lambda^+ \neq 0$. Trivially the degrees of freedom of the pure spinor are $10(\lambda_{ij}) + 1(\lambda^+) = 11$.

1.4.3 Action with solved constraint

Ghost action S_λ given in (1.67) is invariant under local symmetry

$$\delta w_\alpha = \mathcal{Z}_m(\gamma^m)_{\alpha\beta}\lambda^\beta \qquad \delta\lambda^\alpha = 0 \quad , \qquad (1.104)$$

in fact integrating by parts, using $(\gamma^m)_{\alpha\beta} = (\gamma^m)_{\beta\alpha}$ and the pure spinor constraint we have

$$\begin{aligned}
\delta S_\lambda &= \frac{1}{2\pi}\int d^2z\, \mathcal{Z}_m(\gamma^m)_{\alpha\beta}\lambda^\beta\bar\partial\lambda^\alpha \\
&= -\frac{1}{2\pi}\int d^2z\, \bar\partial\mathcal{Z}_m(\gamma^m)_{\alpha\beta}\lambda^\beta\lambda^\alpha - \frac{1}{2\pi}\int d^2z\, \mathcal{Z}_m(\gamma^m)_{\alpha\beta}\bar\partial\lambda^\beta\lambda^\alpha \\
&= -\frac{1}{2\pi}\int d^2z\, \bar\partial\mathcal{Z}_m\lambda^\alpha(\gamma^m)_{\alpha\beta}\lambda^\beta - \frac{1}{2\pi}\int d^2z\, \mathcal{Z}_m(\gamma^m)_{\beta\alpha}\lambda^\alpha\bar\partial\lambda^\beta \\
&= -\delta S_\lambda \qquad \Rightarrow \qquad \delta S_\lambda = 0
\end{aligned}$$

If we decompose $\mathcal{Z}^m$ in (ζ^i, ζ_i), (1.78) gives

$$\delta w_\alpha = \sqrt{2}\left[\zeta^i(a_i\lambda)_\alpha + \zeta_i(a^i\lambda)_\alpha\right] \qquad (1.105)$$

and, remembering that $\delta w_i = \langle 0|a_i|\delta w\rangle$, we have

$$\begin{aligned}
\delta w_i &= \sqrt{2}\left[\zeta^j\langle 0|a_ia_j|\lambda\rangle + \zeta_j\langle 0|a_ia^j|\lambda\rangle\right] \\
&= \sqrt{2}\left[\frac{1}{2}\zeta^j\lambda_{kl}\langle 0|a_ia_ja^la^k|0\rangle + \zeta_j\lambda^+\langle 0|a_ia^j|0\rangle\right] \\
&= \sqrt{2}\left(\lambda_{ij}\zeta^j + \lambda^+\zeta_i\right) \quad . \qquad (1.106)
\end{aligned}$$

Let us choice the $\mathcal{Z}$ parameters:

$$\mathcal{Z}^i = -\frac{w_i}{2\lambda^+} \qquad\qquad \mathcal{Z}^{i+5} = -i\frac{w_i}{2\lambda^+} \quad ; \qquad (1.107)$$

we have immediately

$$\zeta^i = 0 \qquad\qquad \zeta_i = -\frac{w_i}{\sqrt{2}\lambda^+} \qquad\qquad (1.108)$$

and so

$$\delta w_i = -w_i \qquad \Rightarrow \qquad w_i \to w_i + \delta w_i = w_i - w_i = 0 \quad . \qquad (1.109)$$

In this way we demonstrated that it is always right to assume $w_i = 0$. We can note also that in 10 dimension C admits the antidiagonal form

$$C = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \qquad\qquad (1.110)$$

and we have the scalar product

$$_-\langle\omega|C|\lambda\rangle_+ = \begin{pmatrix} 0 & w_\beta \end{pmatrix} \begin{pmatrix} 0 & \delta_\alpha^\beta \\ -\delta_\alpha^\beta & 0 \end{pmatrix} \begin{pmatrix} \lambda^\alpha \\ 0 \end{pmatrix} = -w_\alpha\lambda^\alpha \quad . \qquad (1.111)$$

Now

$$\begin{aligned}
-\langle\omega|C|\lambda\rangle+ &= \left(w_i^*\langle 0|a_i + \frac{1}{2\cdot 3!}w^{ij*}\epsilon_{ijklm}\langle 0|a_m a_l a_k + w_+^*\langle 0|a_5 a_4 a_3 a_2 a_1 \right) \times \\
&\quad \times C\left(|0\rangle\lambda^+ + a^q a^p|0\rangle\frac{1}{2}\lambda_{pq} + a^s a^r a^q a^p|0\rangle\frac{1}{4!}\lambda^t\epsilon_{tpqrs} \right) \\
&= \frac{1}{4!}w_i^*\lambda^t\epsilon_{tpqrs}\langle 0|a_i C a^s a^r a^q a^p|0\rangle + \frac{1}{4\cdot 3!}w^{ij*}\lambda_{pq}\epsilon_{ijklm}\langle 0|a_m a_l a_k C a^q a^p|0\rangle + \\
&\quad + w_+^*\lambda^+\langle 0|a_5 a_4 a_3 a_2 a_1 C|0\rangle \\
&= -w_+^*\lambda^+ - w_i^*\lambda^i - \frac{1}{2}w^{ij*}\lambda_{ij}
\end{aligned}$$

so $w_\alpha\lambda^\alpha = w_+^*\lambda^+ + w_i^*\lambda^i + \frac{1}{2}w^{ij*}\lambda_{ij}$ and obviously

$$w_\alpha\bar\partial\lambda^\alpha = w_+^*\bar\partial\lambda^+ + w_i^*\bar\partial\lambda^i + \frac{1}{2}w^{ij*}\bar\partial\lambda_{ij} \quad . \qquad (1.112)$$

Let us define

$$\lambda^+ = e^s \qquad \lambda_{ij} = u_{ij} \qquad \lambda^i = -\frac{1}{8}\left(e^{-s}\right)^* \epsilon^{ijklm} u_{jk} u_{lm}^* \qquad (1.113)$$

and

$$w_+^* = \partial t e^{-s} \qquad w^{ij*} = v^{ij} \qquad w_i^* = 0 \quad, \tag{1.114}$$

noting that last position is justified by the gauge choice $w_i = 0$. In this way the ghost action (1.67) can be written

$$S_\lambda = \frac{1}{2\pi} \int d^2 z \left(\partial t \bar{\partial} s + \frac{1}{2} v^{ij} \bar{\partial} u_{ij} \right) \quad, \tag{1.115}$$

the ghost OPE are

$$t(z) s(w) = \ln(z-w) \tag{1.116}$$

$$v^{ij}(z) u_{kl}(w) = \frac{\delta_k^{[i} d_l^{j]}}{z-w} \tag{1.117}$$

and we can compute the stress-energy tensor for the ghost action

$$T_\lambda(z) = \frac{1}{2} v^{ij} u_{ij} + \partial t \partial s + \partial^2 s \quad. \tag{1.118}$$

The last term $\partial^2 s$ is necessary in order that the Lorentz currents are primary fields, i.e their OPE with $T(z)$ have at most a double pole. So, using (1.116) and (1.117), we obtain the central charge of the ghosts $c_\lambda = 22$, typical of a bc bosonic system with 11 degrees of freedom. Thus if we add the ghost action S_λ to the Siegel action S_S we obtain the total central charge

$$c = c_X + c_{p\theta} + c_\lambda = 10 - 32 + 22 = 0 \quad : \tag{1.119}$$

this is the first outcome of the ghost term (1.67).

The Lorentz currents for the ghosts are given by (1.67)

$$N^{mn} = \frac{1}{2} w \gamma^{mn} \lambda \tag{1.120}$$

so the U(5) decomposition is, using (1.80)-(1.83), (1.84)-(1.85), (1.90) and (1.92),

$$n^{ij} = -e^s v^{ij} \tag{1.121}$$

$$n_{ij} = e^{-s} \left(2\partial u_{ij} - u_{ij}\partial t - 2u_{ij}\partial s + u_{ik}u_{jl}v^{kl} - \frac{1}{2}u_{ij}u_{kl}v^{kl} \right) \tag{1.122}$$

$$n_j^i = u_{jk}v^{ik} - \frac{1}{5}\delta_j^i u_{kl}v^{kl} \tag{1.123}$$

$$n = -\frac{1}{\sqrt{5}} \left(\frac{1}{4} u_{ij}v^{ij} + \frac{5}{2}\partial t - \frac{5}{2}\partial s \right) \tag{1.124}$$

and we can compute the OPE current-current. The results can be summarized by the expression

$$N^{kl}(z)N^{mn}(w) = \frac{\eta^{m[l}N^{k]n} - \eta^{n[l}N^{k]m}}{z-w} - 3\frac{\eta^{k[n}\eta^{m]l}}{(z-w)^2}\quad . \qquad (1.125)$$

The total spinorial contribute to the Lorentz current is

$$M^{mn} = L^{mn} + N^{mn} \qquad (1.126)$$

and with (1.65) we have

$$
\begin{aligned}
M^{kl}(z)M^{mn}(w) &= \left[L^{kl}(z) + N^{kl}(z)\right]\left[L^{mn}(w) + N^{mn}(w)\right]\\
&= L^{kl}(z)L^{mn}(w) + N^{kl}(z)N^{mn}(w)\\
&= \frac{\eta^{m[l}L^{k]n} - \eta^{n[l}L^{k]m}}{z-w} + \frac{\eta^{m[l}N^{k]n} - \eta^{n[l}N^{k]m}}{z-w} +\\
&\quad + 4\frac{\eta^{k[n}\eta^{m]l}}{(z-w)^2} - 3\frac{\eta^{k[n}\eta^{m]l}}{(z-w)^2}\\
&= \frac{\eta^{m[l}M^{k]n} - \eta^{n[l}M^{k]m}}{z-w} + \frac{\eta^{k[n}\eta^{m]l}}{(z-w)^2} \qquad (1.127)
\end{aligned}
$$

exactly as the RNS superstring (1.66). So the pure spinor ghost term (1.67) makes right the Siegel action (1.51) and the sum of S_S and S_λ constitutes the *Pure Spinor superstring*.

Chapter 2

Superstring in curved space

2.1 The Green-Schwarz superstring in general background

The Green-Schwarz Type II superstring can be extended naturally in a curved background [15]

$$S_{GS} = -\frac{1}{2} \int d^2\sigma \left(\sqrt{h} h^{ij} G_{MN}(Z) + \varepsilon^{ij} B_{NM}(Z) \right) \partial_i Z^M \partial_j Z^N \qquad (2.1)$$

or rather, in complex coordinates (euclidean flat world-sheet)

$$S_{GS} = \frac{1}{2} \int d^2z \left(G_{MN}(Z) + B_{NM}(Z) \right) \partial Z^M \bar{\partial} Z^N \qquad (2.2)$$

where $Z^M = (X^m, \theta_L^\mu, \theta_R^{\widehat{\mu}})$ are the coordinates of the superspace and $M = (m, \mu, \widehat{\mu})$ with $m = 0, \ldots, 9$, $\mu, \widehat{\mu} = 1, \ldots, 16$. The Grassmann variables θ^μ, $\theta^{\widehat{\mu}}$ are Majorana-Weyl spinors of the opposite chirality for type IIA and same chiralty for type IIB. The first term of (2.1) and (2.2) corresponds to the kinetic term of (1.20) and (1.25), while the second one corresponds to the WZ term.

Explicitly, we can see the superspace as a supermanifold and we can define at every point Z the tangent superspace with flat metric η_{ab} and the cotangent superspace. The last one admits (see section (A)) coordinate dual basis $\{dZ^M\}$ or

orthonormal basis $\{E^{\mathcal{A}}\}$, where $\mathcal{A} = (a, \alpha, \widehat{\alpha})$ with $a = 0, \ldots, 9$, $\alpha, \widehat{\alpha} = 1, \ldots, 16$ are indices on tangent superspace. In the same way of the purely bosonic case, the change of basis defines the *supervielbein* $E_M{}^{\mathcal{A}}(Z)$

$$E^{\mathcal{A}} = E_M{}^{\mathcal{A}} dZ^M \tag{2.3}$$

and we have (cfr. (A.10))

$$G_{MN}(Z) = E_M{}^a(Z) E_N{}^b(Z) \eta_{ab} \quad , \tag{2.4}$$

generalization of the metric to the superspace.

In general supergravity background the WZ term is given by the 2-superform

$$B = \frac{1}{2} B_{MN} dZ^M \wedge dZ^N = \frac{1}{2} B_{\mathcal{A}\mathcal{B}} E^{\mathcal{A}} \wedge E^{\mathcal{B}} \tag{2.5}$$

being

$$B_{MN}(Z) = E_M{}^{\mathcal{A}}(Z) E_N{}^{\mathcal{B}}(Z) B_{\mathcal{A}\mathcal{B}}(Z) \quad . \tag{2.6}$$

In fact it is

$$\begin{aligned}
S_{WZ} &= \int B = \frac{1}{2} \int B_{MN} dZ^M \wedge dZ^N \\
&= +\frac{1}{2} \int B_{MN} \partial_i Z^M \partial_j Z^N d\sigma^i \wedge d\sigma^j \\
&= -\frac{1}{2} \int d^2\sigma \varepsilon^{ij} B_{NM} \partial_i Z^M \partial_j Z^N
\end{aligned} \tag{2.7}$$

as in (2.1). We define in a natural way

$$J_i{}^{\mathcal{A}} = E_M{}^{\mathcal{A}} \partial_i Z^M \tag{2.8}$$

and therefore

$$J_z{}^{\mathcal{A}} = E_M{}^{\mathcal{A}} \partial Z^M \qquad J_{\bar{z}}{}^{\mathcal{A}} = E_M{}^{\mathcal{A}} \bar{\partial} Z^M \quad , \tag{2.9}$$

so we can write

$$S_{GS} = -\frac{1}{2} \int d^2\sigma \left(\sqrt{h} h^{ij} \eta_{ab} J_i{}^a J_j{}^b + \varepsilon^{ij} B_{\mathcal{A}\mathcal{B}} J_i{}^{\mathcal{A}} J_j{}^{\mathcal{B}} \right) \tag{2.10}$$

or

$$S_{GS} = \int d^2z \, \frac{1}{2} \left(\eta_{ab} J_z{}^a J_{\bar{z}}{}^b + B_{\mathcal{A}\mathcal{B}} J_z{}^{\mathcal{A}} J_{\bar{z}}{}^{\mathcal{B}} \right) \quad . \tag{2.11}$$

2.2 Coset formulation of a superspace

From the previous section, it is evident that to write the action of superstring in curved space, we have to know the supervielbein $E_M{}^{\mathcal{A}}$ and the 2-superform $B_{\mathcal{AB}}$. The most important case (and also the simplest one) occurs when the superspace $\mathcal{M}$ can be described as the coset manifold of a Lie supergroup G on a bosonic subgroup H (see B)

$$\mathcal{M} \cong \frac{G}{H} \quad . \tag{2.12}$$

We can divide the complete set of generators of G as $T_A = (T_{(ab)}, T_{\mathcal{A}})$, where $T_{(ab)}$ are the generators of H and all the other ones $T_{\mathcal{A}}$ remain in the quotient G/H. In general $\mathcal{G} \setminus \mathcal{H}$ is not a subalgebra of $\mathcal{G}$. As the generators span the tangent space of a group manifold, $T_{\mathcal{A}}$ describe the tangent superspace of $\mathcal{M}$.

To construct the vielbein, we define the *Maurer-Cartan form*

$$J \equiv g^{-1}dg \quad , \quad g \in G \tag{2.13}$$

which takes values in the Lie algebra of G, as can be seen by substituting $g = e^{\alpha^A T_A}$ in (2.13) and using

$$e^{-A}de^A = dA + \frac{1}{2}[dA, A] + \frac{1}{3!}[[dA, A], A] + \cdots \quad . \tag{2.14}$$

Hence J can be decomposed as

$$J = J^A T_A = J^{\mathcal{A}} T_{\mathcal{A}} + J^{(ab)} T_{(ab)} \tag{2.15}$$

i.e.

$$J = J_M{}^A dZ^M T_A = J_M{}^{\mathcal{A}} dZ^M T_{\mathcal{A}} + J_M{}^{(ab)} dZ^M T_{(ab)} \quad : \tag{2.16}$$

$J_M{}^{\mathcal{A}}$ are exactly the supervielbein $E_M{}^{\mathcal{A}}$, while $J_M{}^{(ab)}$ are *spin connections*. By taking $dZ^M = \partial_i Z^M d\sigma^i$, we find

$$J_i{}^A = J_M{}^A \partial_i Z^M \tag{2.17}$$

and similarly

$$J_z{}^A = J_M{}^A \partial Z^M \qquad J_{\bar{z}}{}^A = J_M{}^A \bar{\partial} Z^M \quad , \tag{2.18}$$

as in (2.8) and (2.9).

The WZ term for a supergroup[1] manifold can be obtained generalizing the bosonic analogue on a group manifold [16]. Let us write the 3-form

$$\Omega_3 = \text{STr}(J \wedge [J \wedge J]) = C_{ABC} J^A \wedge J^B \wedge J^C \qquad (2.19)$$

where

$$C_{ABC} = C_{AB}{}^D \eta_{DC} \qquad (2.20)$$

with $C_{AB}{}^C$ structure constants of G and $\eta_{AB} = \text{STr}(T_A T_B)$: the WZ contribution is given by

$$S_{WZ} = \int_{\mathfrak{D}_3} \Omega_3 \qquad (2.21)$$

being $\mathfrak{D}_3$ a 3-dimensional domain whose boundary is the string world-sheet. By means of the Jacobi identity and the *Maurer-Cartan equation*

$$dJ + \frac{1}{2}[J \wedge J] = 0 \qquad (2.22)$$

one can verify that Ω_3 is closed, $d\Omega_3 = 0$. Thus Ω_3 is *locally* exact and we can found a 2-form B *depending on the coordinates* of the supergroup manifold, so that $\Omega_3 = dB$.

Notice that the WZ term for a coset manifold of a supergroup (or simply *super-coset manifold*), cannot be written as (2.19), with J^A restricted to $\mathcal{G} \smallsetminus \mathcal{H}$, because $\mathcal{G} \smallsetminus \mathcal{H}$ is not a superalgebra and we cannot use Jacobi and Maurer-Cartan equations, so $C_{\mathcal{ABC}} J^{\mathcal{A}} \wedge J^{\mathcal{B}} \wedge J^{\mathcal{C}}$ is in general not closed. We will see in the following how this problem solves for the particular cosets we are interested in.

2.2.1 Flat Green-Schwarz string as sigma model on super-coset

A particular case of a supercoset manifold is the flat space: denoting by $\text{SUSY}(\mathcal{N} = 2)$ the supergroup of Poincaré with 2 supersimmetries in 10 dimension and $SO(9,1)$

---

[1]Note that a supergroup manifold is a the particular case of supercoset manifold

its Lorentz subgroup, the flat 10-dimensional superspace with $\mathcal{N} = 2$ is given by SUSY($\mathcal{N} = 2$)/SO(9,1). The bosonic generators in the coset are P_m and the fermionic ones are $Q_{\alpha I}$, with $m = 0, \ldots, 9$ space-time indices, $\alpha = 1, \ldots, 16$ spinorial indices and $I = 1, 2$ corresponding to the two supersymmetries. We can construct the Maurer-Cartan form on the coset taking the group element

$$g = e^{X^m P_m + \theta^{\alpha I} Q_{\alpha I}} \tag{2.23}$$

and recalling the flat superalgebra for P and Q

$$\{Q_{\alpha I}, Q_{\beta J}\} = -2i\delta_{IJ}(\gamma^m)_{\alpha\beta} P_m \tag{2.24}$$

$$[P_m, P_n] = 0 \tag{2.25}$$

$$[Q_{\alpha I}, P_m] = 0 \quad . \tag{2.26}$$

We have

$$[dX^m P_m + d\theta^{\alpha I} Q_{\alpha I}, X^n P_n + \theta^{\beta J} Q_{\beta J}] = -d\theta^{\alpha I} \theta^{\beta J} \{Q_{\alpha I}, Q_{\beta J}\} = -2i\theta^{\alpha I}(\gamma^m)_{\alpha\beta} d\theta^{\beta I} P_m$$

hence only the first two terms of the expansion (2.14) survive, so

$$g^{-1} dg = (dX^m - i\theta^I \gamma^m d\theta^I) P_m + d\theta^{\alpha I} Q_{\alpha I} \quad . \tag{2.27}$$

Noting that indices on the supermanifold and on the tangent superspace are the same in flat case[2], we can write

$$J_i{}^m = \partial_i X^m - i\theta^I \gamma^m \partial_i \theta^I \qquad J_i{}^{\alpha I} = \partial_i \theta^{\alpha I} \quad . \tag{2.28}$$

We recognize immediately that $J_i{}^m$ is $\Pi_i{}^m$ in (1.21) and the first part of (2.10) reproduces the kinetic term of the Green-Schwarz superstring in flat space, up to a normalization constant.

To construct the WZ term [17], we have to find a closed 3-form invariant under SUSY($\mathcal{N} = 2$): the $J^{\mathcal{A}}$ given in (2.28) are invariant under translations and supersymmetry transformations, so we can start from a 3-form like $f_{\mathcal{ABC}} J^{\mathcal{A}} \wedge J^{\mathcal{B}} \wedge J^{\mathcal{C}}$,

---

[2]In particular we use $m = a$ and $\mu, \widehat{\mu} = \alpha, \widehat{\alpha} \to (\alpha; I = 1, 2)$.

where $f_{\mathcal{ABC}}$ are constants. Since $(J^m, J^{\alpha I})$ are respectively a vector and a spinor under Lorentz transformation $SO(9, 1)$, the Lorentz invariance imposes the structure $s^{IJ} J^m \wedge J^{\alpha I} (\gamma_m)_{\alpha\beta} \wedge J^{\beta J}$, with s^{IJ} symmetric matrix. Finally to have closed form we have to choice s^{IJ} traceless:

$$\Omega_3 = i s^{IJ} J^m \wedge J^{\alpha I} (\gamma_m)_{\alpha\beta} \wedge J^{\beta J} \tag{2.29}$$

with $s^{11} = -s^{22} = 1$. It is not difficult to show that the closed 3-form Ω_3 $(d\Omega_3 = 0)$ is also *exact*:

$$\Omega_3 = dB \tag{2.30}$$

with

$$B = -i dX^m \wedge (s^{IJ} \theta^I \gamma_m d\theta^J) + (\theta^1 \gamma^m d\theta^1) \wedge (\theta^2 \gamma_m d\theta^2) \tag{2.31}$$
$$= \varepsilon^{ij} \left[-i \partial_i X^m \left(\theta^1 \gamma_m \partial_j \theta^1 - \theta^2 \gamma_m \partial_j \theta^2 \right) + (\theta^1 \gamma_m \partial_i \theta^1)(\theta^2 \gamma_m \partial_j \theta^2) \right] d^2\sigma \; .$$

Trivially the WZ term in (1.20) is given - up to a constant - by $\displaystyle\int B$.

Notice that the supercoset $SUSY(\mathcal{N} = 2)/SO(9, 1)$ has a peculiar character since the algebra of P and Q is closed, i.e. P and Q span a subalgebra of $SUSY(\mathcal{N} = 2)$: in fact the form (2.27) does not take values in the Lorentz algebra $so(9, 1)$.

2.2.2   The Green-Schwarz string in $\mathrm{AdS}_5 \times \mathrm{S}^5$

The 10-dimensional space $\mathrm{AdS}_5 \times \mathrm{S}^5$ is homeomorphic to the coset (see B.1 and B.2)

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \cong \frac{SO(4, 2)}{SO(4, 1)} \times \frac{SO(6)}{SO(5)} \equiv \frac{SO(4, 2) \times SO(6)}{SO(4, 1) \times SO(5)} \tag{2.32}$$

so the corresponding superspace is given by a supercoset with bosonic part as above. Let us consider the even supermatrix $(4 + 4) \times (4 + 4)$

$$A = \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix} \tag{2.33}$$

with Grassmann even entries for X, Y and Grassmann odd entries for θ, η. If we introduce the $(2,2|4)$ metric

$$K = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix} \qquad \text{with} \qquad \Sigma = \sigma^3 \otimes \mathbb{1}_2 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \quad , \qquad (2.34)$$

and the superadjoint of A

$$A^{\ddagger} = A^{t\sharp} \qquad \text{with} \qquad c^{\sharp} = (c^* \text{ even}, -ic^* \text{ odd}) \quad , \qquad (2.35)$$

we can define the Lie superalgebra $psu(2,2|4)$ of the supergroup $\mathrm{PSU}(2,2|4)$ imposing that A satisfy [18]

$$AK + KA^{\ddagger} = 0 \qquad (2.36)$$

$$\mathrm{Tr}X = \mathrm{Tr}Y = 0 \quad . \qquad (2.37)$$

For the building blocks of A the conditions above give[3]

$$
\begin{aligned}
X\Sigma + \Sigma X^{\dagger} = 0 \qquad & Y + Y^{\dagger} = 0 \\
& \qquad\qquad\qquad\qquad \theta - i\Sigma\eta^{\dagger} = 0 \quad . \qquad (2.39)\\
\mathrm{Tr}X = 0 \qquad & \mathrm{Tr}Y = 0
\end{aligned}
$$

It means that the bosonic (even) blocks (X, Y) are $X \in su(2,2)$, $Y \in su(4)$, i.e.

$$\mathrm{Bos}[psu(2,2|4)] = su(2,2) \oplus su(4) \qquad (2.40)$$

or, in terms of groups

$$\mathrm{Bos}[\mathrm{PSU}(2,2|4)] = \mathrm{SU}(2,2) \times \mathrm{SU}(4) \quad . \qquad (2.41)$$

From the classical group theory we know that [19] $\mathrm{SU}(2,2) \cong \mathrm{SO}(4,2)$ and $\mathrm{SU}(4) \cong \mathrm{SO}(6)$, thus

$$\mathrm{Bos}[\mathrm{PSU}(2,2|4)] \cong \mathrm{SO}(4,2) \times \mathrm{SO}(6) \quad . \qquad (2.42)$$

---

[3]It is interesting to note that the condition (2.37) admits a weaker form given by $\mathrm{STr}A \equiv \mathrm{Tr}X - \mathrm{Tr}Y = 0$. The so-defined superalgebra is $su(2,2|4)$: it differs from $psu(2,2|4)$ essentially because it contains the identity $\mathbb{1}_8$, i.e

$$\mathrm{Bos}[su(2,2|4)] = su(2,2) \oplus su(4) \oplus u(1) \qquad (2.38)$$

or $\mathrm{PSU}(2,2|4) = \mathrm{SU}(2,2|4)/\mathrm{U}(1)$.

The principal characteristic of $psu(2,2|4)$ is given by the existence of an automorphism $\Omega : A \to \Omega(A) \in psu(2,2|4)$ defined as

$$\Omega(A) = \begin{pmatrix} JX^tJ & -J\eta^tJ \\ J\theta^tJ & JY^tJ \end{pmatrix} \qquad \text{with} \qquad J = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad . \tag{2.43}$$

It is simple to show that $\Omega^4(A) = A$ i.e.

$$\Omega^4 = I \quad , \tag{2.44}$$

hence Ω has eigenvalues $\{\pm 1, \pm i\}$: if $\mathcal{H}_p$ is the eigenspace associated to the eigenvalue i^p $(p = 0,1,2,3)$

$$\Omega(H_p) = i^p H_p \qquad H_p \in \mathcal{H}_p \tag{2.45}$$

we can decompose $psu(2,2|4)$ in the direct sum of $\mathcal{H}_p$:

$$psu(2,2|4) = \bigoplus_{p=0}^{3} \mathcal{H}_p \tag{2.46}$$

said $\mathbb{Z}_4$-*grading* of $psu(2,2|4)$. Further one can demonstrate that

$$\Omega\left([H_p, H_q]\right) = [\Omega(H_p), \Omega(H_q)] = i^{p+q}[H_p, H_q] \tag{2.47}$$

thus

$$[H_p, H_q] \in \mathcal{H}_{p+q \,|\, \text{mod}\, 4} \quad . \tag{2.48}$$

Notice that the only closed subalgebra of $psu(2,2|4)$ is $\mathcal{H}_0$.

We want to study the explicit form of H_p; let us start from H_0: the definition $\Omega(H_0) = H_0$ on the building blocks gives

$$X^tJ + JX = 0 \qquad Y^tJ + JY = 0 \qquad \theta = \eta = 0 \tag{2.49}$$

that means $X, Y \in sp(4)$, superalgebra of the symplectic group $\text{Sp}(4)$. On the other hand $X \in su(2,2)$ and $Y \in su(4)$, so we can conclude that in H_0

$$X \in usp(2,2) \qquad Y \in usp(4) \tag{2.50}$$

with $usp(n)$ superalgebra of the unitary-symplectic group $\mathrm{USp}(n) \equiv \mathrm{SU}(n) \cap \mathrm{Sp}(n)$. Because of $\theta = \eta = 0$ $\mathcal{H}_0$ is a bosonic subalgebra of the form

$$\mathcal{H}_0 = usp(2,2) \oplus usp(4) \tag{2.51}$$

and it generates the bosonic subgroup $\mathrm{USp}(2,2) \times \mathrm{USp}(4)$ of $\mathrm{PSU}(2,2|4)$. From classical group theory [19] $\mathrm{USp}(2,2) \cong \mathrm{SO}(4,1)$ and $\mathrm{USp}(4) \cong \mathrm{SO}(5)$, so $\mathcal{H}_0$ generates $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ and can be identified with $so(4,1) \oplus so(5)$.

As regards the $\mathcal{H}_2$ one can observe that $\theta = \eta = 0$, so it is a bosonic eigenspace, while in $\mathcal{H}_1$ and $\mathcal{H}_3$ $X = Y = 0$, so they are fermionic eigenspaces.

Using (2.42) we can write

$$\mathrm{AdS}_5 \times \mathrm{S}^5 \cong \mathrm{Bos}\left[\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(4,1) \times \mathrm{SO}(5)}\right] \quad, \tag{2.52}$$

hence we can study the superstring in the corresponding supermanifold by means of the supercoset $\mathrm{PSU}(2,2|4)/\mathrm{SO}(4,1) \times \mathrm{SO}(5)$. Notice that the bosonic subgroup $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ represents the generalization of the Lorentz group in the space $\mathrm{AdS}_5 \times \mathrm{S}^5$, on analogy of the flat case.

We define the canonical form $J = g^{-1}dg$ with $g \in \mathrm{PSU}(2,2|4)$: since it takes values in $psu(2,2|4)$, we can decompose it in $\mathbb{Z}_4$ components

$$J = \sum_{i=0}^{3} J_i \tag{2.53}$$

where

$$J_0 = J^{(ab)}T_{(ab)} \qquad J_2 = J^a T_a \qquad J_1 = J^\alpha T_\alpha \qquad J_3 = J^{\widehat{\alpha}} T_{\widehat{\alpha}} \tag{2.54}$$

and $(T_{(ab)}, T_a, T_\alpha, T_{\widehat{\alpha}})$ are the generators respectively in $(\mathcal{H}_0, \mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_3)$. Trivially J is invariant under global left $\mathrm{PSU}(2,2|4)$ multiplication

$$g \to g'g \qquad g' \in \mathrm{PSU}(2,2|4) \quad ; \tag{2.55}$$

on other hand it is simple to verify that under a *local* right $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ multiplication

$$g \to gh \qquad h \in \mathrm{SO}(4,1) \times \mathrm{SO}(5) \tag{2.56}$$

J_i transform in the way

$$J_0 \to h^{-1} J_0 h + h^{-1} dh \tag{2.57}$$

$$J_i \to h^{-1} J_i h \qquad i = 1, 2, 3 \quad . \tag{2.58}$$

The transformation of J_0 is the typical of a gauge field, so we can understand J_0 as the $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$-gauge field; J_i transform according to the adjoint representation of $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ as befits matter fields. If we write $h = e^{\mathfrak{h}}$ with $\mathfrak{h} \in so(4,1) \oplus so(5)$ the infinitesimal gauge transformation on g is

$$\delta g = g \mathfrak{h} \tag{2.59}$$

and we obtain the infinitesimal form of (2.57) and (2.58)

$$\delta J_0 = [J_0, \mathfrak{h}] + d\mathfrak{h} \tag{2.60}$$

$$\delta J_i = [J_i, \mathfrak{h}] \quad . \tag{2.61}$$

The kinetic term of the action can be written immediately in the form (2.10) or (2.11), noting that J^a in (2.54) are exactly J^a of these expressions. In particular if we pose[4] $\eta_{ab} = \mathrm{STr}(T_a T_b)$ and $J_z^a \equiv J^a$, $J_{\bar{z}}^a \equiv \overline{J}^a$ we can write

$$S_{kin} = \int d^2 z \, \frac{1}{2} \mathrm{STr}(J_2 \overline{J}_2) \quad . \tag{2.62}$$

To construct the WZ term we use the $\mathbb{Z}_4$-grading and define the 3-form [20]

$$\Omega_3 = \mathrm{STr}\left(J_2 \wedge [J_1 \wedge J_1] - J_2 \wedge [J_3 \wedge J_3]\right) \quad . \tag{2.63}$$

It is simple to note that Ω_3 is $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$-gauge invariant and one can show that it is also closed. By means of the $\mathrm{PSU}(2,2|4)$ structure constants, Ω_3 becomes

$$\Omega_3 = C_{a\alpha\beta} J^a \wedge J^\alpha \wedge J^\beta - C_{a\widehat{\alpha}\widehat{\beta}} J^a \wedge J^{\widehat{\alpha}} \wedge J^{\widehat{\beta}} \tag{2.64}$$

---

[4]We define only a constant, because the trace of two generators is always proportional to the metric. It can be understood noting that only the metric takes appropriate indices (*Killing formalism*).

and introducing the explicit formulas for C one can obtain an expression closely analogous to (2.29) [21]. As the flat case, one can demonstrate that Ω_3 is not only closed, but also exact, $\Omega_3 = dB$ with

$$B = \mathrm{STr}(J_1 \wedge J_3) = \mathrm{STr}(J_1 \overline{J}_3 - \overline{J}_1 J_3) d^2 z \quad . \tag{2.65}$$

To add the WZ term to the action we have to fix a constant in front of Ω_3 i.e. in front of B. The correct choice is $\dfrac{1}{2}$ to have κ-symmetry [21] and conformal invariance [20], and the final expression of GS superstring in $\mathrm{AdS}_5 \times \mathrm{S}^5$ is a *Wess-Zumino non-linear sigma model*

$$S_{GS} = \int d^2 z \, \mathrm{STr} \left[\frac{1}{2} J_2 \overline{J}_2 - \frac{1}{4} \left(J_3 \overline{J}_1 - J_1 \overline{J}_3 \right) \right] \quad . \tag{2.66}$$

2.3 The Pure Spinor superstring in curved background

The pure spinor superstring in curved background can be constructed from GS superstring in a similar way to the flat case. To this purpose, we recall the relation between the GS and the Siegel action (1.52) which using (2.28) can be rewritten as

$$S_S = S_{GS} + \int d^2 z \, d_{\alpha 1} \overline{J}^{\alpha 1} \quad . \tag{2.67}$$

In the case of a type II superstring we have to add a term for the right sector $d_{\alpha 2} J^{\alpha 2}$, where d^1 and d^2 have $(1, 0)$ and $(0, 1)$ conformal weight respectively. This can be immediately generalized in a curved background with $\mathbb{Z}_4$-grading by $d_\alpha \overline{J}^\alpha + d_{\widehat{\alpha}} J^{\widehat{\alpha}}$ with the index convention of the last section. Furthermore we have to add a quadratic term in d_α and $d_{\widehat{\alpha}}$ generalization of the massless vertex operator [22]: hence we have

$$S_{matter} = S_{GS} + \int d^2 z \, \left(d_\alpha \overline{J}^\alpha + d_{\widehat{\alpha}} J^{\widehat{\alpha}} + \eta^{\alpha \widehat{\alpha}} d_\alpha d_{\widehat{\alpha}} \right) \quad . \tag{2.68}$$

The auxiliary fields d_α, $d_{\widehat{\alpha}}$ can integrated out and we obtain trivially

$$S_{matter} = S_{GS} + \int \mathrm{STr}(J_3 \overline{J}_1)$$

$$= \int d^2z \, \mathrm{STr} \left[\frac{1}{2} J_2 \overline{J}_2 + \frac{3}{4} J_3 \overline{J}_1 + \frac{1}{4} J_1 \overline{J}_3 \right] \qquad (2.69)$$

with the position $\mathrm{STr}(T_\alpha T_{\widehat{\alpha}}) = \eta_{\alpha\widehat{\alpha}}$ and $\eta^{\alpha\widehat{\alpha}}$ inverse of $\eta_{\alpha\widehat{\alpha}}$. Notice that S_{matter} can be written

$$S_{matter} = \int d^2z \, \mathrm{STr} \left[\frac{1}{2} \left(J_2 \overline{J}_2 + J_3 \overline{J}_1 + J_1 \overline{J}_3 \right) + \frac{1}{4} \left(J_3 \overline{J}_1 - J_1 \overline{J}_3 \right) \right] \quad : \quad (2.70)$$

the first bracket is a *principal chiral model* and it can be understood as the kinetic term for the matter, while the second term has the structure of a Wess-Zumino action. So we define $S_{matter} = S_{P\chi M} + S_{WZ}$.

The ghost fields have to take values in the fermionic eigenspaces $\mathcal{H}_1$ and $\mathcal{H}_3$, hence we can define them λ_1 and λ_3. In flat space the momentum has opposite chirality respect to its conjugate field, so that the coupling ghost-momentum is Lorentz invariant: analogously in curved space we take each momentum in a different eigenspace respect to its ghost, i.e. $w_3 \in \mathcal{H}_3$ is the conjugate momentum of λ_1 and $w_1 \in \mathcal{H}_1$ is the conjugate one of λ_3. To construct the ghost term of the action we have to substitute the canonical derivative with the covariant one: because we interpreted the J_0 as the gauge field, it is natural to define

$$\nabla \equiv \partial + [J_0, \ ] \qquad (2.71)$$

and analogue for $\overline{\nabla}$. In this way the ghost term can be written

$$S_{ghost} = - \int d^2z \, \mathrm{STr} \left[w_3 \overline{\nabla} \lambda_1 + w_1 \nabla \lambda_3 \right] \quad . \qquad (2.72)$$

To complete the action, we have to add a *current-current term*

$$S_{current} = - \int d^2z \, \mathrm{STr} \left(\{w_3, \lambda_1\} \{w_1, \lambda_3\} \right) \quad : \qquad (2.73)$$

we will see in the next chapter that $N = -\{w_3, \lambda_1\}$ and $\hat{N} = \{w_1, \lambda_3\}$ are gauge currents and $S_{current}$ is necessary to allow the BRST invariance of the action. Now we

can note that $\text{STr}(N\hat{N}) = \eta_{(ab)(cd)}N^{(ab)}\hat{N}^{(cd)}$ and the tensor $\eta_{(ab)(cd)} = \text{STr}(T_{(ab)}T_{(cd)})$ corresponds to the Riemann curvature tensor of the space [23].

Finally we write the complete pure spinor superstring in a curved space described by a $\mathbb{Z}_4$-graded supercoset. We fix a normalization and choose a coupling constant R^2: for spaces with some characteristic length it will be natural to identify R with this length, e.g. in AdS_n space R is the curvature radius. We have

$$S_{PS} = \frac{R^2}{2\pi} \int d^2z\, \text{STr}\left[\frac{1}{2}J_2\overline{J}_2 + \frac{3}{4}J_3\overline{J}_1 + \frac{1}{4}J_1\overline{J}_3 - w_3\overline{\nabla}\lambda_1 - w_1\nabla\lambda_3 - N\hat{N}\right]. \quad (2.74)$$

Let us note that the presence of fermionic fields J_1, J_3 out of the WZ term breaks the κ symmetry typical of GS superstring. However in next chapter we will show that the action is invariant under BRST symmetry, as in the flat case.

Chapter 3

Superstring in $\mathrm{AdS}_4 \times \mathbb{CP}^3$ space

3.1 Superalgebra of $\mathrm{OSP}(4|6)$

As we did in the section 2.2.2, now we have to find a supercoset corresponding to $\mathrm{AdS}_4 \times \mathbb{CP}^3$, noting that (B.11) and (B.14) give

$$\mathrm{AdS}_4 \times \mathbb{CP}^3 \cong \frac{\mathrm{SO}(3,2)}{\mathrm{SO}(3,1)} \times \frac{\mathrm{SU}(4)}{\mathrm{U}(3)} \equiv \frac{\mathrm{SO}(3,2) \times \mathrm{SU}(4)}{\mathrm{SO}(3,1) \times \mathrm{U}(3)} \quad . \tag{3.1}$$

In order to do it, we introduce the even supermatrix $(4+6) \times (4+6)$

$$A = \begin{pmatrix} X & \theta \\ \eta & Y \end{pmatrix} \tag{3.2}$$

with Grassmann even entries for X, Y and Grassmann odd entries for θ, η, we define the supertranspose of A

$$A^{st} = \begin{pmatrix} X^t & -\eta^t \\ \theta^t & Y^t \end{pmatrix} \tag{3.3}$$

and the $(4|6)$ metric

$$K = \begin{pmatrix} C_4 & 0 \\ 0 & \mathbb{1}_6 \end{pmatrix} \tag{3.4}$$

where C_4 is the 4-dimensional charge conjugation matrix, that we can always choose real, antisymmetric and so that $C_4{}^2 = -\mathbb{1}_4$ (see (C.3)). The superalgebra $osp(4|6)$

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of the orthosymplectic supergroup $OSP(4|6)$ is given by the matrices A with the property [18]

$$A^{st}K + KA = 0 \tag{3.5}$$

i.e.

$$X^t C_4 + C_4 X = 0 \qquad Y^t + Y = 0 \qquad \eta^t - C_4 \theta = 0 \quad , \tag{3.6}$$

hence $X \in sp(4)$ and $Y \in so(6)$ that is

$$\mathrm{Bos}[osp(4|6)] = sp(4) \oplus so(6) \quad . \tag{3.7}$$

In group term it means

$$\mathrm{Bos}[OSP(4|6)] \cong \mathrm{Sp}(4) \times \mathrm{SO}(6) \tag{3.8}$$

and because [19] $\mathrm{Sp}(4) \cong \mathrm{SO}(3,2)$ and $\mathrm{SO}(6) \cong \mathrm{SU}(4)$

$$\mathrm{Bos}[OSP(4|6)] \cong \mathrm{SO}(3,2) \times \mathrm{SU}(4) \quad . \tag{3.9}$$

One can show that there exist two real antisymmetric matrices K_4, K_6 respectively 4×4 and 6×6, with the properties

$$[K_4, C_4] = 0 \qquad {K_4}^2 = -\mathbb{1}_4 \qquad {K_6}^2 = -\mathbb{1}_6 \quad ; \tag{3.10}$$

thus we can define the automorphism $\Omega : A \to \Omega(A) \in osp(4|6)$ with

$$\Omega(A) = \begin{pmatrix} K_4 X^t K_4 & K_4 \eta^t K_6 \\ -K_6 \theta^t K_4 & K_6 Y^t K_6 \end{pmatrix} \quad . \tag{3.11}$$

Let us note that if we introduce the matrix

$$\Upsilon = \begin{pmatrix} K_4 C_4 & 0 \\ 0 & -K_6 \end{pmatrix} \tag{3.12}$$

the Ω automorphism can be written

$$\Omega(A) = \Upsilon A \Upsilon^{-1} \tag{3.13}$$

using (3.6). It is simple to show that $\Upsilon^4 = \mathbb{1}_{10}$, therefore

$$\Omega^4(A) = \Upsilon^4 A (\Upsilon^{-1})^4 = A \quad \Rightarrow \quad \Omega^4 = I \tag{3.14}$$

and $osp(4|6)$ admits $\mathbb{Z}_4$-grading

$$osp(4|6) = \bigoplus_{k=0}^{3} \mathcal{H}_k \tag{3.15}$$

with $\mathcal{H}_k = \{H_k \in osp(4|6) : \Omega(H_k) = i^k H_k\}$. Trivially we observe that $[H_p, H_q] \in \mathcal{H}_{p+q \,|\, \mathrm{mod}\, 4}$; further one can demonstrate that $\mathcal{H}_0$, $\mathcal{H}_2$ are bosonic eigenspaces ($\theta = \eta = 0$) and $\mathcal{H}_1$, $\mathcal{H}_3$ are fermionic ones ($X = Y = 0$). In particular one can show that $\mathcal{H}_0 = so(3,1) \oplus u(3)$, so the subgroup of OSP(4|6) is SO(3,1) $\times$ U(3). Consequently

$$\mathrm{AdS}_4 \times \mathbb{CP}^3 \cong \mathrm{Bos}\left[\frac{\mathrm{OSP}(4|6)}{\mathrm{SO}(3,1) \times \mathrm{U}(3)}\right] \tag{3.16}$$

and we can use the supercoset OSP(4|6)/SO(3,1) $\times$ U(3) to study superstring in $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

In addition we note that

$$[\Omega(A)]^* = \left(\Upsilon A \Upsilon^{-1}\right)^* = \Upsilon A^* \Upsilon^{-1} = \Omega(A^*) \tag{3.17}$$

since Υ is real. Thus for all $H_3 \in \mathcal{H}_3$

$$\Omega(H_3{}^*) = [\Omega(H_3)]^* = (-iH_3)^* = iH_3{}^* \qquad \Rightarrow \qquad H_3{}^* \in \mathcal{H}_1 \tag{3.18}$$

and in the same way $H_1{}^* \in \mathcal{H}_3$: we can conclude that

$$\mathcal{H}_3{}^* \equiv \mathcal{H}_1 \tag{3.19}$$

and it means that there is a one-to-one correspondence between $\mathcal{H}_1$ and $\mathcal{H}_3$.

The construction of pure spinor superstring in the supercoset OSP(4|6)/SO(3,1)$\times$ U(3) is totally analogous to the PSU(2,2|4)/SO(4,1) $\times$ SO(5) case we discussed in section 2.2.2 and we do not repeat it.

However it is important to note that the supercoset OSP(4|6)/SO(3,1) $\times$ U(3) contains 24 fermionic degrees of freedom, while the type IIA GS superstring has

32 fermions. Thus the sigma model can be interpreted as GS formulation with a partially fixed κ symmetry [24], i.e. where 8 fermions are gauged away. This interpretation is confirmed by the presence in the GS coset model of a local fermionic symmetry that is able to remove other 8 fermionic degrees of freedom, giving finally 16 fermions, as in the GS superstring with κ symmetry totally fixed. This argument fails for some particular bosonic configurations, corresponding to string moving in the AdS part of the space only, because the number of κ simmetries becomes 12 and the gauge fixed sigma model has less fermionic degrees of freedom than canonical GS string. In pure spinor superstring there is not κ symmetry, so one can hope to solve the problem within this formulation.

Finally we fix our conventions about the generators of $osp(4|6)$:

$$
\begin{aligned}
&\mathcal{H}_0 \left(M^{mn}, V_a{}^b \right) \quad M^{mn} \in so(3,1) \qquad\quad V_a{}^b \in u(3) \\
&\mathcal{H}_2 \left(P^m, V_a, V^a \right) \quad P^m \in so(3,2) \smallsetminus so(3,1) \quad (V_a, V^a) \in su(4) \smallsetminus u(3) \\
&\mathcal{H}_1 \left(\mathcal{O}_{\alpha a}, \mathcal{O}^{\dot\alpha a} \right) \\
&\mathcal{H}_3 \left(\mathcal{O}_\alpha{}^a, \mathcal{O}^{\dot\alpha}{}_a \right)
\end{aligned}
\tag{3.20}
$$

while the expression for OSP(4|6) algebra is given in Appendix C.

3.2   BRST transformation

3.2.1   Nilpotency

The BRST transformation is a gauge transformation with ghost fields as gauge parameters, i.e the BRST charge acts on g element of the supergroup OSP(4|6) as an infinitesimal right multiplication like (2.59)

$$
Q(g) = g\Lambda
\tag{3.21}
$$

where $\Lambda = \lambda_1 + \lambda_3$ is the ghost total field. It is simple to verify that $Q(g^{-1}) = -\Lambda g^{-1}$, so the BRST transformation of $J = g^{-1}\partial g$ is

$$Q(J) = \partial\Lambda + [J, \Lambda] \quad . \tag{3.22}$$

In $\mathbb{Z}_4$ components

$$Q(J_0) = [J_1, \lambda_3] + [J_3, \lambda_1] \tag{3.23}$$

$$Q(J_1) = \partial\lambda_1 + [J_0, \lambda_1] + [J_2, \lambda_3] = \nabla\lambda_1 + [J_2, \lambda_3] \tag{3.24}$$

$$Q(J_2) = [J_1, \lambda_1] + [J_3, \lambda_3] \tag{3.25}$$

$$Q(J_3) = \partial\lambda_3 + [J_0, \lambda_3] + [J_2, \lambda_1] = \nabla\lambda_3 + [J_2, \lambda_1] \quad . \tag{3.26}$$

Then, taking into account the fermionic character of Q, we have

$$Q^2(J) = Q([J, \Lambda]) = \{\partial\Lambda, \Lambda\} + \{[J, \Lambda], \Lambda\} \quad ; \tag{3.27}$$

using Jacobi identities and

$$\{\partial\lambda_1, \lambda_1\} = \frac{1}{2}\partial\{\lambda_1, \lambda_1\} \qquad \{\partial\lambda_3, \lambda_3\} = \frac{1}{2}\partial\{\lambda_3, \lambda_3\} \tag{3.28}$$

$$\{\partial\lambda_1, \lambda_3\} + \{\partial\lambda_3, \lambda_1\} = \partial\{\lambda_1, \lambda_3\} \tag{3.29}$$

we can write also (3.27) in $\mathbb{Z}_4$ components

$$Q^2(J_0) = -\frac{1}{2}[\{\lambda_1, \lambda_1\} + \{\lambda_3, \lambda_3\}, J_2] + \nabla\{\lambda_1, \lambda_3\} \tag{3.30}$$

$$Q^2(J_1) = -\frac{1}{2}[\{\lambda_1, \lambda_1\} + \{\lambda_3, \lambda_3\}, J_3] - [\{\lambda_1, \lambda_3\}, J_1] \tag{3.31}$$

$$Q^2(J_2) = \frac{1}{2}\nabla\{\lambda_1, \lambda_1\} + \frac{1}{2}\nabla\{\lambda_3, \lambda_3\} - [\{\lambda_1, \lambda_3\}, J_2] \tag{3.32}$$

$$Q^2(J_3) = -\frac{1}{2}[\{\lambda_1, \lambda_1\} + \{\lambda_3, \lambda_3\}, J_1] - [\{\lambda_1, \lambda_3\}, J_3] \tag{3.33}$$

As in the flat case, ghost fields need a constraint to give the nilpotency of Q^2: if we impose

$$\{\lambda_1, \lambda_1\} = 0 \qquad \{\lambda_3, \lambda_3\} = 0 \tag{3.34}$$

we obtain

$$Q^2(J_0) = \nabla\{\lambda_1, \lambda_3\} \tag{3.35}$$

$$Q^2(J_1) = -[\{\lambda_1, \lambda_3\}, J_1] \tag{3.36}$$

$$Q^2(J_2) = -[\{\lambda_1, \lambda_3\}, J_2] \tag{3.37}$$

$$Q^2(J_3) = -[\{\lambda_1, \lambda_3\}, J_3] \quad . \tag{3.38}$$

Since $\{\lambda_1, \lambda_3\} \in so(3,1) \oplus u(3)$, equations $(3.35) - (3.38)$ are exactly gauge transformations of the form (2.60)-(2.61) and so $Q^2(J_i) = 0$ just up to gauge transformations. Therefore the conditions (3.34) are the constraints of the ghosts on curved background and correspond to the pure spinor constraint on flat space.

In addition we note that using (3.23)-(3.26) one can obtain the BRST conserved charge for the action in the form

$$Q = \int dz \mathrm{STr}(\lambda_1 J_3) + \int d\bar{z} \mathrm{STr}(\lambda_3 \overline{J}_1) \quad . \tag{3.39}$$

If we reintroduce the auxiliary fields d_α, $d_{\widehat{\alpha}}$, Q can be written

$$Q = -\int dz \lambda^\alpha d_\alpha + \int d\bar{z} \lambda^{\widehat{\alpha}} d_{\widehat{\alpha}} \quad , \tag{3.40}$$

that is analogous to (1.70) in the flat case. In fact solving the equation of motion for d fields, (3.40) gives exactly (3.39).

3.2.2 Invariance of the action

We want verify the BRST invariance for the PS action. For the kinetic term of matter

$$S_{P\chi M} = \frac{R^2}{2\pi} \int d^2z \, \mathrm{STr}\left[\frac{1}{2} J_2 \overline{J}_2 + \frac{1}{2} J_1 \overline{J}_3 + \frac{1}{2} J_3 \overline{J}_1\right] \tag{3.41}$$

we have

$$QS_{P\chi M} = \frac{R^2}{2\pi} \int d^2z \, \mathrm{STr}\frac{1}{2}\left[\nabla\lambda_3 \overline{J}_1 + \overline{\nabla}\lambda_3 J_1 + \nabla\lambda_1 \overline{J}_3 + \overline{\nabla}\lambda_1 J_3\right] \tag{3.42}$$

and for the Wess-Zumino term

$$S_{WZ} = \frac{R^2}{2\pi} \int d^2z \operatorname{STr}\left[\frac{1}{4}J_3\overline{J}_1 - \frac{1}{4}J_1\overline{J}_3\right] \tag{3.43}$$

using the Maurer-Cartan equations

$$\nabla\overline{J}_1 - \overline{\nabla}J_1 = -[J_3, \overline{J}_2] - [J_2, \overline{J}_3] \tag{3.44}$$

$$\nabla\overline{J}_3 - \overline{\nabla}J_3 = -[J_1, \overline{J}_2] - [J_2, \overline{J}_1] \tag{3.45}$$

we have

$$QS_{WZ} = \frac{R^2}{2\pi} \int d^2z \operatorname{STr}\frac{1}{2}\left[\nabla\lambda_3\overline{J}_1 - \overline{\nabla}\lambda_3 J_1 - \nabla\lambda_1\overline{J}_3 + \overline{\nabla}\lambda_1 J_3\right] +$$
$$+ \frac{R^2}{2\pi} \int d^2z \operatorname{STr}\frac{1}{4}\left[\partial(\lambda_1\overline{J}_3 - \lambda_3\overline{J}_1) + \bar{\partial}(\lambda_3 J_1 - \lambda_1 J_3)\right] \quad . \tag{3.46}$$

Canceling the total derivative, BRST transformation for $S_{matter} = S_{P\chi M} + S_{WZ}$ is

$$QS_{matter} = \frac{R^2}{2\pi} \int d^2z \operatorname{STr}\left[\nabla\lambda_3\overline{J}_1 + \overline{\nabla}\lambda_1 J_3\right] \quad . \tag{3.47}$$

This quantity has to be deleted by the BRST variation of the ghost term

$$S_{ghost} = -\frac{R^2}{2\pi} \int d^2z \operatorname{STr}\left(w_3\overline{\nabla}\lambda_1 + w_1\nabla\lambda_3\right) \quad : \tag{3.48}$$

together with the usual transformations for the ghosts

$$Q(\lambda_1) = 0 \qquad Q(\lambda_3) = 0 \quad , \tag{3.49}$$

we assume for the momenta

$$Q(w_3) = J_3 \qquad Q(w_1) = \overline{J}_1 \quad , \tag{3.50}$$

noting that, as required, Q does not modify the $\mathbb{Z}_4$ grading and the conformal weight, but increase the ghost number.

By means of the identity

$$\operatorname{STr}\left(w[J_0, \lambda]\right) = -\operatorname{STr}\left(J_0\{w, \lambda\}\right) \tag{3.51}$$

it is possible to write

$$S_{ghost} = -\frac{R^2}{2\pi} \int d^2z \, \text{STr}\big(w_3 \bar{\partial} \lambda_1 + w_1 \partial \lambda_3\big) +$$
$$+ \frac{R^2}{2\pi} \int d^2z \, \text{STr}\big(\bar{J}_0 \{w_3, \lambda_1\} + J_0 \{w_1, \lambda_3\}\big) \qquad (3.52)$$

and the BRST transformation is

$$QS_{ghost} = -\frac{R^2}{2\pi} \int d^2z \, \text{STr}\big(\nabla \lambda_3 \bar{J}_1 + \overline{\nabla} \lambda_1 J_3\big) +$$
$$+ \frac{R^2}{2\pi} \int d^2z \, \text{STr}\big(Q(\bar{J}_0)\{w_3, \lambda_1\} + Q(J_0)\{w_1, \lambda_3\}\big) \quad . \quad (3.53)$$

The first term cancels exactly QS_{matter}, but now we have to eliminate the second one. By means of (3.23) we have

$$\text{STr}\big(Q(\bar{J}_0)\{w_3, \lambda_1\} + Q(J_0)\{w_1, \lambda_3\}\big) =$$
$$= \text{STr}\big[([\bar{J}_1, \lambda_3] + [\bar{J}_3, \lambda_1])\{w_3, \lambda_1\} + ([J_1, \lambda_3] + [J_3, \lambda_1])\{w_1, \lambda_3\}\big] \quad (3.54)$$

Then, using the the constraint (3.34) into the identity

$$\text{STr}\left([\bar{J}_3, \lambda_1]\{w_3, \lambda_1\}\right) = -\frac{1}{2} \text{STr}\left([\bar{J}_3, w_3]\{\lambda_1, \lambda_1\}\right)$$

we obtain

$$\text{STr}\left([\bar{J}_3, \lambda_1]\{w_3, \lambda_1\}\right) = 0 \qquad (3.55)$$

and in the same way

$$\text{STr}\left([J_1, \lambda_3]\{w_1, \lambda_3\}\right) = 0 \quad , \qquad (3.56)$$

so (3.54) becomes

$$\text{STr}\big(Q(\bar{J}_0)\{w_3, \lambda_1\} + Q(J_0)\{w_1, \lambda_3\}\big) = \text{STr}\big([\bar{J}_1, \lambda_3]\{w_3, \lambda_1\} + [J_3, \lambda_1]\{w_1, \lambda_3\}\big) \quad . \qquad (3.57)$$

Because of the fermionic character of Q, it is

$$Q\{w_3, \lambda_1\} = [J_3, \lambda_1] \qquad Q\{w_1, \lambda_3\} = [\bar{J}_1, \lambda_3] \qquad (3.58)$$

and finally we can write

$$\text{STr}\big(Q(\bar{J}_0)\{w_3, \lambda_1\} + Q(J_0)\{w_1, \lambda_3\}\big) = \text{STr}\, Q(\{w_3, \lambda_1\}\{w_1, \lambda_3\}) \qquad (3.59)$$

i.e.

$$Q(S_{matter} + S_{ghost}) = \frac{R^2}{2\pi} \int d^2z \operatorname{STr} Q(\{w_3, \lambda_1\}\{w_1, \lambda_3\}) \quad . \tag{3.60}$$

This way, to have BRST invariance, we must add to $S_{matter} + S_{ghost}$ the term

$$S_{current} = -\frac{R^2}{2\pi} \int d^2z \operatorname{STr}(\{w_3, \lambda_1\}\{w_1, \lambda_3\}) \quad . \tag{3.61}$$

We note that in (3.52) $\{w_3, \lambda_1\}$ and $\{w_1, \lambda_3\}$ couple linearly with $\overline{J}_0$ and J_0, hence they can be understood as *gauge currents* and it explains the subscript of S.

3.3   Action

The action of pure spinor in coset superspace is

$$S = S_{matter} + S_{ghost} + S_{current} \tag{3.62}$$

with

$$S_{matter} = \frac{R^2}{2\pi} \int d^2z \operatorname{STr}\left[\frac{1}{2}J_2\overline{J}_2 + \frac{3}{4}J_3\overline{J}_1 + \frac{1}{4}J_1\overline{J}_3\right] \tag{3.63}$$

$$S_{ghost} = -\frac{R^2}{2\pi} \int d^2z \operatorname{STr}\left(w_3\overline{\nabla}\lambda_1 + w_1\nabla\lambda_3\right) \tag{3.64}$$

$$S_{current} = -\frac{R^2}{2\pi} \int d^2z \operatorname{STr}(\{w_3, \lambda_1\}\{w_1, \lambda_3\}) \quad . \tag{3.65}$$

To write the explicit form of S we have to expand the fields J_i and the ghosts w_i, λ_i on the generators of the $osp(4|6)$ superalgebra:

$$J_0 = J^{mn}M_{mn} + J^a{}_b V_b{}^a \tag{3.66}$$

$$J_1 = J^{\alpha a}\mathcal{O}_{\alpha a} + J_{\dot{\alpha}a}\mathcal{O}^{\dot{\alpha}a} \tag{3.67}$$

$$J_2 = J^m P_m + J^a V_a + J_a V^a \tag{3.68}$$

$$J_3 = J^{\alpha}{}_a \mathcal{O}_{\alpha}{}^a + J_{\dot{\alpha}}{}^a \mathcal{O}^{\dot{\alpha}}{}_a \tag{3.69}$$

$$\lambda_1 = \lambda^{\alpha a}\mathcal{O}_{\alpha a} + \lambda_{\dot{\alpha}a}\mathcal{O}^{\dot{\alpha}a} \qquad \lambda_3 = \lambda^{\alpha}{}_a \mathcal{O}_{\alpha}{}^a + \lambda_{\dot{\alpha}}{}^a \mathcal{O}^{\dot{\alpha}}{}_a \tag{3.70}$$

and analogue for w_1, w_2. For convenience, we introduce also the gauge currents

$$N \equiv -\{w_3, \lambda_1\} \qquad , \qquad \hat{N} \equiv -\{w_1, \lambda_3\} \tag{3.71}$$

and using (C.41) we obtain

$$
N = \frac{i}{4}\left(w^{\alpha}{}_a(\sigma^{mn})_{\alpha\beta}\lambda^{\beta a} + w_{\dot\alpha}{}^a(\bar\sigma^{\,mn})^{\dot\alpha\dot\beta}\lambda_{\dot\beta a}\right)M_{mn} +
$$
$$
+ \frac{1}{2}\left(\varepsilon_{\alpha\beta}w^{\alpha}{}_b\lambda^{\beta a} - \varepsilon^{\dot\alpha\dot\beta}w_{\dot\alpha}{}^a\lambda_{\dot\beta b}\right)V_a{}^b \tag{3.72}
$$
$$
\hat N = \frac{i}{4}\left(w^{\alpha a}(\sigma^{mn})_{\alpha\beta}\lambda^{\beta}{}_a + w_{\dot\alpha a}(\bar\sigma^{\,mn})^{\dot\alpha\dot\beta}\lambda_{\dot\beta}{}^a\right)M_{mn} +
$$
$$
- \frac{1}{2}\left(\varepsilon_{\alpha\beta}w^{\alpha a}\lambda^{\beta}{}_b - \varepsilon^{\dot\alpha\dot\beta}w_{\alpha b}\lambda_{\beta}{}^a\right)V_a{}^b \tag{3.73}
$$

i.e.

$$
N^{mn} \equiv \frac{i}{4}\left(w^{\alpha}{}_a(\sigma^{mn})_{\alpha\beta}\lambda^{\beta a} + w_{\dot\alpha}{}^a(\bar\sigma^{\,mn})^{\dot\alpha\dot\beta}\lambda_{\dot\beta a}\right) \tag{3.74}
$$
$$
\hat N^{mn} \equiv \frac{i}{4}\left(w^{\alpha a}(\sigma^{mn})_{\alpha\beta}\lambda^{\beta}{}_a + w_{\dot\alpha a}(\bar\sigma^{\,mn})^{\dot\alpha\dot\beta}\lambda_{\dot\beta}{}^a\right) \tag{3.75}
$$
$$
N^a{}_b = +\frac{1}{2}\left(\varepsilon_{\alpha\beta}w^{\alpha}{}_b\lambda^{\beta a} - \varepsilon^{\dot\alpha\dot\beta}w_{\dot\alpha}{}^a\lambda_{\dot\beta b}\right) \tag{3.76}
$$
$$
\hat N^a{}_b = -\frac{1}{2}\left(\varepsilon_{\alpha\beta}w^{\alpha a}\lambda^{\beta}{}_b - \varepsilon^{\dot\alpha\dot\beta}w_{\dot\alpha b}\lambda_{\dot\beta}{}^a\right) \quad . \tag{3.77}
$$

Furthermore using (C.46) and (C.48)

$$
S_{current} = -\frac{R^2}{2\pi}\int d^2z \, \mathrm{STr}(N\hat N)
$$
$$
= \frac{R^2}{2\pi}\int d^2z \, 2\left(-\eta_{km}\eta_{ln}N^{kl}\hat N^{mn} + N^a{}_b\hat N^b{}_a\right) \tag{3.78}
$$

and with all the other traces (C.46)-(C.51), we have

$$
S = \frac{R^2}{2\pi}\int d^2z \left[\frac{1}{2}\eta_{mn}J^m\overline J^n - \frac{1}{2}J_a\overline J^a - \frac{1}{2}J^a\overline J_a + \right.
$$
$$
- \frac{i}{4}\varepsilon_{\alpha\beta}\left(3J^{\alpha}{}_a\overline J^{\beta a} + J^{\alpha a}\overline J^{\beta}{}_a\right) - \frac{i}{4}\varepsilon^{\dot\alpha\dot\beta}\left(3J_{\dot\alpha}{}^a\overline J_{\dot\beta a} + J_{\dot\alpha a}\overline J_{\dot\beta}{}^a\right) +
$$
$$
- i\varepsilon_{\alpha\beta}\left(w^{\alpha}{}_a\overline\nabla\lambda^{\beta a} + w^{\alpha a}\nabla\lambda^{\beta}{}_a\right) - i\varepsilon^{\dot\alpha\dot\beta}\left(w_{\dot\alpha}{}^a\overline\nabla\lambda_{\dot\beta a} + w_{\dot\alpha a}\nabla\lambda_{\dot\beta}{}^a\right) +
$$
$$
+ \frac{1}{8}\eta_{km}\eta_{ln}\left(w^{\alpha}{}_a(\sigma^{kl})_{\alpha\beta}\lambda^{\beta a} + w_{\dot\alpha}{}^a(\bar\sigma^{\,kl})^{\dot\alpha\dot\beta}\lambda_{\dot\beta a}\right)\times
$$
$$
\times\left(w^{\gamma b}(\sigma^{mn})_{\gamma\delta}\lambda^{\delta}{}_b + w_{\dot\gamma b}(\bar\sigma^{\,mn})^{\dot\gamma\dot\delta}\lambda_{\dot\delta}{}^b\right) +
$$
$$
\left. - \frac{1}{2}\left(\varepsilon_{\alpha\beta}w^{\alpha}{}_b\lambda^{\beta a} - \varepsilon^{\dot\alpha\dot\beta}w_{\dot\alpha}{}^a\lambda_{\dot\beta b}\right)\left(\varepsilon_{\alpha\beta}w^{\alpha b}\lambda^{\beta}{}_a - \varepsilon^{\dot\alpha\dot\beta}w_{\dot\alpha a}\lambda_{\dot\beta}{}^b\right)\right] \tag{3.79}
$$

Finally we observe that the action (3.79) has another local invariance:

$$
\delta w_3 = [\lambda_1, \Omega_2] \qquad \delta w_1 = [\lambda_3, \Omega_2] \qquad \delta\lambda_1 = \delta\lambda_3 = 0 \tag{3.80}
$$

with $\Omega_2 \in \mathcal{H}_2$. In fact we have

$$\delta \mathrm{STr}(w_3 \overline{\nabla} \lambda_1) = \mathrm{STr}(\delta w_3 \overline{\nabla} \lambda_1) = \mathrm{STr}([\lambda_1, \Omega_2] \overline{\nabla} \lambda_1) = \mathrm{STr}(\Omega_2 \{\overline{\nabla} \lambda_1, \lambda_1\}) \quad :$$

using the Jacobi identities and (3.28) we can show that

$$\{\overline{\nabla} \lambda_1, \lambda_1\} = \frac{1}{2} \overline{\nabla} \{\lambda_1, \lambda_1\} \tag{3.81}$$

and

$$\delta \mathrm{STr}(w_3 \overline{\nabla} \lambda_1) = \frac{1}{2} \mathrm{STr}(\Omega_2 \overline{\nabla} \{\lambda_1, \lambda_1\}) = 0 \tag{3.82}$$

because of (3.34). In analogous way we could show that $\delta \mathrm{STr}(w_1 \nabla \lambda_3) = 0$, therefore $\delta S_{ghost} = 0$. Then we have, by means of the Jacobi identities,

$$\delta N = -\{\delta w_3, \lambda_1\} = -\{[\lambda_1, \Omega_2], \lambda_1\} = -\frac{1}{2}[\{\lambda_1, \lambda_1\}, \Omega_2] = 0 \tag{3.83}$$

and identically $\delta \hat{N} = 0$, so that $\delta S_{current} = 0$ too. Now, if we write Ω_2 in $osp(4|6)$ components

$$\Omega_2 = \Omega_m P^m + \Omega_a V^a + \Omega^a V_a \quad , \tag{3.84}$$

we obtain the explicit form of (3.80), e.g. for w_3:

$$\delta w^\alpha{}_a = \frac{i}{2} \Omega_m \lambda_{\dot{\alpha} a} (\bar{\sigma}^m)^{\dot{\alpha}\alpha} + \frac{i}{\sqrt{2}} \epsilon_{abc} \Omega^b \lambda^{\alpha c} \tag{3.85}$$

$$\delta w_{\dot{\alpha}}{}^a = \frac{i}{2} \Omega_m \lambda^{\alpha a} (\sigma^m)_{\alpha\dot{\alpha}} - \frac{i}{\sqrt{2}} \epsilon^{abc} \Omega_b \lambda_{\dot{\alpha} c} \quad . \tag{3.86}$$

3.4 Solution of the constraint

We obtained the ghost constraint in Section 3.2.1: our goal is to find a form for the action in which this constraint is already solved, that is the ghost fields have manifestly the right number of degrees of freedom, in a similar way to flat case.

The ghost constraints (3.34) can be explicitly written using the algebra (C.40)

$$\begin{array}{ll}
\epsilon_{abc} \lambda^{\alpha a} \varepsilon_{\alpha\beta} \lambda^{\beta b} = 0 & \epsilon^{abc} \lambda^\alpha{}_a \varepsilon_{\alpha\beta} \lambda^\beta{}_b = 0 \\[2mm]
\epsilon^{abc} \lambda_{\dot{\alpha} a} \varepsilon^{\dot{\alpha}\dot{\beta}} \lambda_{\dot{\beta} b} = 0 & \epsilon_{abc} \lambda_{\dot{\alpha}}{}^a \varepsilon^{\dot{\alpha}\dot{\beta}} \lambda_{\dot{\beta}}{}^b = 0 \\[2mm]
\lambda^{\alpha a} (\sigma^m)_\alpha{}^{\dot{\beta}} \lambda_{\dot{\beta} a} = 0 & \lambda_{\dot{\alpha}}{}^a (\bar{\sigma}^m)^{\dot{\alpha}}{}_\beta \lambda^\beta{}_a = 0 \quad .
\end{array} \tag{3.87}$$

The constraint on λ_1 can be solved setting [25]

$$\lambda^{\alpha a} = \theta^\alpha u^a \qquad\qquad \lambda_{\dot\alpha a} = \psi_{\dot\alpha} v_a \qquad\qquad (3.88)$$

with the condition

$$u^a v_a = 0 \quad . \qquad\qquad (3.89)$$

Moreover we can scale

$$u^a \to c u^a \qquad \theta^\alpha \to \frac{1}{c} \theta^\alpha \qquad\qquad (3.90)$$

and

$$v_a \to d v_a \qquad \psi_{\dot\alpha} \to \frac{1}{d} \psi_{\dot\alpha} \qquad\qquad (3.91)$$

with $c, d \in \mathbb{C}$, so we can impose the further conditions on u and v:

$$|u|^2 \equiv u^a u_a^* = 1 \qquad |v|^2 \equiv v^{a*} v_a = 1 \qquad\qquad (3.92)$$

In this way the constraint on λ_1 - i.e. the first column of (3.87) - becomes the couple of conditions (3.89) and (3.92).

The constraint on λ_3 admits identical solution, however we remember that there is a one-to-one correspondence (3.19) between the eigenspaces $\mathcal{H}_1$ and $\mathcal{H}_3$, so we can construct the fields λ_3 by the same elements of λ_1. We guess

$$\lambda^\alpha{}_a = \bar\psi^\alpha v_a \qquad\qquad \lambda_{\dot\alpha}{}^a = \bar\theta_{\dot\alpha} u^a \qquad\qquad (3.93)$$

with

$$\bar\psi^\alpha = \psi_{\dot\alpha}^* (\bar\sigma^2)^{\dot\alpha\alpha} \qquad\qquad \bar\theta_{\dot\alpha} = \theta^{\alpha*} (\sigma^2)_{\alpha\dot\alpha} \quad . \qquad\qquad (3.94)$$

As far as the field w is concerned, we can use the gauge invariance (3.80) to simplify its decomposition. Let us consider e.g. the field w_3: substituting (3.88) in (3.85)-(3.86) we have

$$\delta w^\alpha{}_a = \frac{i}{2} \Omega_m \psi_{\dot\alpha} (\bar\sigma^m)^{\dot\alpha\alpha} v_a + \frac{i}{\sqrt{2}} \epsilon_{abc} \Omega^b \theta^\alpha u^c \qquad\qquad (3.95)$$

$$\delta w_{\dot\alpha}{}^a = \frac{i}{2} \Omega_m \theta^\alpha (\sigma^m)_{\alpha\dot\alpha} u^a - \frac{i}{\sqrt{2}} \epsilon^{abc} \Omega_b \psi_{\dot\alpha} v_c \quad . \qquad\qquad (3.96)$$

On general grounds, w_3 can be written as

$$w^\alpha{}_a = \omega^\alpha(u_a^* + A\epsilon_{abc}u^b v^{c*} + Bv^a) \tag{3.97}$$

$$w_{\dot\alpha}{}^a = \rho_{\dot\alpha}(u_a + C\epsilon^{abc}u_b^* v_c + Dv^{a*}) \tag{3.98}$$

where ω^α, $\rho_{\dot\alpha}$ are the antighost fields that will play the role of the conjugate momenta of θ^a, $\psi_{\dot\alpha}$ respectively and A, B, C, D are arbitrary functions. However, using the gauge transformation (3.95) and (3.96) we can cancel exactly the second and the third term in (3.97) and (3.98) and set

$$w^\alpha{}_a = \omega^\alpha u_a^* \qquad w_{\dot\alpha}{}^a = \rho_{\dot\alpha}v^{a*} \quad . \tag{3.99}$$

In analogous way

$$w^{\alpha a} = \bar\rho^\alpha v^{a*} \qquad w_{\dot\alpha a} = \bar\omega_{\dot\alpha}u_a^* \tag{3.100}$$

with

$$\bar\rho^\alpha = \rho_{\dot\alpha}^*(\bar\sigma^2)^{\dot\alpha\alpha} \qquad \bar\omega_{\dot\alpha} = \omega^{\alpha*}(\sigma^2)_{\alpha\dot\alpha} \quad . \tag{3.101}$$

3.4.1 Gauge transformations

In this section we discuss the gauge transformation of the ghost, and the u, v variables. Let us recall the gauge transformation for a field F in $\mathcal{G}\smallsetminus\mathcal{H}$ (see (2.58))

$$F \rightarrow h^{-1}Fh \qquad \text{with} \qquad h \in H \tag{3.102}$$

or, in infinitesimal form,

$$h = e^{\xi^{(ab)}T_{(ab)}} \quad \Rightarrow \quad \delta F = -[\xi^{(ab)}T_{(ab)}, F] \quad . \tag{3.103}$$

In our case $\mathcal{H} = so(3,1)\oplus u(3)$ and we can write the transformation for ghost fields:

$$\delta\lambda = -\frac{1}{2}\xi_{mn}[M^{mn}, \lambda] - \xi^a{}_b[V_a{}^b, \lambda] \equiv \delta_{so}\lambda + \delta_u\lambda \tag{3.104}$$

in components (case λ_1)

$$\delta_{so}\lambda^{\alpha a}\mathcal{O}_{\alpha a} + \delta_{so}\lambda_{\dot\alpha a}\mathcal{O}^{\dot\alpha a} = -\frac{1}{2}\xi_{mn}\lambda^{\alpha a}[M^{mn}, \mathcal{O}_{\alpha a}] - \frac{1}{2}\xi_{mn}\lambda_{\dot\alpha a}[M^{mn}, \mathcal{O}^{\dot\alpha a}]$$

$$\delta_u\lambda^{\alpha a}\mathcal{O}_{\alpha a} + \delta_u\lambda_{\dot\alpha a}\mathcal{O}^{\dot\alpha a} = -\xi^a{}_b\lambda^{\alpha c}[V_a{}^b, \mathcal{O}_{\alpha c}] - \xi^a{}_b\lambda_{\dot\alpha c}[V_a{}^b, \mathcal{O}^{\dot\alpha c}] \quad .$$

For clearness we write the (C.44) algebra in the same way of (C.42), introducing

$$(\sigma_a{}^b)_c{}^d = -i\delta_a{}^d\delta_c{}^b \tag{3.105}$$

so that

$$[V_a{}^b, \mathcal{O}_{\alpha c}] = -(\sigma_a{}^b)_c{}^d\mathcal{O}_{\alpha d} \quad [V_a{}^b, \mathcal{O}^{\dot\alpha}{}_c] = -(\sigma_a{}^b)_c{}^d\mathcal{O}^{\dot\alpha}{}_d$$
$$[V_a{}^b, \mathcal{O}^{\dot\alpha c}] = (\sigma_a{}^b)_d{}^c\mathcal{O}^{\dot\alpha d} \quad [V_a{}^b, \mathcal{O}_\alpha{}^c] = (\sigma_a{}^b)_d{}^c\mathcal{O}_\alpha{}^d \ . \tag{3.106}$$

We obtain the transformations under $so(3,1)$

$$\delta_{so}\lambda^{\alpha a} = \frac{1}{4}\lambda^{\beta a}(\xi_{mn}\sigma^{mn})_\beta{}^\alpha \qquad \delta_{so}\lambda_{\dot\alpha a} = \frac{1}{4}\lambda_{\dot\beta a}(\xi_{mn}\bar\sigma^{mn})^{\dot\beta}{}_{\dot\alpha} \tag{3.107}$$

and under $u(3)$

$$\delta_u\lambda^{\alpha a} = \lambda^{\alpha b}(\xi^c{}_d\sigma_c{}^d)_b{}^a \qquad \delta_u\lambda_{\dot\alpha a} = -(\xi^c{}_d\sigma_c{}^d)_a{}^b\lambda_{\dot\alpha b} \ . \tag{3.108}$$

It is evident that latin indices a (up and down) and greek indices $\alpha,\dot\alpha$ transform independently under $so(3,1)$ and $u(3)$, therefore we can write the gauge transformation for the fields θ,ψ and u,v:

$$\delta_{so}\theta^\alpha = \frac{1}{4}\theta^\beta(\xi_{mn}\sigma^{mn})_\beta{}^\alpha \qquad \delta_{so}\psi_{\dot\alpha} = \frac{1}{4}\psi_{\dot\beta}(\xi_{mn}\bar\sigma^{mn})^{\dot\beta}{}_{\dot\alpha} \tag{3.109}$$

$$\delta_u u^a = u^b(\xi^c{}_d\sigma_c{}^d)_b{}^a \qquad \delta_u v_a = -(\xi^c{}_d\sigma_c{}^d)_a{}^b v_b \ , \tag{3.110}$$

while $\delta_u\theta^\alpha = \delta_u\psi_{\dot\alpha} = 0$ and $\delta_{so}u^a = \delta_{so}v_a = 0$. In the following we will omit the subscripts so and u without confusion. Assuming as usual $\xi \in \mathbb{R}$ and noting that $(\sigma^*)_b{}^a = -\sigma_b{}^a$, it is trivial to see that v_a transform with the hermitian conjugate matrix of u^a: then u^a and v_a lie respectively in the representations $\mathbf{3}$ and $\mathbf{3}^*$ of $U(3)$. Now let us study the behaviour of the complex conjugate fields. First

$$\delta u^{a*} = u^{b*}(\xi \cdot \sigma^*)_b{}^a = -u^{b*}(\xi \cdot \sigma)_b{}^a \quad : \tag{3.111}$$

if we transpose this identity, we obtain

$$\delta u_a^* = -(\xi \cdot \sigma)_a{}^b u_b^* \tag{3.112}$$

exactly as the transformation of v_a, then u_a^* transforms in $\mathbf{3}^*$ representation. Identical argument works for v: $\delta v^{a*} = v^{b*}(\xi \cdot \sigma)_b{}^a$ and v^{a*} lies in $\mathbf{3}$. It follows that the conditions (3.89) and (3.92) are U(3) invariant.

For θ^* we have

$$\delta\theta^{\alpha*} = \frac{1}{4}\theta^{\beta*}\xi_{mn}(\sigma^{mn*})_\beta{}^\alpha \quad : \tag{3.113}$$

trivially $\sigma^{m*} = -\sigma^2\bar{\sigma}^m\bar{\sigma}^2 \quad \Rightarrow$

$$(\sigma^{mn*})_\beta{}^\alpha = -(\sigma^2)_{\beta\dot\beta}(\bar{\sigma}^{mn})^{\dot\beta}{}_{\dot\alpha}(\bar{\sigma}^2)^{\dot\alpha\alpha} \tag{3.114}$$

so

$$\delta\theta^{\alpha*} = -\frac{1}{4}\theta^{\beta*}(\sigma^2)_{\beta\dot\beta}\xi_{mn}(\bar{\sigma}^{mn})^{\dot\beta}{}_{\dot\alpha}(\bar{\sigma}^2)^{\dot\alpha\alpha} \tag{3.115}$$

and right-multiplying by σ^2 ($\bar{\sigma}^2\sigma^2 = -\mathbb{1}$) we obtain

$$\delta\theta^{\alpha*}(\sigma^2)_{\alpha\dot\alpha} = \frac{1}{4}\theta^{\beta*}(\sigma^2)_{\beta\dot\beta}\xi_{mn}(\bar{\sigma}^{mn})^{\dot\beta}{}_{\dot\alpha} \tag{3.116}$$

i.e.

$$\delta\bar{\theta}_{\dot\alpha} = \frac{1}{4}\bar{\theta}_{\dot\beta}(\xi_{mn}\bar{\sigma}^{mn})^{\dot\beta}{}_{\dot\alpha} \quad . \tag{3.117}$$

In this way we proved that $\bar{\theta}_{\dot\alpha}$ in (3.94) transforms in the right way under $so(3,1)$, that is in the same representation of $\psi_{\dot\alpha}$. An analogous computation shows that $\bar{\psi}^\alpha$ transforms like θ^α. This property is a fundamental one, because the independency of latin and greek index imposes λ_3 (3.93) transform under SO(3, 1) just like λ_1 (3.88), as we could prove directly by (3.104).

3.4.2 Covariant derivative

We introduced the covariant derivative in Section 3.1. Now we want to see how it work on θ, ψ and u, v. By definition

$$\nabla = \partial + [J_0, \quad] \tag{3.118}$$

with $J_0 = J_{mn}M^{mn} + J^a{}_b V_a{}^b$, therefore in components (case λ_1)

$$\nabla\lambda^{\alpha a} = \partial\lambda^{\alpha a} - \frac{1}{2}\lambda^{\beta a}(J_{mn}\sigma^{mn})_\beta{}^\alpha - \lambda^{\alpha b}(J^c{}_d\sigma_c{}^d)_b{}^a \tag{3.119}$$

$$\nabla\lambda_{\dot\alpha a} = \partial\lambda_{\dot\alpha a} - \frac{1}{2}\lambda_{\dot\beta a}(J_{mn}\bar{\sigma}^{mn})^{\dot\beta}{}_{\dot\alpha} + (J^c{}_d\sigma_c{}^d)_a{}^b\lambda_{\alpha b} \quad , \tag{3.120}$$

using obviously (C.42) and (C.44). By $\lambda^{\alpha a} = \theta^\alpha u^a$

$$\nabla(\theta^\alpha u^a) = [\partial\theta^\alpha - \frac{1}{2}\theta^\beta(J_{mn}\sigma^{mn})_\beta{}^\alpha]u^a + \theta^\alpha[\partial u^a - u^b(J^c_d\sigma_c{}^d)_b{}^a]$$

$$\equiv \nabla\theta^\alpha u^a + \theta^\alpha \nabla u^a \tag{3.121}$$

and analogue for $\lambda_{\dot\alpha a} = \psi_{\dot\alpha}v_a$. So

$$\nabla\theta^\alpha = \partial\theta^\alpha - \frac{1}{2}\theta^\beta(J_{mn}\sigma^{mn})_\beta{}^\alpha \tag{3.122}$$

$$\nabla\psi_{\dot\alpha} = \partial\psi_{\dot\alpha} - \frac{1}{2}\psi_{\dot\beta}(J_{mn}\bar\sigma^{mn})^{\dot\beta}{}_{\dot\alpha} \tag{3.123}$$

and

$$\nabla u^a = \partial u^a - u^b(J^c_d\sigma_c{}^d)_b{}^a \tag{3.124}$$

$$\nabla v_a = \partial v_a + (J^c_d\sigma_c{}^d)_a{}^b v_b \tag{3.125}$$

3.5   The revised action

The second step towards our proposal is to write the action (3.79) by means of the new fields $(\theta, \psi, \omega, \rho\,; u, v)$. The ghost term gives

$$S_{ghost} = \frac{R^2}{2\pi}\int d^2z\,(-i)\left[\varepsilon_{\alpha\beta}\omega^\alpha\overline\nabla\theta^\beta + \varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\overline\nabla\psi_{\dot\beta} + \varepsilon_{\alpha\beta}\bar\rho^\alpha\nabla\bar\psi^\beta + \varepsilon^{\dot\alpha\dot\beta}\bar\omega_{\dot\alpha}\nabla\bar\theta_{\dot\beta}\,+\right.$$

$$+\,(\varepsilon_{\alpha\beta}\omega^\alpha\theta^\beta)u_a^*\overline\nabla u^a + (\varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\psi_{\dot\beta})v^{a*}\overline\nabla v_a\,+$$

$$\left.+(\varepsilon_{\alpha\beta}\bar\rho^\alpha\bar\psi^\beta)v^{a*}\nabla v_a + (\varepsilon^{\dot\alpha\dot\beta}\bar\omega_{\dot\alpha}\bar\theta_{\dot\beta})u_a^*\nabla u^a\right]\quad. \tag{3.126}$$

In the following it is useful to develop the covariant derivative:

$$\varepsilon_{\alpha\beta}\omega^\alpha\overline\nabla\theta^\beta = \varepsilon_{\alpha\beta}\omega^\alpha(\bar\partial\theta^\beta - \frac{1}{2}\theta^\gamma(\overline J_{mn}\sigma^{mn})_\gamma{}^\beta)$$

$$= \varepsilon_{\alpha\beta}\omega^\alpha\bar\partial\theta^\beta + \frac{1}{2}\omega^\alpha(\overline J_{mn}\sigma^{mn})_{\alpha\beta}\theta^\beta \tag{3.127}$$

remembering that $(\sigma^{mn})_\gamma{}^\beta\varepsilon_{\beta\alpha} = (\sigma^{mn})_{\gamma\alpha} = (\sigma^{mn})_{\alpha\gamma}$ (see Appendix C). In analogous way

$$\varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\overline\nabla\psi_{\dot\beta} = \varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}(\bar\partial\psi_{\dot\beta} - \frac{1}{2}\psi_{\dot\gamma}(\overline J_{mn}\bar\sigma^{mn})^{\dot\gamma}{}_{\dot\beta})$$

$$= \varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\bar\partial\psi_{\dot\beta} + \frac{1}{2}\rho_{\dot\alpha}(\overline J_{mn}\bar\sigma^{mn})^{\dot\alpha\dot\beta}\psi_{\dot\beta} \tag{3.128}$$

and the action becomes

$$S_{ghost} = \frac{R^2}{2\pi} \int d^2z \, (-i) \left[\varepsilon_{\alpha\beta}\omega^\alpha\bar\partial\theta^\beta + \varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\bar\partial\psi_{\dot\beta} + \varepsilon_{\alpha\beta}\bar\rho^\alpha\partial\bar\psi^\beta + \varepsilon^{\dot\alpha\dot\beta}\bar\omega_{\dot\alpha}\partial\bar\theta_{\dot\beta} + \right.$$

$$+ \frac{1}{2}\bar{J}_{mn}\left(\omega^\alpha(\sigma^{mn})_{\alpha\beta}\theta^\beta + \rho_{\dot\alpha}(\bar\sigma^{mn})^{\dot\alpha\dot\beta}\psi_{\dot\beta} \right) +$$

$$+ \frac{1}{2}J_{mn}\left(\bar\rho^\alpha(\sigma^{mn})_{\alpha\beta}\bar\psi^\beta + \bar\omega_{\dot\alpha}(\bar\sigma^{mn})^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta} \right)$$

$$+ (\varepsilon_{\alpha\beta}\omega^\alpha\theta^\beta)u_a^*\overline\nabla u^a + (\varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\psi_{\dot\beta})v^{a*}\overline\nabla v_a +$$

$$\left. + (\varepsilon_{\alpha\beta}\bar\rho^\alpha\bar\psi^\beta)v^{a*}\nabla v_a + (\varepsilon^{\dot\alpha\dot\beta}\bar\omega_{\dot\alpha}\bar\theta_{\dot\beta})u_a^*\nabla u^a \right] \quad . \quad (3.129)$$

Then let us substitute (3.88), (3.93), (3.99) and (3.100) into the currents (3.74)-(3.77):

$$N^{mn} = \frac{i}{4}\left(\omega^\alpha(\sigma^{mn})_{\alpha\beta}\theta^\beta + \rho_{\dot\alpha}(\bar\sigma^{\,mn})^{\dot\alpha\dot\beta}\psi_{\dot\beta} \right) \qquad (3.130)$$

$$\hat{N}^{mn} = \frac{i}{4}\left(\bar\rho^\alpha(\sigma^{mn})_{\alpha\beta}\bar\psi^\beta + \bar\omega_{\dot\alpha}(\bar\sigma^{\,mn})^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta} \right) \qquad (3.131)$$

$$N^a_{\ b} = +\frac{1}{2}\left((\varepsilon_{\alpha\beta}\omega^\alpha\theta^\beta)u_b^* u^a - (\varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\psi_{\dot\beta})v^{a*}v_b \right) \qquad (3.132)$$

$$\hat{N}^a_{\ b} = -\frac{1}{2}\left((\varepsilon_{\alpha\beta}\bar\rho^\alpha\bar\psi^\beta)v^{a*}v_b - (\varepsilon^{\dot\alpha\dot\beta}\bar\omega_{\dot\alpha}\bar\theta_{\dot\beta})u_b^* u^a \right) \quad ; \qquad (3.133)$$

because of

$$(\sigma_{mn})_{\alpha\beta}(\sigma^{mn})_{\gamma\delta} = 4(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}) \qquad (3.134)$$

$$(\bar\sigma_{mn})^{\dot\alpha\dot\beta}(\bar\sigma^{\,mn})^{\dot\gamma\dot\delta} = 4(\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\dot\beta\dot\delta} + \varepsilon^{\dot\alpha\dot\delta}\varepsilon^{\dot\beta\dot\gamma}) \qquad (3.135)$$

$$(\sigma_{mn})_{\alpha\beta}(\bar\sigma^{\,mn})^{\dot\alpha\dot\beta} = 0 \qquad (3.136)$$

we have

$$N_{mn}\hat{N}^{mn} = -\frac{1}{4}\left[(\varepsilon_{\alpha\beta}\omega^\alpha\bar\rho^\beta)(\varepsilon_{\gamma\delta}\theta^\gamma\bar\psi^\delta) + (\varepsilon_{\alpha\beta}\omega^\alpha\bar\psi^\beta)(\varepsilon_{\gamma\delta}\theta^\gamma\bar\rho^\delta) + \right.$$

$$\left. + (\varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\bar\omega_{\dot\beta})(\varepsilon^{\dot\gamma\dot\delta}\psi_{\dot\gamma}\bar\theta_{\dot\delta}) + (\varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\bar\theta_{\dot\beta})(\varepsilon^{\dot\gamma\dot\delta}\psi_{\dot\gamma}\bar\omega^{\dot\delta}) \right] \qquad (3.137)$$

and because of (3.89) (3.92) we have

$$N^a_{\ b}\hat{N}^b_{\ a} = \frac{1}{4}\left[(\varepsilon_{\alpha\beta}\omega^\alpha\theta^\beta)(\varepsilon^{\dot\alpha\dot\beta}\bar\omega_{\dot\alpha}\bar\theta_{\dot\beta}) + (\varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\psi_{\dot\beta})(\varepsilon_{\alpha\beta}\bar\rho^\alpha\bar\psi^\beta) \right] \, , \qquad (3.138)$$

so we can write immediately $S_{current}$:

$$S_{current} = \frac{R^2}{2\pi} \int d^2 z \, \frac{1}{2} \left[(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}) \omega^\alpha \theta^\beta \bar\rho^\gamma \bar\psi^\delta + \right.$$

$$+ (\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\dot\beta\dot\delta} + \varepsilon^{\dot\alpha\dot\delta}\varepsilon^{\dot\beta\dot\gamma}) \rho_{\dot\alpha} \psi_{\dot\beta} \bar\omega_{\dot\gamma} \bar\theta_{\dot\delta} +$$

$$\left. + \varepsilon_{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta} \left(\omega^\alpha \theta^\beta \bar\omega_{\dot\alpha} \bar\theta_{\dot\beta} + \rho_{\dot\alpha} \psi_{\dot\beta} \bar\rho^\alpha \bar\psi^\beta \right) \right] \quad . \quad (3.139)$$

3.6 Kinematics of u and v

Last step of our formulation is to add to the action we just revised a kinetic term which contains the conditions on u and v .

It is possible to give a matricial aspect to the vectorial constraints (3.89) and (3.92). We already considered column and row vectors

$$u^a = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \qquad v^{a*} = \begin{pmatrix} v^{1*} \\ v^{2*} \\ v^{3*} \end{pmatrix} \qquad u_a^* = (u_1^* \; u_2^* \; u_3^*) \qquad v_a = (v_1 \; v_2 \; v_3)$$

and we can arrange these in the matrix

$$U = \begin{pmatrix} u^a & \epsilon^{abc} v_b u_c^* & v^{a*} \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} u_a^* \\ \epsilon_{abc} v^{b*} u^c \\ v_a \end{pmatrix} \quad : \qquad (3.140)$$

using $u^a v_a = 0$ and $|u|^2 = |v|^2 = 1$ it is simple to verify that

$$U^\dagger U = \mathbb{1}_{3\times 3} \qquad , \qquad \det U = 1 \quad , \qquad (3.141)$$

hence $U \in \mathrm{SU}(3)$. Furthermore we noted that it is still possible to choose two different phase factors $e^{i\phi_{u,v}} \in \mathrm{U}(1)_{u,v}$ in front of u and v. It means that the space of the constrained variables (u,v) corresponds to the coset $\dfrac{\mathrm{SU}(3)}{\mathrm{U}(1)_u \times \mathrm{U}(1)_v}$. So we can introduce a covariant canonical form

$$j \equiv U^{-1}\nabla U = U^\dagger \nabla U \qquad (3.142)$$

and build the non-linear sigma model

$$S_\chi = \frac{R^2}{2\pi} \int d^2z \operatorname{Tr}(\bar{j}^\dagger j) \quad . \tag{3.143}$$

Explicitly

$$j = \begin{pmatrix} d_1 & -{j_1}^* & -{j_2}^* \\ j_1 & -d_1 + d_2 & -{j_3}^* \\ j_2 & j_3 & -d_2 \end{pmatrix} \tag{3.144}$$

where

$$d_1 = u_a^* \nabla u^a \qquad d_2 = v^{a*} \nabla v_a \tag{3.145}$$

$$j_1 = \epsilon_{abc} v^{a*} u^b \nabla u^c \qquad j_2 = v_a \nabla u^a \qquad j_3 = \epsilon^{abc} u_a^* v_b \nabla v_c \ . \tag{3.146}$$

As usual, j takes values in the Lie algebra $su(3)$, i.e. it can be express in the Gell-Mann matrices basis: if we want to restrict it into the coset $su(3) \diagdown [u(1) \oplus u(1)]$, we have to take only the off-diagonal generators, that is to omit the diagonal elements $d_{1,2}$ in (3.144).

We obtain

$$S_\chi = \frac{R^2}{2\pi} \int d^2z \, [\sum_{k=1}^{3} \bar{j}_k^{\ *} j_k + c.c.] \quad , \tag{3.147}$$

assuming the complex conjugation and the bar on j (i.e. on the derivative) independent operations.

We have to study the behaviour of S_χ under $\mathrm{SO}(3,1) \times \mathrm{U}(3)$ and BRST transformation. Trivially S_χ is $\mathrm{SO}(3,1)$-invariant. Let us consider the finite form of $\mathrm{U}(3)$ transformation: by (3.110) we have

$$u^a \rightarrow u^b M_b^{\ a} \qquad v_a \rightarrow M_a^{*\,b} v_b \tag{3.148}$$

and analogous ones for v^{a*} and u_a^*, with $M = e^{\xi \cdot \sigma} \in \mathrm{U}(3)$. j_2 is invariant by definition, while for j_1 we have

$$\epsilon_{abc} v^{a*} u^b \nabla u^c \rightarrow \epsilon_{abc} v^{d*} M_d^{\ a} u^e M_e^{\ b} \nabla u^f M_f^{\ c} = (\det M) \epsilon_{def} v^{d*} u^e \nabla u^f \tag{3.149}$$

that is $j_1 \rightarrow (\det M) j_1$, so

$$\bar{j}_1^{\ *} j_1 \rightarrow |\det M|^2 \bar{j}_1^{\ *} j_1 = \bar{j}_1^{\ *} j_1 \tag{3.150}$$

and the same for $\bar{j}_1 j_1{}^*$. In similar way we could show $j_3 \rightarrow (\det M^*) j_3$ and the invariance of the $j_3 j_3$-terms.

We assume the fields (u, v) BRST invariant,

$$Q(u) = 0 \qquad\qquad Q(v) = 0 \quad , \tag{3.151}$$

so the BRST generator acts only on the J_0 field in the covariant derivative of u and v. By (3.23) we have, using (C.41) algebra,

$$Q(J^{mn}) = -\frac{i}{4} \left[\left(J^{\alpha a} \lambda^\beta{}_a + J^\alpha{}_a \lambda^{\beta a} \right) (\sigma^{mn})_{\alpha\beta} + \left(J_{\dot{\alpha} a} \lambda_{\dot{\beta}}{}^a + J_{\dot{\alpha}}{}^a \lambda_{\dot{\beta} a} \right) (\bar{\sigma}^{mn})^{\dot{\alpha}\dot{\beta}} \right]$$

$$Q(J^a{}_b) = +\frac{1}{2} \left[\varepsilon_{\alpha\beta} \left(J^{\alpha a} \lambda^\beta{}_b - J^\alpha{}_b \lambda^{\beta a} \right) - \varepsilon^{\dot{\alpha}\dot{\beta}} \left(J_{\dot{\alpha} b} \lambda_{\dot{\beta}}{}^a - J_{\dot{\alpha}}{}^a \lambda_{\dot{\beta} b} \right) \right]$$

and we can rewrite $Q(J^a{}_b)$ by means of (3.88) and (3.93)

$$Q(J^a{}_b) = \frac{1}{2} \left[\left(\varepsilon_{\alpha\beta} J^{\alpha a} \bar{\psi}^\beta + \varepsilon^{\dot{\alpha}\dot{\beta}} J_{\dot{\alpha}}{}^a \psi_{\dot{\beta}} \right) v_b - \left(\varepsilon_{\alpha\beta} J^\alpha{}_b \theta^\beta + \varepsilon^{\dot{\alpha}\dot{\beta}} J_{\dot{\alpha} b} \bar{\theta}_{\dot{\beta}} \right) u^a \right]$$

$$\equiv \mathcal{F}^a v_b + \mathcal{F}_b u^a \quad . \tag{3.152}$$

It is convenient to use the explicit form of the covariant derivative, putting (3.105) into (3.124) and (3.125):

$$\nabla u^a = \partial u^a + i J^a{}_b u^b \tag{3.153}$$

$$\nabla v_a = \partial v_a - i v_b J^b{}_a \tag{3.154}$$

so

$$Q(j_2) = Q(v_a \nabla u^a) = Q(v_a \partial u^a + i v_a J^a{}_b u^b)$$

$$= i v_a Q(J^a{}_b) u^b = i v_a (\mathcal{F}^a v_b + \mathcal{F}_b u^a) u^b = 0 \quad . \tag{3.155}$$

Then

$$Q(j_1) = Q(\epsilon_{abc} v^{a*} u^b \nabla u^c) = Q(\epsilon_{abc} v^{a*} u^b \partial u^c + i \epsilon_{abc} v^{a*} u^b J^c{}_d u^d)$$

$$= i \epsilon_{abc} v^{a*} u^b Q(J^c{}_d) u^d = i \epsilon_{abc} v^{a*} u^b (\mathcal{F}^c v_d + \mathcal{F}_d u^c) u^d = 0 \tag{3.156}$$

and

$$Q(j_3) = Q(\epsilon^{abc} u_a^* v_b \nabla v_c) = Q(\epsilon^{abc} u_a^* v_b \partial v_c - i \epsilon^{abc} u_a^* v_b v_d J^d{}_c)$$

$$= -i \epsilon^{abc} u_a^* v_b v_d Q(J^d{}_c) = -i \epsilon^{abc} u_a^* v_b v_d (\mathcal{F}^d v_c + \mathcal{F}_c u^d) = 0 \quad , \tag{3.157}$$

because of $u^a v_a = 0$ and antisymmetry of ϵ indices. Thus the BRST invariance of S_χ is demonstrated.

3.7 Final form of the action

We can now write the complete action of pure spinor superstring in $\text{AdS}_4 \times \mathbb{CP}^3$ adding S_{matter} (first two line of (3.79)), S_{ghost} (3.129), $S_{current}$ (3.139) and S_χ (3.147):

$$
\begin{aligned}
S = \frac{R^2}{2\pi} \int d^2 z \bigg[& \frac{1}{2} \eta_{mn} J^m \overline{J}^n - \frac{1}{2} J_a \overline{J}^a - \frac{1}{2} J^a \overline{J}_a + \\
& - \frac{i}{4} \varepsilon_{\alpha\beta} \left(3 J^\alpha{}_a \overline{J}^{\beta a} + J^{\alpha a} \overline{J}^\beta{}_a \right) - \frac{i}{4} \varepsilon^{\dot\alpha\dot\beta} \left(3 J_{\dot\alpha}{}^a \overline{J}_{\dot\beta a} + J_{\dot\alpha a} \overline{J}_{\dot\beta}{}^a \right) + \\
& - i \bigg(\varepsilon_{\alpha\beta} \omega^\alpha \overline{\partial} \theta^\beta + \varepsilon^{\dot\alpha\dot\beta} \rho_{\dot\alpha} \overline{\partial} \psi_{\dot\beta} + \varepsilon_{\alpha\beta} \overline{\rho}^\alpha \partial \overline{\psi}^\beta + \varepsilon^{\dot\alpha\dot\beta} \overline{\omega}_{\dot\alpha} \partial \overline{\theta}_{\dot\beta} + \\
& \qquad + \frac{1}{2} \omega^\alpha (\overline{J}_{mn} \sigma^{mn})_{\alpha\beta} \theta^\beta + \frac{1}{2} \rho_{\dot\alpha} (\overline{J}_{mn} \overline{\sigma}^{mn})^{\dot\alpha\dot\beta} \psi_{\dot\beta} + \\
& \qquad + \frac{1}{2} \overline{\rho}^\alpha (J_{mn} \sigma^{mn})_{\alpha\beta} \overline{\psi}^\beta + \frac{1}{2} \overline{\omega}_{\dot\alpha} (J_{mn} \overline{\sigma}^{mn})^{\dot\alpha\dot\beta} \overline{\theta}_{\dot\beta} \bigg) + \\
& + \frac{1}{2} \bigg((\varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta} \varepsilon_{\beta\gamma}) \omega^\alpha \theta^\beta \overline{\rho}^\gamma \overline{\psi}^\delta + \\
& \qquad + (\varepsilon^{\dot\alpha\dot\gamma} \varepsilon^{\dot\beta\dot\delta} + \varepsilon^{\dot\alpha\dot\delta} \varepsilon^{\dot\beta\dot\gamma}) \rho_{\dot\alpha} \psi_{\dot\beta} \overline{\omega}_{\dot\gamma} \overline{\theta}_{\dot\delta} + \\
& \qquad + \varepsilon_{\alpha\beta} \varepsilon^{\dot\alpha\dot\beta} \left(\omega^\alpha \theta^\beta \overline{\omega}_{\dot\alpha} \overline{\theta}_{\dot\beta} + \rho_{\dot\alpha} \psi_{\dot\beta} \overline{\rho}^\alpha \overline{\psi}^\beta \right) \bigg) + \\
& + (\varepsilon_{\alpha\beta} \omega^\alpha \theta^\beta) u_a^* \overline{\nabla} u^a + (\varepsilon^{\dot\alpha\dot\beta} \rho_{\dot\alpha} \psi_{\dot\beta}) v^{a*} \overline{\nabla} v_a + \\
& + (\varepsilon_{\alpha\beta} \overline{\rho}^\alpha \overline{\psi}^\beta) v^{a*} \nabla v_a + (\varepsilon^{\dot\alpha\dot\beta} \overline{\omega}_{\dot\alpha} \overline{\theta}_{\dot\beta}) u_a^* \nabla u^a \bigg] + S_\chi \quad .
\end{aligned}
\tag{3.158}
$$

This formulation for pure spinor superstring presents explicitly solved ghost constraint and in that way it becomes easy to handle: in particular we will be able to compute the central charge and the beta function up to one loop. It will be the matter of next chapter.

Chapter 4

Properties of the action in $\mathrm{AdS}_4 \times \mathbb{CP}^3$

4.1 The background field method

We want to quantize the model using the background field method [26], i.e. we have to expand the fields around a classical configuration, just named the *background field*. First we consider the matter part of the action: because we are in a coset manifold, it is natural to expand around an element of the group, so

$$g = \widetilde{g}e^{X/R} \tag{4.1}$$

where $\widetilde{g}$ is in OSP(4|6), X are the quantistic fluctuations and R is some scale which can be identified with the radius of AdS_4. As we know, the gauge transformation is $g \rightarrow gh$ with $h \in \mathrm{SO}(3,1) \times \mathrm{U}(3)$, that is $g \rightarrow ge^{\mathfrak{h}}$ with $\mathfrak{h} \in so(3,1) \oplus u(3)$, then it is always possible to take away the $so(3,1) \oplus u(3)$ component from X and to choice $X \in osp(4|6) \smallsetminus [so(3,1) \oplus u(3)]$. In $\mathbb{Z}_4$ terms

$$X = \sum_{i=1}^{3} X_i \qquad \text{with} \qquad X_i \in \mathcal{H}_i \quad . \tag{4.2}$$

The Maurer-Cartan forms become

$$
\begin{aligned}
J = g^{-1}dg &= e^{-X/R}\widetilde{g}^{-1}d(\widetilde{g}e^{X/R}) \\
&= e^{-X/R}(\widetilde{g}^{-1}d\widetilde{g})e^{X/R} + e^{-X/R}de^{X/R} \\
&\equiv e^{-X/R}\widetilde{J}e^{X/R} + e^{-X/R}de^{X/R} \\
&= \widetilde{J} + \frac{1}{R}(dX + [\widetilde{J}, X]) + \frac{1}{2R^2}[dX + [\widetilde{J}, X], X] + O\left(\frac{1}{R^3}\right) \quad ; \quad (4.3)
\end{aligned}
$$

in components, up to the second order in the fluctuations,

$$
\begin{aligned}
J_0 = {}& \widetilde{J}_0 + [\widetilde{J}_2, X_2] + [\widetilde{J}_1, X_3] + [\widetilde{J}_3, X_1] + \frac{1}{2}\left([\nabla X_2, X_2] + [\nabla X_1, X_3] + [\nabla X_3, X_1]\right) + \\
&+ \frac{1}{2}\Big([[\widetilde{J}_1, X_1], X_2] + [[\widetilde{J}_1, X_2], X_1] + [[\widetilde{J}_2, X_1], X_1] + [[\widetilde{J}_2, X_3], X_3] + \\
&+ [[\widetilde{J}_3, X_2], X_3] + [[\widetilde{J}_3, X_3], X_2]\Big) \tag{4.4} \\
J_1 = {}& \widetilde{J}_1 + \nabla X_1 + [\widetilde{J}_2, X_3] + [\widetilde{J}_3, X_2] + \frac{1}{2}\left([\nabla X_2, X_3] + [\nabla X_3, X_2]\right) + \\
&+ \frac{1}{2}\Big([[\widetilde{J}_1, X_2], X_2] + [[\widetilde{J}_1, X_1], X_3] + [[\widetilde{J}_1, X_3], X_1] + \\
&+ [[\widetilde{J}_2, X_1], X_2] + [[\widetilde{J}_2, X_2], X_1] + [[\widetilde{J}_3, X_1], X_1] + [[\widetilde{J}_3, X_3], X_3]\Big) \tag{4.5} \\
J_2 = {}& \widetilde{J}_2 + \nabla X_2 + [\widetilde{J}_1, X_1] + [\widetilde{J}_3, X_3] + \frac{1}{2}\left([\nabla X_1, X_1] + [\nabla X_3, X_3]\right) + \\
&+ \frac{1}{2}\Big([[\widetilde{J}_2, X_2], X_2] + [[\widetilde{J}_2, X_1], X_3] + [[\widetilde{J}_2, X_3], X_1] + \\
&+ [[\widetilde{J}_1, X_2], X_3] + [[\widetilde{J}_1, X_3], X_2] + [[\widetilde{J}_3, X_1], X_2] + [[\widetilde{J}_3, X_2], X_1]\Big) \tag{4.6} \\
J_3 = {}& \widetilde{J}_3 + \nabla X_3 + [\widetilde{J}_1, X_2] + [\widetilde{J}_2, X_1] + \frac{1}{2}\left([\nabla X_1, X_2] + [\nabla X_2, X_1]\right) + \\
&+ \frac{1}{2}\Big([[\widetilde{J}_3, X_2], X_2] + [[\widetilde{J}_3, X_1], X_3] + [[\widetilde{J}_3, X_3], X_1] + \\
&+ [[\widetilde{J}_2, X_2], X_3] + [[\widetilde{J}_2, X_3], X_2] + [[\widetilde{J}_1, X_1], X_1] + [[\widetilde{J}_1, X_3], X_3]\Big) \tag{4.7}
\end{aligned}
$$

where for simplicity $R = 1$ and $\nabla = \partial + [\widetilde{J}_0, \ ]$. In this way we can write S_{matter} in terms of background and fluctuations: at order 0 in X we have the classical action with $\widetilde{J}$ instead of J; the first order in X does not contribute to the effective action and can be put zero on shell. The second order in X gives, omitting for simplicity

the *tilde* over J and reintroducing the R constant,

$$S_{matter} \to \frac{1}{2\pi} \int d^2z \, \mathrm{STr}\left[\frac{1}{2}\nabla X_2 \overline{\nabla} X_2 + \frac{3}{4}\nabla X_3 \overline{\nabla} X_1 + \frac{1}{4}\nabla X_1 \overline{\nabla} X_3 + L_J + L_{JJ}^{(1)} + L_{JJ}^{(2)}\right]$$

(4.8)

with

$$\begin{aligned}
L_J = \ & \frac{3}{8}J_1[X_1, \overline{\nabla} X_2] + \frac{5}{8}J_1[X_2, \overline{\nabla} X_1] + \frac{1}{8}\overline{J}_1[X_1, \nabla X_2] - \frac{1}{8}\overline{J}_1[X_2, \nabla X_1] + \\
& + \frac{1}{8}J_3[X_3, \overline{\nabla} X_2] - \frac{1}{8}J_3[X_2, \overline{\nabla} X_3] + \frac{3}{8}\overline{J}_3[X_3, \nabla X_2] + \frac{5}{8}\overline{J}_3[X_2, \nabla X_3] + \\
& + \frac{1}{2}J_2[X_1, \overline{\nabla} X_1] + \frac{1}{2}\overline{J}_2[X_3, \nabla X_3]
\end{aligned}$$

(4.9)

$$\begin{aligned}
L_{JJ}^{(1)} = \ & -\frac{1}{2}[J_2, X_2][\overline{J}_2, X_2] + \frac{1}{4}[J_2, X_1][\overline{J}_2, X_3] - \frac{1}{4}[J_2, X_3][\overline{J}_2, X_1] + \\
& + \frac{1}{2}[J_1, X_2][\overline{J}_3, X_2] + \frac{1}{4}[J_1, X_1][\overline{J}_3, X_3] - \frac{1}{4}[J_1, X_3][\overline{J}_3, X_1] + \\
& - \frac{1}{2}[J_3, X_2][\overline{J}_1, X_2] - \frac{3}{4}[J_3, X_1][\overline{J}_1, X_3] - \frac{1}{4}[J_3, X_3][\overline{J}_1, X_1]
\end{aligned}$$

(4.10)

$$\begin{aligned}
L_{JJ}^{(2)} = \ & \frac{3}{8}[J_1, X_2][\overline{J}_2, X_3] - \frac{3}{8}[J_1, X_3][\overline{J}_2, X_2] + \\
& - \frac{3}{8}[\overline{J}_1, X_2][J_2, X_3] - \frac{5}{8}[\overline{J}_1, X_3][J_2, X_2] + \\
& - \frac{5}{8}[J_3, X_1][\overline{J}_2, X_2] - \frac{3}{8}[J_3, X_2][\overline{J}_2, X_1] + \\
& - \frac{3}{8}[\overline{J}_3, X_1][J_2, X_2] + \frac{3}{8}[\overline{J}_3, X_2][J_2, X_1] + \\
& - \frac{1}{2}[J_1, X_3][\overline{J}_1, X_3] - \frac{1}{2}[J_3, X_1][\overline{J}_3, X_1] \quad ,
\end{aligned}$$

(4.11)

where we used

$$\mathrm{STr}(A[B, C]) = -\mathrm{STr}(B[A, C]) \quad .$$

(4.12)

Let us consider the first line of the expression above: we can write

$$\begin{aligned}
& \mathrm{STr}\left[\frac{1}{2}\nabla X_2 \overline{\nabla} X_2 + \frac{3}{4}\nabla X_3 \overline{\nabla} X_1 + \frac{1}{4}\nabla X_1 \overline{\nabla} X_3\right] = \\
& = \mathrm{STr}\Big[\frac{1}{2}\left(\partial X_2 \bar{\partial} X_2 - \overline{J}_0[\partial X_2, X_2] - J_0[\bar{\partial} X_2, X_2] + [J_0, X_2][\overline{J}_0, X_2]\right) + \\
& \qquad + \frac{3}{4}\left(\partial X_3 \bar{\partial} X_1 - \overline{J}_0[\partial X_3, X_1] - J_0[\bar{\partial} X_1, X_3] + [J_0, X_3][\overline{J}_0, X_1]\right) \\
& \qquad + \frac{1}{4}\left(\partial X_1 \bar{\partial} X_3 - \overline{J}_0[\partial X_1, X_3] - J_0[\bar{\partial} X_3, X_1] + [J_0, X_1][\overline{J}_0, X_3]\right)\Big]
\end{aligned}$$

.(4.13)

If we write X in $osp(4|6)$ components

$$X_1 = X^{\alpha a}\mathcal{O}_{\alpha a} + X_{\dot\alpha a}\mathcal{O}^{\dot\alpha a} \tag{4.14}$$

$$X_2 = X^m P_m + X^a V_a + X_a V^a \tag{4.15}$$

$$X_3 = X^\alpha{}_a \mathcal{O}_\alpha{}^a + X_{\dot\alpha}{}^a \mathcal{O}^{\dot\alpha}{}_a \quad , \tag{4.16}$$

integrating for parts we obtain the kinetic term for X:

$$
\begin{aligned}
S_{XX} &= \frac{1}{2\pi}\int d^2z\, \mathrm{STr}\left[\frac{1}{2}\partial X_2\bar\partial X_2 + \frac{3}{4}\partial X_3\bar\partial X_1 + \frac{1}{4}\partial X_1\bar\partial X_3\right] \tag{4.17}\\
&= \frac{1}{2\pi}\int d^2z\, \mathrm{STr}\left[\frac{1}{2}\bar\partial X_2\partial X_2 + \bar\partial X_1\partial X_3\right]\\
&= \frac{1}{2\pi}\int d^2z\left[\frac{1}{2}\bar\partial X^m\partial X_m - \bar\partial X^a\partial X_a - i\varepsilon_{\alpha\beta}\bar\partial X^{\alpha a}\partial X^\beta{}_a - i\varepsilon^{\dot\alpha\dot\beta}\bar\partial X_{\dot\alpha a}\partial X_{\dot\beta}{}^a\right] \quad .
\end{aligned}
$$

Now let us consider the coupling of X with J_0: remembering (3.66)

$$J_0 = J^{mn}M_{mn} + J^a{}_b V_b{}^a \tag{4.18}$$

we have

$$\mathrm{STr}(\overline{J}_0[\partial X_2, X_2]) = -2\overline{J}_{mn}(\partial X^m X^n) + \tag{4.19}$$
$$- i\overline{J}^a{}_b\left[\left(\partial X^b X_a - \partial X_a X^b\right) - \delta_a{}^b\left(\partial X^c X_c - \partial X_c X^c\right)\right]$$

$$\mathrm{STr}(\overline{J}_0[\partial X_1, X_3]) = \frac{i}{2}\overline{J}_{mn}\left(\partial X^{\alpha a}(\sigma^{mn})_{\alpha\beta}X^\beta{}_a + \partial X_{\dot\alpha a}(\bar\sigma^{mn})^{\dot\alpha\dot\beta}X_{\dot\beta}{}^a\right) + \tag{4.20}$$
$$+ \overline{J}^a{}_b\left(\varepsilon_{\alpha\beta}\partial X^{\alpha b}X^\beta{}_a - \varepsilon^{\dot\alpha\dot\beta}\partial X_{\dot\alpha a}X_{\dot\beta}{}^b\right)$$

$$\mathrm{STr}(\overline{J}_0[\partial X_3, X_1]) = \frac{i}{2}\overline{J}_{mn}\left(\partial X^\alpha{}_a(\sigma^{mn})_{\alpha\beta}X^{\beta a} + \partial X_{\dot\alpha}{}^a(\bar\sigma^{mn})^{\dot\alpha\dot\beta}X_{\dot\beta a}\right) + \tag{4.21}$$
$$- \overline{J}^a{}_b\left(\varepsilon_{\alpha\beta}\partial X^\alpha{}_a X^{\beta b} - \varepsilon^{\dot\alpha\dot\beta}\partial X_{\dot\alpha}{}^b X_{\dot\beta a}\right)$$

$$\mathrm{STr}\left([\overline{J}_0, X_2][J_0, X_2]\right) = 4\overline{J}_{mn}J^{mk}X^n X_k + \tag{4.22}$$
$$- \left(\overline{J}^a{}_c J^c{}_b + \overline{J}^c{}_b J^a{}_c - 2\overline{J}^c{}_c J^a{}_b - 2\overline{J}^a{}_b J^c{}_c + 2\overline{J}^c{}_c J^d{}_d\delta_b{}^a\right)X_a X^b$$

$$\mathrm{STr}\left([\overline{J}_0, X_1][J_0, X_3]\right) = \frac{i}{4}J_{mn}\overline{J}_{kl}\left((\sigma^{mn})_\gamma{}^\alpha(\sigma^{kl})_\alpha{}^\beta\varepsilon_{\beta\delta}X^{\gamma a}X^\delta{}_a + \tag{4.23}\right.$$
$$+ (\bar\sigma^{mn})^{\dot\gamma}{}_{\dot\alpha}(\bar\sigma^{kl})^{\dot\alpha}{}_{\dot\beta}\varepsilon^{\dot\beta\dot\delta}X_{\dot\gamma a}X_{\dot\delta}{}^a\Big) +$$
$$- i\left(\overline{J}^b{}_c J^c{}_a\varepsilon_{\alpha\beta}X^{\alpha a}X^\beta{}_b + \overline{J}^c{}_a J^b{}_c\varepsilon^{\dot\alpha\dot\beta}X_{\dot\alpha b}X_{\dot\beta}{}^a\right) +$$
$$+ \frac{1}{2}\left(J_{mn}\overline{J}^b{}_a + \overline{J}_{mn}J^b{}_a\right)\left(X^{\alpha a}(\sigma^{mn})_{\alpha\beta}X^\beta{}_b - X_{\dot\alpha b}(\bar\sigma^{mn})^{\dot\alpha\dot\beta}X_{\dot\beta}{}^a\right)$$

In this way we can write

$$
\begin{aligned}
S_{J_0 XX} = \frac{1}{2\pi} \int d^2 z \Bigg\{ & \overline{J}_{mn} \bigg[\frac{1}{2} \left(\partial X^m X^n - \partial X^n X^m \right) + \\
& - \frac{3}{8} i \left(\partial X^\alpha{}_a (\sigma^{mn})_{\alpha\beta} X^{\beta a} + \partial X_{\dot\alpha}{}^a (\bar\sigma^{\ mn})^{\dot\alpha\dot\beta} X_{\dot\beta a} \right) + \\
& - \frac{1}{8} i \left(\partial X^{\alpha a} (\sigma^{mn})_{\alpha\beta} X^\beta{}_a + \partial X_{\dot\alpha a} (\bar\sigma^{\ mn})^{\dot\alpha\dot\beta} X_{\dot\beta}{}^a \right) \bigg] + \\
& + J_{mn} \bigg[\frac{1}{2} \left(\bar\partial X^m X^n - \bar\partial X^n X^m \right) + \\
& - \frac{3}{8} i \left(\bar\partial X^\alpha{}_a (\sigma^{mn})_{\alpha\beta} X^{\beta a} + \bar\partial X_{\dot\alpha}{}^a (\bar\sigma^{\ mn})^{\dot\alpha\dot\beta} X_{\dot\beta a} \right) + \\
& - \frac{1}{8} i \left(\bar\partial X^{\alpha a} (\sigma^{mn})_{\alpha\beta} X^\beta{}_a + \bar\partial X_{\dot\alpha a} (\bar\sigma^{\ mn})^{\dot\alpha\dot\beta} X_{\dot\beta}{}^a \right) \bigg] + \\
& + \overline{J}^a{}_b \bigg[\frac{i}{2} \left[\left(\partial X^b X_a - \partial X_a X^b \right) - \delta_a{}^b \left(\partial X^c X_c - \partial X_c X^c \right) \right] + \\
& + \frac{3}{4} \left(\varepsilon_{\alpha\beta} \partial X^\alpha{}_a X^{\beta b} - \varepsilon^{\dot\alpha\dot\beta} \partial X_{\dot\alpha}{}^b X_{\dot\beta a} \right) + \\
& - \frac{1}{4} \left(\varepsilon_{\alpha\beta} \partial X^{\alpha b} X^\beta{}_a - \varepsilon^{\dot\alpha\dot\beta} \partial X_{\dot\alpha a} X_{\dot\beta}{}^b \right) \bigg] + \\
& + J^a{}_b \bigg[\frac{i}{2} \left[\left(\bar\partial X^b X_a - \bar\partial X_a X^b \right) - \delta_a{}^b \left(\bar\partial X^c X_c - \bar\partial X_c X^c \right) \right] + \\
& + \frac{3}{4} \left(\varepsilon_{\alpha\beta} \bar\partial X^\alpha{}_a X^{\beta b} - \varepsilon^{\dot\alpha\dot\beta} \bar\partial X_{\dot\alpha}{}^b X_{\dot\beta a} \right) + \\
& - \frac{1}{4} \left(\varepsilon_{\alpha\beta} \bar\partial X^{\alpha b} X^\beta{}_a - \varepsilon^{\dot\alpha\dot\beta} \bar\partial X_{\dot\alpha a} X_{\dot\beta}{}^b \right) \bigg] \Bigg\}
\end{aligned}
\tag{4.24}
$$

and

$$
\begin{aligned}
S_{J_0 J_0 XX} = \frac{1}{2\pi} \int d^2 z \Bigg\{ & 2\overline{J}_{mn} J_{kl} \eta^{mk} X^n X^l + \frac{i}{16} \left(\overline{J}_{mn} J_{kl} + 3 J_{mn} \overline{J}_{kl} \right) \times \\
& \times \Big[(\sigma^{mn})_\gamma{}^\alpha (\sigma^{kl})_\alpha{}^\beta \varepsilon_{\beta\delta} X^{\gamma a} X^\delta{}_a + (\bar\sigma^{\ mn})^{\dot\gamma}{}_{\dot\alpha} (\bar\sigma^{\ kl})^{\dot\alpha}{}_{\dot\beta} \varepsilon^{\dot\beta\dot\delta} X_{\dot\gamma a} X_{\dot\delta}{}^a \Big] + \\
& - \left(\frac{1}{2} \overline{J}^b{}_c J^c{}_a + \frac{1}{2} J^b{}_c \overline{J}^c{}_a - \overline{J}^c{}_c J^b{}_a - J^c{}_c \overline{J}^b{}_a + \overline{J}^c{}_c J^d{}_d \delta_a{}^b \right) X_b X^a + \\
& - \frac{i}{4} \overline{J}^b{}_c J^c{}_a \left[3\varepsilon_{\alpha\beta} X^{\alpha a} X^\beta{}_b + \varepsilon^{\dot\alpha\dot\beta} X_{\dot\alpha b} X_{\dot\beta}{}^a \right] + \\
& - \frac{i}{4} J^b{}_c \overline{J}^c{}_a \left[\varepsilon_{\alpha\beta} X^{\alpha a} X^\beta{}_b + 3\varepsilon^{\dot\alpha\dot\beta} X_{\dot\alpha b} X_{\dot\beta}{}^a \right] + \\
& + \frac{1}{2} \left(\overline{J}_{mn} J^b{}_a + J_{mn} \overline{J}^b{}_a \right) \left[X^{\alpha a} (\sigma^{mn})_{\alpha\beta} X^\beta{}_b - X_{\dot\alpha b} (\bar\sigma^{\ mn})^{\dot\alpha\dot\beta} X_{\dot\beta}{}^a \right] \Bigg\}
\end{aligned}
\tag{4.25}
$$

4.2   The ghost term

The ghost kinetic term for (ω, θ) and (ρ, ψ) does not require background expansion:

$$S_{\omega\theta} = \frac{R^2}{2\pi} \int d^2 z \,(-i) \left[\varepsilon_{\alpha\beta} \omega^{\alpha} \bar{\partial} \theta^{\beta} + \varepsilon^{\dot{\alpha}\dot{\beta}} \rho_{\dot{\alpha}} \bar{\partial} \psi_{\dot{\beta}} + \varepsilon_{\alpha\beta} \bar{\rho}^{\alpha} \partial \bar{\psi}^{\beta} + \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\omega}_{\dot{\alpha}} \partial \bar{\theta}_{\dot{\beta}} \right] \quad ; \quad (4.26)$$

on the other hand, we apply the background method to the coupling term of (ω, θ) and (ρ, ψ) with J_0:

$$S_{J_0\omega\theta} = \frac{R^2}{2\pi} \int d^2 z \,(-i) \left[\frac{1}{2} \overline{J}_{mn} \omega^{\alpha} (\sigma^{mn})_{\alpha\beta} \theta^{\beta} + \frac{1}{2} \overline{J}_{mn} \rho_{\dot{\alpha}} (\bar{\sigma}^{\,mn})^{\dot{\alpha}\dot{\beta}} \psi_{\dot{\beta}} + \right.$$
$$\left. + \frac{1}{2} J_{mn} \bar{\rho}^{\alpha} (\sigma^{mn})_{\alpha\beta} \bar{\psi}^{\beta} + \frac{1}{2} J_{mn} \bar{\omega}_{\dot{\alpha}} (\bar{\sigma}^{\,mn})^{\dot{\alpha}\dot{\beta}} \bar{\theta}_{\dot{\beta}} \right] . (4.27)$$

For our purposes we consider the expansion of J_0 in two X fields plus the background $\widetilde{J}_0$:

$$J_0 \to \widetilde{J}_0 + \frac{1}{2R^2} \left([\partial X_2, X_2] + [\partial X_1, X_3] + [\partial X_3, X_1] \right) \quad . \tag{4.28}$$

The interaction between the ghost fields and $\widetilde{J}_0$ is identical to (4.27) with $\widetilde{J}_0$ instead of J_0. Then the $\mathrm{SO}(3,1)$ components of the $[\partial X, X]$ part is

$$2R^2 J^{mn} \to -\partial X^m X^n + \frac{i}{4} \left(\partial X^{\alpha a} (\sigma^{mn})_{\alpha\beta} X^{\beta}{}_a + \partial X_{\dot{\alpha} a} (\bar{\sigma}^{\,mn})^{\dot{\alpha}\dot{\beta}} X_{\dot{\beta}}{}^a \right) +$$
$$+ \frac{i}{4} \left(\partial X^{\alpha}{}_a (\sigma^{mn})_{\alpha\beta} X^{\beta a} + \partial X_{\dot{\alpha}}{}^a (\bar{\sigma}^{\,mn})^{\dot{\alpha}\dot{\beta}} X_{\dot{\beta} a} \right) , (4.29)$$

so, using (3.134)-(3.136), we can write

$$2R^2 J_{mn} (\sigma^{mn})_{\alpha\beta} \to -\partial X_m X_n (\sigma^{mn})_{\alpha\beta} +$$
$$+ \frac{i}{4} \left(\partial X^{\gamma a} X^{\delta}{}_a + \partial X^{\gamma}{}_a X^{\delta a} \right) (\sigma_{mn})_{\gamma\delta} (\sigma^{mn})_{\alpha\beta}$$
$$\to -\partial X_m X_n (\sigma^{mn})_{\alpha\beta} +$$
$$+ i (\varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta} \varepsilon_{\beta\gamma}) \left(\partial X^{\gamma a} X^{\delta}{}_a + \partial X^{\gamma}{}_a X^{\delta a} \right) \tag{4.30}$$

and in analogous way

$$2R^2 J_{mn} (\bar{\sigma}^{\,mn})^{\dot{\alpha}\dot{\beta}} \to -\partial X_m X_n (\bar{\sigma}^{\,mn})^{\dot{\alpha}\dot{\beta}} +$$
$$+ i (\varepsilon^{\dot{\alpha}\dot{\gamma}} \varepsilon^{\dot{\beta}\dot{\delta}} + \varepsilon^{\dot{\alpha}\dot{\delta}} \varepsilon_{\dot{\beta}\dot{\gamma}}) \left(\partial X_{\dot{\gamma} a} X_{\dot{\delta}}{}^a + \partial X_{\dot{\gamma}}{}^a X_{\dot{\delta} a} \right) \quad . (4.31)$$

So we can write

$$
\begin{aligned}
S_{XX\omega\theta} = \frac{1}{2\pi} \int d^2z \Bigg[&\frac{i}{4} \bar{\partial} X_m X_n \left(\omega^\alpha (\sigma^{mn})_{\alpha\beta} \theta^\beta + \rho_{\dot\alpha} (\bar\sigma^{\;mn})^{\dot\alpha\dot\beta} \psi_{\dot\beta} \right) + \\
&+ \frac{i}{4} \partial X_m X_n \left(\bar\rho^\alpha (\sigma^{mn})_{\alpha\beta} \bar\psi^\beta + \bar\omega_{\dot\alpha} (\bar\sigma^{\;mn})^{\dot\alpha\dot\beta} \bar\theta_{\dot\beta} \right) + \\
&+ \frac{1}{4} (\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}) \left(\bar\partial X^{\gamma a} X^\delta_{\;a} + \bar\partial X^\gamma_{\;a} X^{\delta a} \right) \omega^\alpha \theta^\beta + \\
&+ \frac{1}{4} (\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\dot\beta\dot\delta} + \varepsilon^{\dot\alpha\dot\delta}\varepsilon_{\dot\beta\dot\gamma}) \left(\bar\partial X_{\dot\gamma a} X_{\dot\delta}^{\;a} + \bar\partial X_{\dot\gamma}^{\;a} X_{\dot\delta a} \right) \rho_{\dot\alpha} \psi_{\dot\beta} + \\
&+ \frac{1}{4} (\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}) \left(\partial X^{\gamma a} X^\delta_{\;a} + \partial X^\gamma_{\;a} X^{\delta a} \right) \bar\rho^\alpha \bar\psi^\beta + \\
&+ \frac{1}{4} (\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\dot\beta\dot\delta} + \varepsilon^{\dot\alpha\dot\delta}\varepsilon_{\dot\beta\dot\gamma}) \left(\partial X_{\dot\gamma a} X_{\dot\delta}^{\;a} + \partial X_{\dot\gamma}^{\;a} X_{\dot\delta a} \right) \bar\omega_{\dot\alpha} \bar\theta_{\dot\beta} \Bigg] (4.32)
\end{aligned}
$$

Finally, for completeness, we remember the current action (3.139), that gives the coupling ghost-ghost:

$$
\begin{aligned}
S_{current} = \frac{R^2}{2\pi} \int d^2z \, \frac{1}{2} \Big[&(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma}) \omega^\alpha \theta^\beta \bar\rho^\gamma \bar\psi^\delta + \\
&+ (\varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\dot\beta\dot\delta} + \varepsilon^{\dot\alpha\dot\delta}\varepsilon^{\dot\beta\dot\gamma}) \rho_{\dot\alpha} \psi_{\dot\beta} \bar\omega_{\dot\gamma} \bar\theta_{\dot\delta} + \\
&+ \varepsilon_{\alpha\beta} \varepsilon^{\dot\alpha\dot\beta} \left(\omega^\alpha \theta^\beta \bar\omega_{\dot\alpha} \bar\theta_{\dot\beta} + \rho_{\dot\alpha} \psi_{\dot\beta} \bar\rho^\alpha \bar\psi^\beta \right) \Big] \quad . \quad (4.33)
\end{aligned}
$$

4.3 The *uv* term

The background expansion for (u, v) fields takes origin from the position

$$
U = \widetilde{U} e^{x/R} \tag{4.34}
$$

where $\widetilde{U} \in \mathrm{SU}(3)$ is the background and $x \in su(3) \backslash [u(1) \oplus u(1)]$ is the fluctuation. Trivially, noting that the covariant derivative acts on x like a canonical one,

$$
j = U^\dagger \nabla U = e^{-x/R} \mathrm{j}\, e^{x/R} + e^{-x/R} \partial e^{x/R} \quad , \tag{4.35}
$$

with $\mathrm{j} = \widetilde{U}^\dagger \nabla \widetilde{U}$. In this way

$$
\begin{aligned}
\mathrm{Tr}(\bar{j}^\dagger j) &= -\mathrm{Tr}(\bar{j} j) = -\mathrm{Tr}(e^{-x/R} \bar{\mathrm{j}}\, e^{x/R} + e^{-x/R} \bar\partial e^{x/R})(e^{-x/R} \mathrm{j}\, e^{x/R} + e^{-x/R} \partial e^{x/R}) \\
&= -\mathrm{Tr}(\bar{\mathrm{j}}\,\mathrm{j} - \bar{\mathrm{j}} e^{x/R} \partial e^{-x/R} - \mathrm{j} e^{x/R} \bar\partial e^{-x/R} - \bar\partial e^{x/R} \partial e^{-x/R}) \\
&= \mathrm{Tr}(\bar{\mathrm{j}}^\dagger \mathrm{j}) + \mathrm{Tr}(\bar{\mathrm{j}} e^{x/R} \partial e^{-x/R} + \mathrm{j} e^{x/R} \bar\partial e^{-x/R}) + \mathrm{Tr}(\bar\partial e^{x/R} \partial e^{-x/R}) \quad , \quad (4.36)
\end{aligned}
$$

using the cyclical properties of the trace and $e^{-x}\partial e^x = -\partial e^{-x} \cdot e^x$. Let us consider the background free term

$$\mathrm{Tr}(\bar{\partial}e^{x/R}\partial e^{-x/R}) = \mathrm{Tr}\left[-\frac{1}{R^2}\bar{\partial}x\partial x + O\left(\frac{1}{R^4}\right)\right] \quad : \tag{4.37}$$

explicitly the matrix x is

$$x = \begin{pmatrix} 0 & -x_1^* & -x_2^* \\ x_1 & 0 & -x_3^* \\ x_2 & x_3 & 0 \end{pmatrix} \tag{4.38}$$

so

$$-\frac{1}{R^2}\mathrm{Tr}(\bar{\partial}x\partial x) = \frac{1}{R^2}\sum_{k=1}^{3}\left(\bar{\partial}x_k^*\partial x_k + c.c.\right) \tag{4.39}$$

and

$$\mathrm{Tr}(\bar{\partial}e^{x/R}\partial e^{-x/R}) = \frac{1}{R^2}\sum_{k=1}^{3}\left(\bar{\partial}x_k^*\partial x_k + c.c.\right) + O\left(\frac{1}{R^4}\right) \quad . \tag{4.40}$$

We can write the kinetic action for the fluctuations x_k from S_χ (3.147):

$$S_\chi \to \frac{1}{2\pi}\int d^2z \left[\sum_{k=1}^{3}\left(\bar{\partial}x_k^*\partial x_k + c.c.\right) + O\left(\frac{1}{R^2}\right)\right] \quad \Rightarrow$$

$$S_{xx} = \frac{1}{2\pi}\int d^2z \sum_{k=1}^{3}\left(\bar{\partial}x_k^*\partial x_k + c.c.\right) = \frac{1}{2\pi}\int d^2z\, 2\sum_{k=1}^{3}\bar{\partial}x_k^*\partial x_k \quad . \tag{4.41}$$

4.4   The central charge

We want to compute the central charge for the action (3.158) using the background field method. We will prove that it is zero at tree level (i.e. at order $1/R^0$) and at one loop (i.e. at order $1/R^2$).

4.4.1   Matter sector

Let us start with the matter term: the stress-energy tensor can be obtained directly by (3.63)

$$T_{matter} = -R^2\mathrm{STr}\left[\frac{1}{2}J_2J_2 + \frac{3}{4}J_3J_1 + \frac{1}{4}J_1J_3\right]$$

$$= -R^2\mathrm{STr}\left[\frac{1}{2}J_2J_2 + J_1J_3\right] \quad ; \tag{4.42}$$

to our aim, since the central charge cannot contain fields, we consider only the background free expansion of J in (4.3) and we can interrupt it at the order $1/R^2$:

$$J \to \frac{1}{R}\partial X + \frac{1}{2R^2}[\partial X, X] \tag{4.43}$$

i.e.

$$J_1 \to \frac{1}{R}\partial X_1 + \frac{1}{2R^2}\left([\partial X_2, X_3] + [\partial X_3, X_2]\right) \tag{4.44}$$

$$J_2 \to \frac{1}{R}\partial X_2 + \frac{1}{2R^2}\left([\partial X_1, X_1] + [\partial X_3, X_3]\right) \tag{4.45}$$

$$J_3 \to \frac{1}{R}\partial X_3 + \frac{1}{2R^2}\left([\partial X_1, X_2] + [\partial X_2, X_1]\right) \quad . \tag{4.46}$$

In this way we have

$$
\begin{aligned}
T_{matter} \to -R^2 \mathrm{STr}\Bigg[&\frac{1}{2}\left(\frac{1}{R}\partial X_2 + \frac{1}{2R^2}[\partial X_1, X_1] + \frac{1}{2R^2}[\partial X_3, X_3]\right)^2 + \\
&+ \left(\frac{1}{R}\partial X_1 + \frac{1}{2R^2}[\partial X_2, X_3] + \frac{1}{2R^2}[\partial X_3, X_2]\right) \times \\
&\times \left(\frac{1}{R}\partial X_3 + \frac{1}{2R^2}[\partial X_1, X_2] + \frac{1}{2R^2}[\partial X_2, X_1]\right)\Bigg] \\
\to -\mathrm{STr}&\left[\frac{1}{2}\partial X_2 \partial X_2 + \partial X_1 \partial X_3\right] + \\
-\frac{1}{2R}\mathrm{STr}&\Big[\partial X_2[\partial X_1, X_1] + \partial X_2[\partial X_3, X_3] + \\
&+ \partial X_1[\partial X_1, X_2] + \partial X_1[\partial X_2, X_1] + \partial X_3[\partial X_3, X_2] + \partial X_3[\partial X_2, X_3]\Big] \quad .
\end{aligned}
$$

Because of (4.12), we have

$$\mathrm{STr}(\partial X_1[\partial X_2, X_1]) = -\mathrm{STr}(\partial X_2[\partial X_1, X_1]) \quad , \tag{4.47}$$

$$\mathrm{STr}(\partial X_1[\partial X_1, X_2]) = -\mathrm{STr}(\partial X_1[\partial X_1, X_2]) \quad \Rightarrow \quad \mathrm{STr}(\partial X_1[\partial X_1, X_2]) = 0 \tag{4.48}$$

and analogues with ∂X_3 instead of ∂X_1. So the background free tensor is

$$T_{matter} \to -\mathrm{STr}\left[\frac{1}{2}\partial X_2 \partial X_2 + \partial X_1 \partial X_3\right] + O\left(\frac{1}{R^2}\right) \quad . \tag{4.49}$$

Obviously the first term of T_{matter} above is the stress-energy tensor of the kinetic action for X fields S_{XX} (4.17). In components

$$T_{matter} \to -\left[\frac{1}{2}\eta_{mn}\partial X^m \partial X^n - \partial X^a \partial X_a - i\varepsilon_{\alpha\beta}\partial X^{\alpha a}\partial X^\beta{}_a - i\varepsilon^{\dot\alpha\dot\beta}\partial X_{\dot\alpha a}\partial X_{\dot\beta}{}^a\right] \tag{4.50}$$

By S_{XX} we obtain the fundamental OPE

$$X^m(z)X^n(w) = -\eta^{mn}\ln|z-w|^2 \tag{4.51}$$

$$X^a(z)X_b(w) = \delta_b{}^a\ln|z-w|^2 \tag{4.52}$$

$$X^{\alpha a}(z)X^\beta{}_b(w) = -i\varepsilon^{\alpha\beta}\delta_b{}^a\ln|z-w|^2 \tag{4.53}$$

$$X_{\dot\alpha a}(z)X_{\dot\beta}{}^b(w) = -i\varepsilon_{\dot\alpha\dot\beta}\delta_a{}^b\ln|z-w|^2 \tag{4.54}$$

and thus the terms $1/(z-w)^4$ of the OPE $T_{matter}(z)T_{matter}(w)$ are

$$\langle \frac{1}{2}\eta_{mn}\partial X^m\partial X^n\Big|_z\ \frac{1}{2}\eta_{kl}\partial X^k\partial X^l\Big|_w\rangle \to \frac{1}{4}\eta_{mn}\eta_{kl}\left(\eta^{mk}\eta^{nl}+\eta^{ml}\eta^{nk}\right)\frac{1}{(z-w)^4}$$
$$= \frac{1}{4}\frac{2\delta_k{}^k}{(z-w)^4} = \frac{2}{(z-w)^4} \tag{4.55}$$

$$\langle \partial X^a\partial X_a|_z\ \partial X^b\partial X_b|_w\rangle \to \frac{\delta_b{}^a\delta_a{}^b}{(z-w)^4} = \frac{\delta_a{}^a}{(z-w)^4} = \frac{3}{(z-w)^4} \tag{4.56}$$

$$\langle i\varepsilon_{\alpha\beta}\partial X^{\alpha a}\partial X^\beta{}_a\Big|_z\ i\varepsilon_{\gamma\delta}\partial X^{\gamma b}\partial X^\delta{}_b\Big|_w\rangle \to i^2\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}(-i)^2\varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma}\delta_b{}^a\delta_a{}^b\frac{1}{(z-w)^4}$$
$$= -\frac{\delta_\alpha{}^\alpha\delta_a{}^a}{(z-w)^4} = -\frac{6}{(z-w)^4} \tag{4.57}$$

$$\langle i\varepsilon^{\dot\alpha\dot\beta}\partial X_{\dot\alpha a}\partial X_{\dot\beta}{}^a\Big|_z\ i\varepsilon^{\dot\gamma\dot\delta}\partial X_{\dot\gamma b}\partial X_{\dot\delta}{}^b\Big|_w\rangle \to i^2\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\dot\gamma\dot\delta}(-i)^2\varepsilon_{\dot\alpha\dot\delta}\varepsilon_{\dot\beta\dot\gamma}\delta_a{}^b\delta_b{}^a\frac{1}{(z-w)^4}$$
$$= -\frac{\delta_{\dot\alpha}{}^{\dot\alpha}\delta_a{}^a}{(z-w)^4} = -\frac{6}{(z-w)^4}\quad . \tag{4.58}$$

The central charge for different sectors is

$$c_{bos.\ matter} = 2\cdot 2+3\cdot 2 = 10 \qquad\qquad c_{ferm.\ matter} = -6\cdot 2-6\cdot 2 = -24\quad . \tag{4.59}$$

The absence of a term $1/R$ in T_{matter} (4.49) implies that matter does not give contribution to the central charge of the order $1/R^2$.

4.4.2 Ghost sector

The stress-energy tensor for the ghost is[1]

$$T_{ghost} = iR^2 \left(\varepsilon_{\alpha\beta}\omega^\alpha \nabla\theta^\beta + \varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\nabla\psi_{\dot\beta} \right) \tag{4.60}$$

and by (4.26) we have the fundamental OPE

$$\omega^\alpha(z)\theta^\beta(w) = -\frac{i}{R^2}\varepsilon^{\alpha\beta}\frac{1}{z-w} \tag{4.61}$$

$$\rho_{\dot\alpha}(z)\psi_{\dot\beta}(w) = -\frac{i}{R^2}\varepsilon_{\dot\alpha\dot\beta}\frac{1}{z-w} \quad . \tag{4.62}$$

Thus the terms $1/(z-w)^4$ of the OPE $T_{ghost}(z)T_{ghost}(w)$ at the order $1/R^0$ are

$$(iR^2)^2 \left\langle \varepsilon_{\alpha\beta}\omega^\alpha\partial\theta^\beta \big|_z \, \varepsilon_{\gamma\delta}\omega^\gamma\partial\theta^\delta \big|_w \right\rangle \to (iR^2)^2 \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} \left(\frac{-i}{R^2}\right)^2 \frac{\varepsilon^{\alpha\delta}\varepsilon^{\gamma\beta}}{(z-w)^4}$$

$$= \frac{\delta_\alpha{}^\alpha}{(z-w)^4} = \frac{2}{(z-w)^4} \tag{4.63}$$

$$(iR^2)^2 \left\langle \varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\partial\psi_{\dot\beta} \big|_z \, \varepsilon^{\dot\gamma\dot\delta}\rho_{\dot\gamma}\partial\psi_{\dot\delta} \big|_w \right\rangle \to (iR^2)^2 \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\dot\gamma\dot\delta} \left(\frac{-i}{R^2}\right)^2 \frac{\varepsilon_{\dot\alpha\dot\delta}\varepsilon_{\dot\gamma\dot\beta}}{(z-w)^4}$$

$$= \frac{\delta_{\dot\alpha}{}^{\dot\alpha}}{(z-w)^4} = \frac{2}{(z-w)^4} \tag{4.64}$$

therefore the ghost central charge is

$$c_{ghost} = 2 \cdot 2 + 2 \cdot 2 = 8 \quad . \tag{4.65}$$

If we expand the covariant derivative, we would have to add a contribution like $\partial XX\omega\theta$ to T_{ghost} and it would give a term proportional to $1/R^4$ in the c expression above. It means that the ghost sector does not give contribution $1/R^2$ to c, analogously to the matter sector.

4.4.3 uv sector

The stress-energy tensor for u and v is given by (3.143)

$$T_\chi = -R^2 \text{Tr}\left(j^\dagger j\right) \tag{4.66}$$

---

[1] We consider only the holomorphic component of T: the anti-holomorphic one is given by $(\bar\omega, \bar\theta)$ and $(\bar\rho, \bar\psi)$.

and the fundamental OPE is

$$x_k^*(z)x_l(w) = -\frac{1}{2}\delta_{kl}\ln|z-w|^2 \quad , \tag{4.67}$$

so at order $1/R^0$ we have

$$T_{xx} = -2\sum_{k=1}^{3} \partial x_k^* \partial x_k \tag{4.68}$$

and

$$\langle T_{xx}|_z \, T_{xx}|_w\rangle = (-2)^2 \sum_{k,l=1}^{3} \langle \partial x_k^* \partial x_k|_z \, \partial x_l^* \partial x_l|_w\rangle$$

$$\to 4\sum_{k,l=1}^{3} \left(-\frac{1}{2}\right)^2 \frac{\delta_{kl}\delta_{lk}}{(z-w)^4} = \frac{3}{(z-w)^4} \quad . \tag{4.69}$$

The central charge is

$$c_{uv} = 3 \cdot 2 = 6 \quad ; \tag{4.70}$$

in analogous way to the matter sector, the correction of c_{uv} proportional to $1/R^2$ is zero.

4.4.4   Ghost-uv sector

The last terms in S_{ghost} (3.129) give a tensor

$$T_{ghost+uv} = iR^2 \left[(\varepsilon_{\alpha\beta}\omega^\alpha\theta^\beta)u_a^*\nabla u^a + (\varepsilon^{\dot\alpha\dot\beta}\rho_{\dot\alpha}\psi_{\dot\beta})v^{a*}\nabla v_a \right] \quad . \tag{4.71}$$

The lower order in the background free expansion of $u_a^*\nabla u^a$ and $v^{a*}\nabla v_a$ is $1/R^2$: it can be understood noting that these ones are the diagonal elements of j while in the expansion

$$j \to \frac{1}{R}\partial x + \frac{1}{2R^2}[\partial x, x] + O\left(\frac{1}{R^3}\right) \tag{4.72}$$

the first diagonal terms are $[\partial x, x]/2R^2$, since x has only extra-diagonal components. It is simple to verify that the coefficient of $1/(z-w)^4$ in the OPE of T with itself is proportional to $1/R^4$, so that the ghost-uv sector does not give contribution to c

up to this order.

Finally we can collect the results above:

$$c = c_{bos.\,matter} + c_{ferm.\,matter} + c_{\omega\theta} + c_{uv} = 10 - 24 + 8 + 6 = 0 \qquad (4.73)$$

noting it is true up to the $1/R^4$ order.

4.5   The operator product expansion of the Lorentz current

As final check of the consistency of our pure spinor action, we compute the OPE of the Lorentz current with itself. In our case from the definition of the gauge-current coupling

$$S_{coupling} = \frac{1}{2\pi} \int d^2 z \, L^{mn} \overline{J}_{mn} \qquad (4.74)$$

we obtain the Lorentz current for the action (3.158)[2]

$$L^{mn} = -\frac{i}{2} R^2 \left(\omega^\alpha (\sigma^{mn})_{\alpha\beta} \theta^\beta + \rho_{\dot\alpha} (\bar\sigma^{mn})^{\dot\alpha\dot\beta} \psi_{\dot\beta} \right) \quad . \qquad (4.75)$$

Notice that respect to the flat case, the Lorentz current is not given by the sum of a matter contribute and a ghost one, but is completely provided by the ghost fields.

---

[2]The Lorentz current L^{mn} differs from N^{mn} (3.130) in a constant. Although the physical content is the same, we have to fix the normalization so that the simple pole of the OPE current-current reproduce the $so(3,1)$ algebra.

Using the ghost OPE (4.61) and (4.62) we have

$$\langle L^{mn}(z)L^{kl}(w)\rangle = \left(-\frac{i}{2}R^2\right)^2 \langle \omega^\alpha(\sigma^{mn})_{\alpha\beta}\theta^\beta + \rho_{\dot\alpha}(\bar\sigma^{mn})^{\dot\alpha\dot\beta}\psi_{\dot\beta}\Big|_z \times$$

$$\times \ \omega^\gamma(\sigma^{kl})_{\gamma\delta}\theta^\delta + \rho_{\dot\gamma}(\bar\sigma^{kl})^{\dot\gamma\dot\delta}\psi_{\dot\delta}\Big|_w\rangle$$

$$= \left(\frac{i}{2}\right)^2 R^4 \left[\frac{i}{R^2}\left(\omega^\alpha[\sigma^{mn},\sigma^{kl}]_\alpha{}^\beta \varepsilon_{\beta\delta}\theta^\delta + \rho_{\dot\alpha}[\bar\sigma^{mn},\bar\sigma^{kl}]^{\dot\alpha}{}_{\dot\beta}\varepsilon^{\dot\beta\dot\delta}\psi_{\dot\delta}\right)\frac{1}{z-w} + \right.$$

$$\left. - \left(\frac{i}{R^2}\right)^2 \left((\sigma^{mn})_\alpha{}^\beta(\sigma^{kl})_\beta{}^\alpha + (\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta}(\bar\sigma^{kl})^{\dot\beta}{}_{\dot\alpha}\right)\frac{1}{(z-w)^2}\right]$$

$$= -\frac{i}{4}R^2\left[2\eta^{nk}\left(\omega^\alpha(\sigma^{ml})_{\alpha\beta}\theta^\beta + \rho_{\dot\alpha}(\bar\sigma^{ml})^{\dot\alpha\dot\beta}\psi_{\dot\beta}\right) + \text{permutations}\right]\frac{1}{z-w} +$$

$$- \left(\eta^{ml}\eta^{nk} - \eta^{mk}\eta^{nl}\right)\frac{1}{(z-w)^2} \tag{4.76}$$

having used

$$[\sigma^{mn},\sigma^{kl}] = 2\left(\eta^{nk}\sigma^{ml} + \text{permutations}\right) \tag{4.77}$$

$$(\sigma^{mn})_\alpha{}^\beta(\sigma^{kl})_\beta{}^\alpha = 2\left(\eta^{ml}\eta^{nk} - \eta^{mk}\eta^{nl}\right) \tag{4.78}$$

and analogous ones for $\bar\sigma$. Thus we have

$$\langle L^{mn}(z)L^{kl}(w)\rangle = \frac{\eta^{k[n}L^{m]l} - \eta^{l[n}L^{m]k}}{z-w} + \frac{\eta^{[mn][kl]}}{(z-w)^2} \tag{4.79}$$

i.e. the Lorentz currents form a current algebra with level $k=1$, as in the flat case.

Conclusions

In this thesis we presented a formulation of the Pure Spinor superstring in $AdS_4 \times \mathbb{CP}^3$ with unconstrainted ghost fields.

We started from the Pure Spinor superstring in a supercoset manifold and we studied the BRST invariance. Imposing the nilpotency of the BRST charge, we derived the general form of the ghost constraint. To solve this constraint in the $OSP(4|6)/SO(3,1) \times U(3)$ coset, corresponding to the $AdS_4 \times \mathbb{CP}^3$ superspace, we chose a convenient realization of the superalgebra of $OSP(4|6)$. In this way we were able to write the solutions of the constraint as a direct product of new ghosts and bosonic variables. We noted that $OSP(4|6)/SO(3,1) \times U(3)$ coset admits a $\mathbb{Z}_4$-grading and in particular that there is a one-to-one correspondence between the two fermionic eigenspaces $\mathcal{H}_1$ and $\mathcal{H}_3$: so the ghosts $\lambda_3 \in \mathcal{H}_3$ are given by the same field content of the ghosts $\lambda_1 \in \mathcal{H}_1$. Then we extended the ghost decomposition to the conjugate momenta w_1, w_3 of λ_1, λ_3, using a residual gauge invariance of the action to make the most convenient choice.

We replaced the so-determined ghosts and momenta in the original action: by the choice we did, the bosonic variables have not a kinetic term. To give a kinematics to these variables we observed that they lie in the $SU(3)/U(1) \times U(1)$ coset, thus we added to the action a $SO(3,1) \times U(3)$ and BRST invariant non-linear sigma model on this coset.

Working with unconstrainted ghosts presents a lot of advantages, first of all the possibility of computing directly the ghost propagators (as in flat case). We

used these propagators to compute the operator product expansion of the Lorentz currents; moreover, using also the background field method, we proved that the central charge of the action vanishes up to one loop, i.e. $1/R^2$ order. The results we obtained confirm the correctness of the action proposed.

Using the background field expansion derived in Chapter 4, one could compute the effective action and check for instance the vanishing of the beta function. Preliminary results indicate the absence of one-loop divergent contributions.

Appendix A

Vielbein formalism

Be $\mathcal{M}$ a m-dimension differentiable manifold and φ a local chart[1]

$$\varphi : U \subset \mathcal{M} \rightarrow \mathbb{R}^m \quad . \tag{A.1}$$

If $p \in U$, the tangent space $T_p\mathcal{M}$ admits *coordinate basis* $\left\{ \dfrac{\partial}{\partial\varphi^i} \right\}_{i=1,...,m}$ and the cotangent space $T_p^*\mathcal{M}$ admits *dual basis* $\{d\varphi^i\}_{i=1,...,m}$ so that

$$d\varphi^i \left(\frac{\partial}{\partial\varphi^j} \right) = \delta_j{}^i \quad . \tag{A.2}$$

It is possible to chose a different, non-coordinate, basis $\{e_a\}_{a=1,...,m}$ for $T_p\mathcal{M}$ and its dual basis $\{e^a\}_{a=1,...,m}$, related in the usual way

$$e^a(e_b) = \delta_b{}^a \quad . \tag{A.3}$$

Obviously they are linear combinations of the old bases

$$e_a = e_a{}^i \frac{\partial}{\partial\varphi^i} \qquad , \qquad e^a = e_i{}^a d\varphi^i \quad , \tag{A.4}$$

so

$$\delta_b{}^a \equiv e_i{}^a d\varphi^i \left(e_b{}^j \frac{\partial}{\partial\varphi^j} \right) = e_i{}^a e_b{}^j \delta_j{}^i = e_b{}^i e_i{}^a \tag{A.5}$$

i.e. $e_i{}^a = (e_a{}^i)^{-1}$ and viceversa.

---

[1] For simplicity we chose real manifold, but the the complex case is a trivial generalization.

Metric 2-form is an intrinsic property of $\mathcal{M}$ and does not depend from the coordinates:

$$g = g_{ij}d\varphi^i \otimes d\varphi^j = g_{ab}d\varphi^a \otimes d\varphi^b \tag{A.6}$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial\varphi^i}, \frac{\partial}{\partial\varphi^j}\right) \qquad g_{ab} = g(e_a, e_b) \quad . \tag{A.7}$$

We can chose the basis vectors e_a Lorentz-orthogonal, that is

$$g_{ab} = \eta_{ab} \equiv diag(+, -, \ldots, -) \quad , \tag{A.8}$$

provided that the components of e_a on the coordinate basis change point by point, $e_a{}^i = e_a{}^i(p)$. On the contrary $\partial/\partial\varphi^i$ usually not have fixed angles, then $g_{ij} = g_{ij}(p)$. It is simple to compute the relation between g_{ij} and g_{ab}:

$$g_{ab} = g\left(e_a{}^i\frac{\partial}{\partial\varphi^i}, e_b{}^j\frac{\partial}{\partial\varphi^j}\right) = e_a{}^i e_b{}^j g_{ij} \tag{A.9}$$

and inverting $e_a{}^i$

$$g_{ij}(p) = e_i{}^a(p)e_j{}^b(p)\eta_{ab} \quad . \tag{A.10}$$

Let us consider now a diffeomorphism on $\mathbb{R}^m$, $f : x \to x'$; obviously $\varphi' \equiv f \circ \varphi$ is still a local chart of $\mathcal{M}$ with coordinate basis $\partial/\partial\varphi'^i$. By definition, if $F : \mathcal{M} \to \mathbb{R}^m$

$$\left.\frac{\partial F}{\partial\varphi^i}\right|_p = \left.\frac{\partial}{\partial x^i}(F \circ \varphi^{-1})\right|_{x=\varphi(p)} \tag{A.11}$$

and analogue for φ'; using

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} \tag{A.12}$$

and $\varphi'^{-1} = \varphi^{-1} \circ f^{-1}$, we have

$$\left.\frac{\partial F}{\partial\varphi'^i}\right|_p = \left.\frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}(F \circ \varphi^{-1} \circ f^{-1})\right|_{x'} = \left.\frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j}(F \circ \varphi^{-1})\right|_x = \left.\frac{\partial x^j}{\partial x'^i}\frac{\partial F}{\partial\varphi^j}\right|_p \tag{A.13}$$

i.e.

$$\frac{\partial}{\partial\varphi'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial\varphi^j} \quad . \tag{A.14}$$

Vectors e_a are not affected by diffeomorphism f, so

$$e_a{}^i\frac{\partial}{\partial\varphi^i} \equiv e'_a{}^j\frac{\partial}{\partial\varphi'^j} = e'_a{}^j\frac{\partial x^i}{\partial x'^j}\frac{\partial}{\partial\varphi^i} \quad \Rightarrow \tag{A.15}$$

$$e_a{}^i = \frac{\partial x^i}{\partial x'^j} e'_a{}^j \qquad \text{or} \qquad e'_a{}^i = \frac{\partial x'^i}{\partial x^j} e_a{}^j \tag{A.16}$$

and trivially

$$e_i{}^a = \frac{\partial x'^j}{\partial x^i} e'_j{}^a \qquad \text{or} \qquad e'_i{}^a = \frac{\partial x^j}{\partial x'^i} e_j{}^a \quad . \tag{A.17}$$

Then let us consider a Lorentz transformation $\widetilde{\Lambda}$ sending the orthogonal basis e_a in another $\widetilde{e}_a$

$$\widetilde{e}_a = \widetilde{\Lambda}^b{}_a e_b \quad . \tag{A.18}$$

It is

$$\eta_{ab} = g(\widetilde{e}_a, \widetilde{e}_b) = \widetilde{\Lambda}^c{}_a \widetilde{\Lambda}^d{}_b \, g(e_c, e_d) = \widetilde{\Lambda}^c{}_a \widetilde{\Lambda}^d{}_b \eta_{cd} \tag{A.19}$$

i.e. $\widetilde{\Lambda}$ is a pseudo-rotation $\mathrm{SO}(m-1, 1)$. In components

$$\widetilde{e}_a{}^i \frac{\partial}{\partial \varphi^i} = \widetilde{\Lambda}^b{}_a e_b{}^i \frac{\partial}{\partial \varphi^i} \quad ; \tag{A.20}$$

posing $\Lambda = \widetilde{\Lambda}^{-1}$

$$e_a{}^i = \Lambda^b{}_a \widetilde{e}_b{}^i \qquad \text{or} \qquad \widetilde{e}_a{}^i = (\Lambda^{-1})^b{}_a e_b{}^i \tag{A.21}$$

and trivially

$$e_i{}^a = (\Lambda^{-1})^a{}_b \widetilde{e}_i{}^b \qquad \text{or} \qquad \widetilde{e}_i{}^a = \Lambda^a{}_b e_i{}^b \quad . \tag{A.22}$$

The components $e_i{}^a$ are called *vielbein* and have the demonstrated properties: they locally generate the metric (A.10), transform under diffeomorphism in the i index (A.17) and under Lorentz-pseudo rotation in the a index (A.22).

Appendix B

Algebraic properties of the manifolds

If $\mathcal{M}$ is a differentiable manifold and G a Lie group with identity e, we can define [27] *action* of G on $\mathcal{M}$ the application $(g, p) \in G \times \mathcal{M} \to gp \in \mathcal{M}$ so that

$$
\begin{aligned}
ep &= pe = p \\
g_1(g_2 p) &= (g_1 g_2)p
\end{aligned}
\qquad \forall p \in \mathcal{M} \quad \forall g_1, g_2 \in G
\tag{B.1}
$$

The action is *transitive* if $\forall p_1, p_2 \in \mathcal{M}$ there is $g \in G$ so that $gp_1 = p_2$. The *orbit* of $p \in \mathcal{M}$ under the action of G is the subset Gp of $\mathcal{M}$ given by

$$
Gp = \{gp : g \in G\} \quad .
\tag{B.2}
$$

Trivially if G acts transitively on $\mathcal{M}$, $Gp = \mathcal{M}$. The *little group* (or *isotropy group*) of $p \in \mathcal{M}$ is the subgroup H_p of G so that

$$
H_p = \{g \in G : gp = p\} \quad .
\tag{B.3}
$$

If $H \subset G$ is a subgroup and $g \in G$, the subset $gH = \{gh \in G : h \in H\}$ is the *left coset* of H; analogously we can define the *right coset Hg*. The set of all gH in G is the *quotient space*

$$
\frac{G}{H} = \{gH \subset G : g \in G\}
\tag{B.4}
$$

and it admits the structure of group only if H in a *normal* subgroup, i.e. if $gH = Hg$ $\forall g$. However if G is a Lie group, G/H admits differentiable manifold structure, called *coset manifold.*

If a group of Lie G acts on $\mathcal{M}$ transitively and we choose as subgroup of G the little group H_p of some $p \in \mathcal{M}$, the *coset manifold G/H_p is homeomorphic to $\mathcal{M}$*, i.e. there is a continuous one-to-one map between G/H_p and $\mathcal{M}$. To see that, we can start identifying H_p with p : if $q \neq p$ is another point of $\mathcal{M}$, there is $g \in G$ so that $gp = q$ and we can identify gH_p with q. This identification is the only one, in fact if there is $g' \neq g$, so that $g'p = q$, surely $g^{-1}g' = h \in H_p$ that is $g'H_p \equiv gH_p$.

Note that the choice of p is completely free. If we start from a point $p' \neq p$, obviously there is a $g \in G$ so that $gp = p'$ and it is straightforward to prove that $H_{p'} = gH_pg^{-1}$, so we can repeat the argument above. The equivalence of all $p \in \mathcal{M}$ means that the coset manifold describes an *homogeneous space*. In the present work we are interested in three fundamental spaces:

B.1   Sphere

If we consider a $(n+1)$-dimensional flat bulk of coordinates $(y^\mu, y^n)_{\mu=0,...,n-1}$ with metric

$$ds_{bulk}^2 = \eta_{\mu\nu}dy^\mu dy^\nu + (dy^n)^2 \qquad \eta_{\mu\nu} = diag(\overbrace{+, \cdots, +}^{n}) \quad , \tag{B.5}$$

the n-dimensional *Sphere* is defined by

$$\eta_{\mu\nu}y^\mu y^\nu + (y^n)^2 = R^2 \tag{B.6}$$

with $R \in \mathbb{R}$ named *curvature radius*. Trivially the action of the group $\mathrm{SO}(n+1)$ is transitive on the Sphere and the rotations of $\mathrm{SO}(n)$ around a point do not shift it, hence

$$\mathrm{S}^n \cong \frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n)} \quad . \tag{B.7}$$

It is simple to note that

$$dim\left(\frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n)}\right) = dim\,\mathrm{SO}(n+1) - dim\,\mathrm{SO}(n)$$

$$= \frac{1}{2}(n+1)n - \frac{1}{2}n(n-1) = n \quad . \tag{B.8}$$

B.2 Anti-de Sitter Space

If the $(n+1)$-dimensional flat bulk has metric

$$ds^2_{bulk} = \eta_{\mu\nu} dy^\mu dy^\nu + (dy^n)^2 \qquad \eta_{\mu\nu} = diag(+, \overbrace{-, \cdots, -}^{n-1}) \quad , \qquad (B.9)$$

the n-dimensional *Anti-de Sitter Space* is defined as the hyperboloid

$$\eta_{\mu\nu} y^\mu y^\nu + (y^n)^2 = R^2 \quad . \qquad (B.10)$$

It corresponds in lorentzian signature to the Lobačevskij Space [28] in euclidean signature. It is simple to see that AdS_n is the orbit of the group $\text{SO}(n-1,2)$ - i.e. this group acts transitively on AdS - and that $\text{SO}(n-1,1)$ is the little group respect to any point of AdS, so

$$\text{AdS}_n \cong \frac{\text{SO}(n-1,2)}{\text{SO}(n-1,1)} \quad . \qquad (B.11)$$

Obviously

$$dim\left(\frac{\text{SO}(n-1,2)}{\text{SO}(n-1,1)}\right) = dim\,\text{SO}(n-1,2) - dim\,\text{SO}(n-1,1)$$

$$= \frac{1}{2}(n+1)n - \frac{1}{2}n(n-1) = n \quad . \qquad (B.12)$$

B.3 Complex Projective Space

Be $\mathbf{z}_1, \mathbf{z}_2 \in \mathbb{C}^{n+1} - \{0\}$: we can define the equivalence $\mathbf{z}_1 \sim \mathbf{z}_2$ if there is a complex number $\lambda \neq 0$ so that $\mathbf{z}_2 = \lambda \mathbf{z}_1$. The *Complex Projective Space* is the set of all the classes in $\mathbb{C}^{n+1} - \{0\}$ [27]

$$\mathbb{C}\text{P}^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\sim} \qquad (B.13)$$

and it represents the space of the direction of $\mathbb{C}^{n+1}$: defining $|\mathbf{z}|^2 = \mathbf{z}^\dagger \mathbf{z}$, we have $\mathbb{C}\text{P}^n = \{\mathbf{z} \in \mathbb{C}^{n+1} - \{0\} : |\mathbf{z}| = 1\}$. Let us consider $g \in \text{U}(n+1)$ acting on $\mathbf{z}$. If $\mathbf{z}_{1,2} \to \mathbf{z}'_{1,2} = g\mathbf{z}_{1,2}$ and $\mathbf{z}_1 \sim \mathbf{z}_2$, surely $\mathbf{z}'_1 \sim \mathbf{z}'_2$, hence the group acts on $\mathbb{C}\text{P}^n$ too. In analogy with $\text{O}(n+1)$ on $\mathbb{R}^{n+1}$, $\text{U}(n+1)$ can be seen as the group of complex rotations of $\mathbb{C}^{n+1}$ and its action is trivially transitive. Choosing an element

of $\mathbb{CP}^n$ by means of an homogeneous coordinate $\mathbf{z} \neq 0$, its little group is given by the complex rotations $U(n)$ around the $\mathbf{z}$-direction and by the phase scaling $U(1)$ $\mathbf{z} \to e^{i\phi}\mathbf{z}$. This way

$$\mathbb{CP}^n \cong \frac{U(n+1)}{U(n) \times U(1)} \cong \frac{SU(n+1)}{U(n)} \tag{B.14}$$

because $SU(n+1) \cong U(n+1)/U(1)$. As above

$$dim\left(\frac{SU(n+1)}{U(n)}\right) = dim\,SU(n+1) - dim\,U(n) = [(n+1)^2 - 1] - n^2 = 2n \quad . \tag{B.15}$$

Appendix C

OSP(4|6) algebra

C.1 Preliminary definitions

Antisymmetric 2-dimensional tensor

$$\varepsilon^{12} = -\varepsilon^{21} = 1 \qquad \varepsilon_{12} = -\varepsilon_{21} = -1 \tag{C.1}$$

$$\varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta_\alpha{}^\beta \qquad \varepsilon^{\dot\alpha\dot\gamma}\varepsilon_{\dot\gamma\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta} \tag{C.2}$$

Charge conjugation matrix and inverse

$$C_{\mu\nu} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & -1 & & 0 \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & 0 & -1 & 0 \end{array} \right) \tag{C.3}$$

$$C_{\mu\rho}C^{\rho\nu} = \delta_\mu{}^\nu \tag{C.4}$$

Dirac matrices in $4 + 1$ dimensions, $m = 0', 0, 1, \ldots, 4$

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn} \quad \text{with} \quad \eta^{mn} = (+ + - - -) \tag{C.5}$$

explicitly

$$(\gamma^m)_\mu{}^\nu = \begin{pmatrix} 0 & (\sigma^m)_{\alpha\dot\alpha} \\ (\bar\sigma^m)^{\dot\alpha\alpha} & 0 \end{pmatrix} \tag{C.6}$$

with $\sigma^m = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3)$ and $\bar{\sigma}^m = (\mathbf{1}, -\sigma^1, -\sigma^2, -\sigma^3)$, noting that

$$(\bar{\sigma}^{\,m})^{\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}(\sigma^m)_{\beta\dot{\beta}} \quad . \tag{C.7}$$

Definition

$$(\gamma^{0'm})_\mu{}^\nu = i\gamma^m \tag{C.8}$$

$$(\gamma^{mn})_\mu{}^\nu = \frac{1}{2}[\gamma^m, \gamma^n] = \begin{pmatrix} (\sigma^{mn})_\alpha{}^\beta & 0 \\ 0 & (\bar{\sigma}^{\,mn})^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} \tag{C.9}$$

with

$$(\sigma^{mn})_\alpha{}^\beta = \frac{1}{2}\left((\sigma^m)_{\alpha\dot{\alpha}}(\bar{\sigma}^{\,n})^{\dot{\alpha}\beta} - (\sigma^n)_{\alpha\dot{\alpha}}(\bar{\sigma}^{\,m})^{\dot{\alpha}\beta}\right) \tag{C.10}$$

$$(\bar{\sigma}^{\,mn})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{2}\left((\bar{\sigma}^{\,m})^{\dot{\alpha}\alpha}(\sigma^n)_{\alpha\dot{\beta}} - (\bar{\sigma}^{\,n})^{\dot{\alpha}\alpha}(\sigma^m)_{\alpha\dot{\beta}}\right) \tag{C.11}$$

Definition

$$(\gamma^{\underline{mn}})^{\mu\nu} \equiv C^{\mu\rho}(\gamma^{\underline{mn}})_\rho{}^\nu \tag{C.12}$$

$$(\gamma^{\underline{mn}})_{\mu\nu} \equiv (\gamma^{\underline{mn}})_\mu{}^\rho C_{\rho\nu} \tag{C.13}$$

explicitly

$$(\gamma^{0'm})_{\mu\nu} = i\begin{pmatrix} 0 & (\sigma^m)_{\alpha\dot{\alpha}}\,\varepsilon^{\dot{\alpha}\dot{\beta}} \\ (\bar{\sigma}^{\,m})^{\dot{\alpha}\alpha}\varepsilon_{\alpha\beta} & 0 \end{pmatrix} \equiv i\begin{pmatrix} 0 & (\sigma^m)_\alpha{}^{\dot{\beta}} \\ (\bar{\sigma}^{\,m})^{\dot{\alpha}}{}_\beta & 0 \end{pmatrix} \tag{C.14}$$

$$(\gamma^{mn})_{\mu\nu} = \begin{pmatrix} (\sigma^{mn})_\alpha{}^\gamma\,\varepsilon_{\gamma\beta} & 0 \\ 0 & (\bar{\sigma}^{\,mn})^{\dot{\alpha}}{}_{\dot{\gamma}}\,\varepsilon^{\dot{\gamma}\dot{\beta}} \end{pmatrix} \equiv \begin{pmatrix} (\sigma^{mn})_{\alpha\beta} & 0 \\ 0 & (\bar{\sigma}^{\,mn})^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{C.15}$$

Antisymmetric chiral matrices in 6 dimensions

$$(\rho^M)_{\underline{ac}}(\widetilde{\rho}^N)^{\underline{cb}} + (\rho^N)_{\underline{ac}}(\widetilde{\rho}^M)^{\underline{cb}} = 2\delta^{MN}\delta_{\underline{a}}{}^b \tag{C.16}$$

$$(\widetilde{\rho}^M)^{\underline{ab}} = \frac{1}{2}\epsilon^{\underline{abcd}}(\rho^M)_{\underline{cd}} \qquad (\rho^M)_{\underline{ab}} = \frac{1}{2}\epsilon_{\underline{abcd}}(\widetilde{\rho}^M)^{\underline{cd}} \tag{C.17}$$

$$(\rho^{MN})_{\underline{a}}{}^b = \frac{1}{2}\left((\rho^M)_{\underline{ac}}(\widetilde{\rho}^N)^{\underline{cb}} - (\rho^N)_{\underline{ac}}(\widetilde{\rho}^M)^{\underline{cb}}\right) \tag{C.18}$$

C.2 Algebras

OSP(4|6) algebra in Sp(4) × SO(6) basis

$$[O_{\mu\nu}, O_{\rho\sigma}] = C_{\mu\rho}O_{\nu\sigma} + C_{\mu\sigma}O_{\nu\rho} + C_{\nu\rho}O_{\mu\sigma} + C_{\nu\sigma}O_{\mu\rho} \tag{C.19}$$

$$[O_{MN}, O_{KL}] = \delta_{MK}O_{NL} + \delta_{ML}O_{NK} + \delta_{NK}O_{ML} + \delta_{NL}O_{MK} \tag{C.20}$$

$$\{O_{\mu M}, O_{\rho L}\} = i(-\delta_{ML}O_{\mu\rho} + C_{\mu\rho}O_{ML}) \tag{C.21}$$

$$[O_{\mu\nu}, O_{\rho L}] = C_{\mu\rho}O_{\nu L} + C_{\nu\rho}O_{\mu L} \tag{C.22}$$

$$[O_{MN}, O_{\rho L}] = \delta_{ML}O_{\rho N} - \delta_{NL}O_{\rho M} \tag{C.23}$$

with $\mu, \nu = 1, \ldots, 4$ and $M, N = 1, \ldots, 6$.

Change of basis

$$M^{\underline{mn}} = \frac{1}{4}(\gamma^{\underline{mn}})^{\mu\nu}O_{\mu\nu} \qquad O_{\mu\nu} = -\frac{1}{2}(\gamma^{\underline{mn}})_{\mu\nu}M_{\underline{mn}}$$

$$U_{\underline{a}}{}^{\underline{b}} = -\frac{i}{4}(\rho^{MN})_{\underline{a}}{}^{\underline{b}}O_{MN} \qquad O_{MN} = -\frac{i}{2}(\rho^{MN})_{\underline{a}}{}^{\underline{b}}U_{\underline{b}}{}^{\underline{a}} \tag{C.24}$$

$$\mathcal{O}_{\mu\,\underline{ab}} = \frac{1}{2}O_{\mu M}(\rho^M)_{\underline{ab}} \qquad O_{\mu M} = -\frac{1}{2}(\widetilde{\rho}^M)^{\underline{ab}}\mathcal{O}_{\mu\,\underline{ab}}$$

$$\mathcal{O}_{\mu}{}^{\underline{ab}} = \frac{1}{2}O_{\mu M}(\widetilde{\rho}^M)^{\underline{ab}} \qquad O_{\mu M} = -\frac{1}{2}(\rho^M)_{\underline{ab}}\mathcal{O}_{\mu}{}^{\underline{ab}}$$

OSP(4|6) algebra in SO(3, 2) × SU(4) basis

$$[M^{\underline{mn}}, M^{\underline{kl}}] = \eta^{\underline{nk}}M^{\underline{ml}} - \eta^{\underline{mk}}M^{\underline{nl}} - \eta^{\underline{nl}}M^{\underline{mk}} + \eta^{\underline{ml}}M^{\underline{nk}} \tag{C.25}$$

$$[U_{\underline{a}}{}^{\underline{b}}, U_{\underline{c}}{}^{\underline{d}}] = i\left(\delta_{\underline{c}}{}^{\underline{b}}U_{\underline{a}}{}^{\underline{d}} - \delta_{\underline{a}}{}^{\underline{d}}U_{\underline{c}}{}^{\underline{b}}\right) \tag{C.26}$$

$$\{\mathcal{O}_{\mu\,\underline{ab}}, \mathcal{O}_{\nu}{}^{\underline{cd}}\} = \frac{i}{4}\left(\delta_{\underline{a}}{}^{\underline{d}}\delta_{\underline{b}}{}^{\underline{c}} - \delta_{\underline{a}}{}^{\underline{c}}\delta_{\underline{b}}{}^{\underline{d}}\right)(\gamma^{\underline{mn}})_{\mu\nu}M_{\underline{mn}} +$$
$$+ \frac{1}{2}C_{\mu\nu}\left(\delta_{\underline{a}}{}^{\underline{c}}U_{\underline{b}}{}^{\underline{d}} - \delta_{\underline{b}}{}^{\underline{c}}U_{\underline{a}}{}^{\underline{d}} - \delta_{\underline{a}}{}^{\underline{d}}U_{\underline{b}}{}^{\underline{c}} + \delta_{\underline{b}}{}^{\underline{d}}U_{\underline{a}}{}^{\underline{c}}\right) \tag{C.27}$$

$$[M^{\underline{mn}}, \mathcal{O}_{\mu}{}^{\underline{cd}}] = -\frac{1}{2}(\gamma^{\underline{mn}})_{\mu}{}^{\nu}\mathcal{O}_{\nu}{}^{\underline{cd}} \tag{C.28}$$

$$[U_{\underline{a}}{}^{\underline{b}}, \mathcal{O}_{\mu}{}^{\underline{cd}}] = -i\left(\delta_{\underline{a}}{}^{\underline{c}}\mathcal{O}_{\mu}{}^{\underline{bd}} - \delta_{\underline{a}}{}^{\underline{d}}\mathcal{O}_{\mu}{}^{\underline{bc}} - \frac{1}{2}\delta_{\underline{a}}{}^{\underline{b}}\mathcal{O}_{\mu}{}^{\underline{cd}}\right) \tag{C.29}$$

$$[U_{\underline{a}}{}^{\underline{b}}, \mathcal{O}_{\mu\,\underline{cd}}] = i\left(\delta_{\underline{c}}{}^{\underline{b}}\mathcal{O}_{\mu\,\underline{ad}} - \delta_{\underline{d}}{}^{\underline{b}}\mathcal{O}_{\mu\,\underline{ac}} - \frac{1}{2}\delta_{\underline{a}}{}^{\underline{b}}\mathcal{O}_{\mu\,\underline{cd}}\right) \tag{C.30}$$

with $\underline{m}, \underline{n}, \underline{k}, \underline{l} = 0', 0, 1, \ldots, 4$ and $\underline{a}, \underline{b}, \underline{c}, \underline{d} = 1, \ldots, 4$.

Properties

$$M^{\underline{mn}} = (M^{0'm}, M^{mn}) \quad \text{with} \quad \underline{m} = (0', m) \quad \text{and} \quad m = 0, 1, 2, 3$$

$$U_{\underline{a}}{}^{\underline{b}} = (U_a{}^b, U_a{}^4, U_4{}^a) \quad \text{with} \quad \underline{a} = (a, 4) \quad \text{and} \quad a = 1, 2, 3$$

$$\mathcal{O}_\mu{}^{\underline{ab}} = \frac{1}{2} \epsilon^{\underline{abcd}} \mathcal{O}_{\mu\,\underline{cd}}$$

$$\mathcal{O}_{\mu\,\underline{ab}} = \frac{1}{2} \epsilon_{\underline{abcd}} \mathcal{O}_\mu{}^{\underline{cd}} \tag{C.31}$$

$$\mathcal{O}_{\mu\,4a} = -\frac{1}{2} \epsilon_{abc} \mathcal{O}_\mu{}^{bc} \qquad\qquad \mathcal{O}_\mu{}^{ab} = -\epsilon^{abc} \mathcal{O}_{\mu\,4c} \tag{C.32}$$

$$\mathcal{O}_\mu{}^{4a} = -\frac{1}{2} \epsilon^{abc} \mathcal{O}_{\mu\,bc} \qquad\qquad \mathcal{O}_{\mu\,ab} = -\epsilon^{abc} \mathcal{O}_\mu{}^{4c} \tag{C.33}$$

Definitions

$$P^m \equiv M^{0'm} \tag{C.34}$$

$$V_a{}^b \equiv U_a{}^b - \delta_a{}^b U_c{}^c \quad , \qquad V_a \equiv \frac{1}{\sqrt{2}} U_a{}^4 \quad , \qquad V^a \equiv \frac{1}{\sqrt{2}} U_4{}^a \tag{C.35}$$

$$\mathcal{O}_{\alpha a}\Big|_{\alpha=1,2} \equiv \mathcal{O}_{\mu\,4a}\Big|_{\mu=1,2} \quad , \quad \mathcal{O}_\alpha{}^a\Big|_{\alpha=1,2} \equiv \mathcal{O}_\mu{}^{4a}\Big|_{\mu=1,2} \tag{C.36}$$

$$\mathcal{O}^{\dot\alpha}{}_a\Big|_{\dot\alpha=1,2} \equiv \mathcal{O}_{\mu\,4a}\Big|_{\mu=3,4} \quad , \quad \mathcal{O}^{\dot\alpha a}\Big|_{\dot\alpha=1,2} \equiv \mathcal{O}_\mu{}^{4a}\Big|_{\mu=3,4} \tag{C.37}$$

OSP(4|6) algebra in chiral basis

$$[M^{mn}, M^{kl}] = \eta^{nk}M^{ml} - \eta^{mk}M^{nl} - \eta^{nl}M^{mk} + \eta^{ml}M^{nk}$$

$$[M^{mn}, P^{k}] = \eta^{nk}P^{m} - \eta^{mk}P^{n} \tag{C.38}$$

$$[P^{m}, P^{n}] = -M^{mn}$$

$$[V_{a}{}^{b}, V_{c}{}^{d}] = i\left(\delta_{c}{}^{b}V_{a}{}^{d} - \delta_{a}{}^{d}V_{c}{}^{b}\right)$$

$$[V_{a}{}^{b}, V_{c}] = i\left(\delta_{c}{}^{b}V_{a} - \delta_{a}{}^{b}V_{c}\right) \qquad [V_{a}{}^{b}, V^{c}] = -i\left(\delta_{a}{}^{c}V^{b} - \delta_{a}{}^{b}V^{c}\right) \tag{C.39}$$

$$[V_{a}, V^{b}] = \tfrac{i}{2}\left(V_{a}{}^{b} - \delta_{a}{}^{b}V_{c}{}^{c}\right)$$

$$\{\mathcal{O}_{\alpha a}, \mathcal{O}_{\beta b}\} = -\tfrac{1}{\sqrt{2}}\varepsilon_{\alpha\beta}\epsilon_{abc}V^{c} \quad \{\mathcal{O}_{\alpha}{}^{a}, \mathcal{O}_{\beta}{}^{b}\} = \tfrac{1}{\sqrt{2}}\varepsilon_{\alpha\beta}\epsilon^{abc}V_{c}$$

$$\{\mathcal{O}^{\dot\alpha a}, \mathcal{O}^{\dot\beta b}\} = \tfrac{1}{\sqrt{2}}\varepsilon^{\dot\alpha\dot\beta}\epsilon^{abc}V_{c} \quad \{\mathcal{O}^{\dot\alpha}_{a}, \mathcal{O}^{\dot\beta}_{b}\} = -\tfrac{1}{\sqrt{2}}\varepsilon^{\dot\alpha\dot\beta}\epsilon_{abc}V^{c} \tag{C.40}$$

$$\{\mathcal{O}_{\alpha a}, \mathcal{O}^{\dot\beta b}\} = \tfrac{1}{2}\delta_{a}{}^{b}(\sigma^{m})_{\alpha}{}^{\dot\beta}P_{m} \quad \{\mathcal{O}_{\alpha}{}^{a}, \mathcal{O}^{\dot\beta}_{b}\} = \tfrac{1}{2}\delta_{b}{}^{a}(\bar\sigma^{m})^{\dot\alpha}{}_{\beta}P_{m}$$

$$\{\mathcal{O}_{\alpha a}, \mathcal{O}_{\beta}{}^{b}\} = -\tfrac{i}{4}\delta_{a}{}^{b}(\sigma^{mn})_{\alpha\beta}M_{mn} + \tfrac{1}{2}\varepsilon_{\alpha\beta}V_{a}{}^{b}$$

$$\{\mathcal{O}^{\dot\alpha}_{a}, \mathcal{O}^{\dot\beta b}\} = -\tfrac{i}{4}\delta_{a}{}^{b}(\bar\sigma^{mn})^{\dot\alpha\dot\beta}M_{mn} + \tfrac{1}{2}\varepsilon^{\dot\alpha\dot\beta}V_{a}{}^{b} \tag{C.41}$$

$$\{\mathcal{O}_{\alpha a}, \mathcal{O}^{\dot\beta}_{b}\} = 0 \quad \{\mathcal{O}_{\alpha}{}^{a}, \mathcal{O}^{\dot\beta b}\} = 0$$

$$[M^{mn}, \mathcal{O}_{\alpha a}] = -\tfrac{1}{2}(\sigma^{mn})_{\alpha}{}^{\beta}\mathcal{O}_{\beta a} \quad [M^{mn}, \mathcal{O}_{\alpha}{}^{a}] = -\tfrac{1}{2}(\sigma^{mn})_{\alpha}{}^{\beta}\mathcal{O}_{\beta}{}^{a}$$

$$[M^{mn}, \mathcal{O}^{\dot\alpha a}] = -\tfrac{1}{2}(\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta}\mathcal{O}^{\dot\beta a} \quad [M^{mn}, \mathcal{O}^{\dot\alpha}_{a}] = -\tfrac{1}{2}(\bar\sigma^{mn})^{\dot\alpha}{}_{\dot\beta}\mathcal{O}^{\dot\beta}_{a} \tag{C.42}$$

$$[P^{m}, \mathcal{O}_{\alpha a}] = -\tfrac{i}{2}(\sigma^{m})_{\alpha\dot\beta}\mathcal{O}^{\dot\beta}_{a} \quad [P^{m}, \mathcal{O}_{\alpha}{}^{a}] = -\tfrac{i}{2}(\sigma^{m})_{\alpha\dot\beta}\mathcal{O}^{\dot\beta a}$$

$$[P^{m}, \mathcal{O}^{\dot\alpha a}] = -\tfrac{i}{2}(\bar\sigma^{m})^{\dot\alpha\beta}\mathcal{O}_{\beta}{}^{a} \quad [P^{m}, \mathcal{O}^{\dot\alpha}_{a}] = -\tfrac{i}{2}(\bar\sigma^{m})^{\dot\alpha\beta}\mathcal{O}_{\beta a} \tag{C.43}$$

$$[V_{a}{}^{b}, \mathcal{O}_{\alpha c}] = i\delta_{c}{}^{b}\mathcal{O}_{\alpha a} \quad [V_{a}{}^{b}, \mathcal{O}^{\dot\alpha}_{c}] = i\delta_{c}{}^{b}\mathcal{O}^{\dot\alpha}_{a}$$

$$[V_{a}{}^{b}, \mathcal{O}^{\dot\alpha c}] = -i\delta_{a}{}^{c}\mathcal{O}^{\dot\alpha b} \quad [V_{a}{}^{b}, \mathcal{O}_{\alpha}{}^{c}] = -i\delta_{a}{}^{c}\mathcal{O}_{\alpha}{}^{b} \tag{C.44}$$

$$[V_{a}, \mathcal{O}_{\alpha b}] = -\tfrac{i}{\sqrt{2}}\epsilon_{abc}\mathcal{O}_{\alpha}{}^{c} \quad [V_{a}, \mathcal{O}_{\alpha}{}^{b}] = 0$$

$$[V_{a}, \mathcal{O}^{\dot\alpha b}] = 0 \qquad\qquad [V_{a}, \mathcal{O}^{\dot\alpha}_{b}] = -\tfrac{i}{\sqrt{2}}\epsilon_{abc}\mathcal{O}^{\dot\alpha c}$$

$$[V^{a}, \mathcal{O}_{\alpha b}] = 0 \qquad\qquad [V^{a}, \mathcal{O}_{\alpha}{}^{b}] = \tfrac{i}{\sqrt{2}}\epsilon^{abc}\mathcal{O}_{\alpha c} \tag{C.45}$$

$$[V^{a}, \mathcal{O}^{\dot\alpha b}] = \tfrac{i}{\sqrt{2}}\epsilon^{abc}\mathcal{O}^{\dot\alpha}_{c} \quad [V^{a}, \mathcal{O}^{\dot\alpha}_{b}] = 0$$

C.3   Super-Traces

$$\text{STr}(M_{kl}M_{mn}) = \eta_{[kl][mn]} \equiv \eta_{km}\eta_{ln} - \eta_{kn}\eta_{lm} \tag{C.46}$$

$$\text{STr}(P_m P_n) = \eta_{mn} \tag{C.47}$$

$$\text{STr}(V_a{}^b V_c{}^d) = -2\delta_a{}^d \delta_c{}^b \tag{C.48}$$

$$\text{STr}(V_a V^b) = -\delta_a{}^b \tag{C.49}$$

$$\text{STr}(\mathcal{O}_{\alpha a}\mathcal{O}_\beta{}^b) = i\varepsilon_{\alpha\beta}\delta_a{}^b \tag{C.50}$$

$$\text{STr}(\mathcal{O}^{\dot\alpha a}\mathcal{O}^{\dot\beta}{}_b) = i\varepsilon^{\dot\alpha\dot\beta}\delta_b{}^a \tag{C.51}$$

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